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On the Yamabe problem and the scalar curvature problems under boundary conditions

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1. Introduction

In this paper we prove some existence results concerning a problem arising in conformal differential geometry. Consider a smooth metric g on $B = \{x \in \mathbb{R}^n : |x| < 1\}$, the unit ball on $\mathbb{R}^n$, $n \geq 3$, and let Δ_g, R_g, ν_g, h_g denote, respectively, the Laplace-Beltrami operator, the scalar curvature of (B, g), the outward unit normal to $\partial B = S^{n-1}$ with respect to g and the mean curvature of (S^{n-1}, g). Given two smooth functions R' and h', we will be concerned with the existence of positive solutions $u \in H^1(B)$ of

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta_g u + R_g u = R' u^{\frac{n+2}{n-2}}, \text{ in } B; \\ \dfrac{2}{(n-2)}\partial_{\nu_g} u + h_g u = h' u^{\frac{n}{n-2}}, \text{ on } \partial B = S^{n-1}. \end{cases} \tag{1}$$

It is well known that such a solution is C^∞ provided g, R' and h' are, see [10]. If $u > 0$ is a smooth solution of (1) then $g' = u^{4/(n-2)}g$ is a metric, conformally equivalent to g, such that R' and h' are, respectively, the scalar curvature of (B, g') and the mean curvature of (S^{n-1}, g'). Up to a stereographic projection, this is equivalent to finding a conformal metric on the upper half sphere $S^n_+ = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} > 0\}$ such that the scalar curvature of S^n_+ and the mean curvature of $\partial S^n_+ = S^{n-1}$ are prescribed functions.

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In the first part of the paper we deal with the the case in which R' and h' are constant, say $R' \equiv 1$ and $h' \equiv c$, when (1) becomes

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta_g u + R_g u = u^{\frac{n+2}{n-2}}, \text{ in } B; \\ \dfrac{2}{(n-2)}\partial_{\nu_g} u + h_g u = c u^{\frac{n}{n-2}}, \text{ on } \partial B = S^{n-1}. \end{cases} \tag{Y}$$

This will be referred as the *Yamabe like problem* and was first studied in [10–12]. More recently, the existence of a solution of (1) has been proved in [14,15] under the assumption that (B, g) is of positive type (for a definition see [14]) and satisfies one of the following assumptions:

(i) (B, g) is locally conformally flat and ∂B is umbilical;
(ii) $n \geq 5$ and ∂B is not umbilical.

Our main result concerning the *Yamabe like problem* shows that none of (i) or (ii) is required when g is close to the standard metric g_0 on B. Precisely, consider the following class $\mathcal{G}_\varepsilon$ of bilinear forms

$$\mathcal{G}_\varepsilon = \{g \in C^\infty(B) : \|g - g_0\|_{L^\infty(B)} \leq \varepsilon, \|\nabla g\|_{L^n(B)} \leq \varepsilon, \|\nabla g\|_{L^{n-1}(S^{n-1})} \leq \varepsilon\}. \tag{2}$$

Inequalities in (2) hold if for example $\|g - g_0\|_{C^1(B)} \leq \varepsilon$, or if $\|g - g_0\|_{W^{2,n}(B)} \leq \varepsilon$. We will show:

Theorem 1. *Given $M > 0$ there exists $\varepsilon_0 > 0$ such that for every ε with $\varepsilon \in (0, \varepsilon_0)$, for every $c > -M$ and for every metric $g \in \mathcal{G}_\varepsilon$ problem (Y) possesses a positive solution.*

In the second part of the paper we will take $g = g_0$, $R' = 1 + \varepsilon K(x)$, $h' = c + \varepsilon h(x)$ and consider the *Scalar Curvature like problem*

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta u = (1 + \varepsilon K(x)) u^{\frac{n+2}{n-2}}, \text{ in } B; \\ \dfrac{2}{(n-2)}\dfrac{\partial u}{\partial \nu} + u = (c + \varepsilon h(x))\, u^{\frac{n}{n-2}}, \text{ on } S^{n-1}, \end{cases} \tag{P_ε}$$

where $\nu = \nu_{g_0}$. The *Scalar Curvature like problem* has been studied in [16] where a non perturbative problem like

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta u = R'(x) u^{\frac{n+2}{n-2}}, \text{ in } B; \\ \dfrac{2}{(n-2)}\dfrac{\partial u}{\partial \nu} + u = 0, \text{ on } S^{n-1}, \end{cases}$$

has been considered. We also mention the paper [9] dealing with the existence of solutions of

$$\begin{cases} \Delta u = 0, \text{ in } B; \\ \dfrac{2}{(n-2)}\dfrac{\partial u}{\partial \nu} + u = (1+\varepsilon h(x))\, u^{\frac{n}{n-2}}, \text{ on } S^{n-1}, \end{cases} \tag{3}$$

a problem similar in nature to (P_ε).
To give an idea of the existence results we can prove, let us consider the particular cases that either $h \equiv 0$ or $K \equiv 0$. In the former, problem (P_ε) becomes

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta u = (1+\varepsilon K(x))u^{\frac{n+2}{n-2}}, \text{ in } B; \\ \dfrac{2}{(n-2)}\dfrac{\partial u}{\partial \nu} + u = c\, u^{\frac{n}{n-2}}, \text{ on } S^{n-1}, \end{cases} \tag{$P_{\varepsilon,K}$}$$

Theorem 2. *Suppose that K satisfies*

(K_1) *there exists an absolute maximum (resp. minimum) p of $K|_{S^{n-1}}$ such that $K'(p)\cdot p < 0$, resp. $K'(p)\cdot p > 0$.*

Then for $|\varepsilon|$ sufficiently small, $(P_{\varepsilon,K})$ has a positive solution.

Another kind of result is the following

Theorem 3. *Let $K|_{S^{n-1}}$ be a Morse function and satisfies*

$$K'(x)\cdot x \neq 0, \qquad \forall x \in Crit(K|_{S^{n-1}}) \tag{K_2}$$

$$\sum_{x\in Crit(K|_{S^{n-1}}):K'(x)\cdot x<0} (-1)^{m(x,K)} \neq 1, \tag{K_3}$$

where $m(x, K)$ is the Morse index of $K|_{S^{n-1}}$ at x. Then for $|\varepsilon|$ sufficiently small, problem $(P_{\varepsilon,K})$ has a positive solution.

When $K \equiv 0$ problem (P_ε) becomes

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta u = u^{\frac{n+2}{n-2}}, & \text{in } B; \\ \dfrac{2}{(n-2)}\dfrac{\partial u}{\partial \nu} + u = (c+\varepsilon h(x))\, u^{\frac{n}{n-2}}, & \text{on } S^{n-1}. \end{cases} \tag{$P_{\varepsilon,h}$}$$

Theorem 4. *Let $h \in C^\infty(S^{n-1})$ be a Morse function satisfying:*

$$\Delta_T h(x) \neq 0, \qquad \forall\, x \in Crit(h); \tag{h_1}$$

$$\sum_{x \in Crit(h): \Delta_T h(x) < 0} (-1)^{m(x,h)} \neq 1, \tag{h_2}$$

Then for $|\varepsilon|$ sufficiently small, problem ($P_{\varepsilon,h}$) has a positive solution.

The preceding results are particular cases of more general ones, dealing with problem (P_ε), where assumptions on a suitable combination of K and h are made. See Theorems 6 and 7 later on. For a comparison with the results of [9, 16], we refer to Remarks 5 and 6 in Sect. 4.

Solutions of the preceding problems are critical points of the energy functional $I^c = I^c_g : H^1(B) \to \mathbb{R}$,

$$\begin{aligned} I^c(u) = {} & 2\frac{(n-1)}{(n-2)} \int_B |\nabla_g u|^2 dV_g + \frac{1}{2} \int_B R_g u^2 dV_g - \frac{1}{2^*} \int_B R' u^{2^*} dV_g \\ & + (n-1) \int_{\partial B} h_g u^2 d\sigma_g - c(n-2) \int_{\partial B} h' |u|^{2\frac{n-1}{n-2}} d\sigma_g. \end{aligned} \tag{4}$$

In all the cases we will deal with, I^c can be written in the form $I^c(u) = I^c_0(u) + O(\varepsilon)$, where

$$\begin{aligned} I^c_0(u) = {} & 2\frac{(n-1)}{(n-2)} \int_B |\nabla u|^2 dx + (n-1) \\ & \times \int_{\partial B} u^2 d\sigma - \frac{1}{2^*} \int_B |u|^{2^*} dx - c(n-2) \int_{S^{n-1}} |u|^{2\frac{n-1}{n-2}} d\sigma \end{aligned}$$

and can be faced by means of a perturbation method in critial point theory discussed in [1]. First, in Sect. 2, we show that I^c_0 has a finite dimensional manifold $Z^c \simeq B$ of critical points that is *non degenerate*, in the sense of [1], see Lemma 3. This allows us to perform a finite dimensional reduction (uniformly with respect to $c \geq -M$) that leads to seeking the critical points of I^c constrained to Z^c. The proof of Theorem 1 is carried out in Sect. 3 and is mainly based upon the study of $I^c_{|Z^c}$. The lack of compactness inherited by I^c is reflected on the fact that Z^c is not closed. This difficulty is overcome using arguments similar to those emploied in [3,7]: we show that I^c can be extended to the boundary ∂Z^c and there results $I^c_{|\partial Z^c} \equiv const.$, see Proposition 2.

In Sect. 4 we deal with the *Scalar Curvature like* problem. In this case there results $I^c(u) = I^c_0(u) + \varepsilon G(u)$, where G depends upon K and h only, and one is lead to study the finite dimensional auxiliary functional $\Gamma = G_{|Z^c}$. More precisely, following the approach of [2], we evaluate Γ on ∂Z^c, together with its first and second derivative. This permits to prove some general existence results which contain as particular cases Theorems 2, 3 and 4. The last part of Sect. 4 is devoted to a short discussion of the case in which K, h inherit a simmetry. For example, if K and h are even functions, (P_ε) has always a solution provided ε is small, without any further assumption, see Theorem 8.

Finally, in the Appendix we prove some technical Lemmas.

The main results of this paper has been annouced in [5].

Notation

B denotes the unit ball in $\mathbb{R}^n$, centered at $x = 0$.

We will work mainly in the functional space $H^1(B)$. In some cases it will be convenient to equip $H^1(B)$ with the scalar product

$$(u, v)_1 = 4\frac{(n-1)}{(n-2)} \int_B \nabla u \cdot \nabla v dx + 2(n-1) \int_{\partial B} uv d\sigma,$$

that gives rise to the norm $\|u\|_1^2 = (u, u)_1$, equivalent to the standard one.

If E is an Hilbert space and $f \in C^2(E, \mathbb{R})$ is a functional, we denote by f' or ∇f its gradient; $f''(u) : E \to E$ is the linear operator defined by duality in the following way

$$(f''(u)v, w) = D^2 f(u)[v, w], \qquad \forall v, w \in E.$$

σ_S denotes the stereographic projection $\sigma_S : S^n = \left\{x \in \mathbb{R}^{n+1} |\ |x| = 1\right\} \to \mathbb{R}^n$ trough the south pole, where we identify $\mathbb{R}^n$ with $\left\{x \in \mathbb{R}^{n+1} | x_{n+1} = 0\right\}$.

More in general, given $p \in S^n$, we denote by $\sigma_p : \mathbb{R}^n \to S^n$ the stereographic projection trough the point p.

The stereographic projections give rise to some isometries in the following way. The projection trough the south pole S of S^n gives rise to the isometry $\tau_S : H^1(S^n) \to H^1(B)$

$$\tau_S u(x) = \frac{2}{1 + |x|^2} u(\sigma_S^{-1} x), \qquad x \in B.$$

Moreover, given $p \in \partial S^n_+$, the stereographic projection trough p gives rise to the isometry $\tau_p : H^1(S^n_+) \to E = \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ given by

$$\tau_p u(x) = \frac{2}{1 + |x|^2} u(\sigma_p^{-1} x), \qquad x \in \mathbb{R}^n_+.$$

2. The unperturbed problem

When $\varepsilon = 0$, resp. $g = g_0$, problem (P_ε), resp. (Y), coincides with the unperturbed problem

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta u = u^{\frac{n+2}{n-2}}, & \text{in } B; \\ \dfrac{2}{(n-2)}\partial_\nu u + u = cu^{\frac{n}{n-2}}, & \text{on } \partial B = S^{n-1}. \end{cases} \tag{UP}$$

Solutions of problem (UP) can be found as critical points of the functional $I_0^c : H^1(B) \to \mathbb{R}$ defined as

$$I_0^c(u) = \frac{1}{2}\|u\|_1^2 - \frac{1}{2^*}\int_B |u|^{2^*} dx - c(n-2)\int_{S^{n-1}} |u|^{2\frac{n-1}{n-2}} d\sigma.$$

Consider the function $z_0 : \mathbb{R}^n \to \mathbb{R}$,

$$z_0(x) = \left(\frac{\kappa}{1+|x|^2}\right)^{\frac{n-2}{2}}; \qquad \kappa = \kappa_n = (4n(n-1))^{\frac{1}{2}}.$$

The function z_0 is the unique solution (up to translation and dilation) to the problem in $\mathbb{R}^n$

$$-4\frac{(n-1)}{(n-2)}\Delta u = u^{\frac{n+2}{n-2}}, \quad \text{in } \mathbb{R}^n; \qquad u > 0.$$

We also set

$$z_{\mu,\xi} = \mu^{-\frac{n-2}{2}} z_0((x-\xi)/\mu), \quad z_\mu = \mu^{-\frac{n-2}{2}} z_0(x/\mu).$$

By a stright calculation it follows that $z_{\mu,\xi}$ is a critical points of I_0^c, namely solutions of the problem (UP), iff

$$\mu^2 + |\xi|^2 - c\kappa\mu - 1 = 0, \quad \mu > 0. \tag{5}$$

The set

$$Z^c = \{z_{\mu,\xi} : \mu^2 + |\xi|^2 - c\kappa\mu - 1 = 0\} \tag{6}$$

is an n-dimensional manifold, diffeomorphic to a ball in $\mathbb{R}^n$, with boundary ∂Z^c corresponding to the parameter values $\mu = 0$, $|\xi| = 1$.

We need to study the eigenvalues of $I_0''(z_{\mu,\xi})$, with $z_{\mu,\xi} \in Z^c$. Recall that, by definition, $\lambda \in \mathbb{R}$ is an eigenvalue of $I_0''(z_{\mu,\xi})$ if there exists $v \in H^1(B)$, $v \neq 0$ such that $I_0''(z_{\mu,\xi})[v] = \lambda v$ and this means that v is solution of the linear problem

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}(1-\lambda)\,\Delta v = \dfrac{n+2}{n-2} z_{\mu,\xi}^{\frac{4}{n-2}} v, & \text{in } B; \\ 4\dfrac{(n-1)}{(n-2)}(1-\lambda)\,\partial_\nu v = 2(n-1)\left(c\dfrac{n}{(n-2)} z_{\mu,\xi}^{\frac{2}{n-2}} + \lambda - 1\right) v, & \text{on } S^{n-1}. \end{cases} \tag{7}$$

The following lemma is well known.

Lemma 1. (a) $\lambda = 0$ *is an eigenvalue of (7) and the corresponding eigenspace is n dimensional and coincides with the tangent space to Z^c at $z_{\mu,\xi}$, namely is spanned by $Dz_{\mu,\xi}$.*
(b) *(7) has precisely one negative eigenvalue $\lambda_1(c)$; all the remaining eigenvalues are positive.*

Item (a) is proved in [14]. Item (b) easily follows from the fact that $z_{\mu,\xi}$ is a Mountain Pass critical point of I_0^c.

Let $\lambda_2(c)$ denote the smallest positive eigenvalue of $I_0''(z_{\mu,\xi})$.

The main result of this section is the following one:

Lemma 2. *For all $M > 0$ there exists a positive constant C_M such that*

$$\frac{1}{C_M} \le |\lambda_i(c)| \le C_M, \quad \forall\, c \ge -M, \quad i = 1, 2.$$

Remark. There is a numerical evidence that $\lambda_2(c) \downarrow 0$ as $c \downarrow -\infty$.

Proof. We will prove separately that $|\lambda_i(c)| \le C_M$ and that $\frac{1}{C_M} \le |\lambda_i(c)|$. For symmetry reasons it is sufficient to take $z_{\mu,\xi} = z_\mu$, namely to take $\xi = 0$. In such a case μ depends only on ξ and (5) yields

$$\mu(c) = \frac{1}{2}\left(\kappa c + \sqrt{\kappa^2 c^2 + 4}\right).$$

Case 1. $|\lambda_i(c)| \le C_M$. By contradiction suppose there exists a sequence $c_j \to +\infty$ such that $|\lambda_i(c_j)| \to +\infty$, $i = 1, 2$. Let v_j denote an eigenfunction of (7) with $\lambda = \lambda_i(c_j)$. Then v_j solves the problem

$$\begin{cases} \Delta v_j = a_j(x) v_j, & \text{in } B; \\ \partial_\nu v_j = b_j(x) v_j, & \text{on } S^{n-1}, \end{cases} \tag{8}$$

where

$$a_j(x) = \frac{1}{(\lambda_i(c_j)-1)}\frac{n+2}{4(n-1)} z_{\mu(c_j)}^{\frac{4}{n-2}}(x), \quad x \in B$$

$$b_j(x) = \frac{n-2}{2(1-\lambda_i(c_j))}\left(c_j \frac{n}{(n-2)} z_{\mu(c_j)}^{\frac{2}{n-2}}(x) + \lambda_i(c_j) - 1\right), \quad x \in S^{n-1}.$$

Above, it is worth pointing out that b_j is constant on S^{n-1}. Actually, there results

$$z_{\mu}^{\frac{2}{n-2}}(x) = \kappa\,\mu^{-1}\left(1+\frac{1}{\mu^2}\right)^{-1}, \quad \forall\, x \in S^{n-1},$$

and hence

$$b_j \equiv \frac{n-2}{2(1-\lambda_i(c_j))}\left(c_j \frac{n}{(n-2)} \cdot \kappa\,\mu^{-1}(c_j)\left(1+\frac{1}{\mu^2(c_j)}\right)^{-1} + \lambda_i(c_j) - 1\right), \quad \forall\, x \in S^{n-1}.$$

Moreover, since $\mu \sim \kappa c$ as $c \to +\infty$, it turns out that

$$b_j \to -\frac{(n-2)}{2}. \tag{9}$$

Now, integrating by parts we deduce from (8)

$$\int_B |\nabla v_j|^2 dx + \int_B a_j v_j^2 dx = b_j \int_{S^{n-1}} v_j^2 d\sigma. \tag{10}$$

Using (9) and a Poincaré-like inequality, we find there exists $C > 0$ [1]

$$-\int_B a_j v_j^2 dx \geq C \int_B v_j^2 dx.$$

This leads to a contradiction because $a_j(x) \to 0$ in $C^0(\overline{B})$ and $v_j \not\equiv 0$.

Case 2. $\frac{1}{C_M} \leq |\lambda_i(c)|$. Arguing again by contradiction, let $c_j \to +\infty$ and suppose that $|\lambda_i(c_j)| \to 0$. As before, the corresponding eigenfunctions v_j satisfy (10), where now $b_j \to 1$, because $\mu \sim \kappa c$ and $|\lambda_i(c_j)| \to 0$. Choosing v_j is such a way that $\sup_B |v_j| = 1$, then (10) yields that v_j is bounded in $H^1(B)$ and hence $v_j \rightharpoonup v_0$ weakly in $H^1(B)$. Passing to the limit in

$$\int_B \nabla v_j \cdot \nabla w + \int_B a_j v_j w - \int_{S^{n-1}} b_j v_j w = 0, \qquad \forall w \in H^1(B),$$

[1] in the sequel we will use the same symbol C to denote possibly different positive constants.

it immedately follows that v_0 satisfies

$$\begin{cases} \Delta v_0 = 0, & \text{in } B; \\ \partial_\nu v_0 = v_0, & \text{on } S^{n-1}. \end{cases} \tag{P_3}$$

The solutions of problem (P_3) are explicitly known, namely they are the linear functions an B. We denote by X the vector space of these solutions, which is n-dimensional. To complete the proof we will show that $v_0 \in X$ leads to a contradiction. We know that $\lambda = 0$ is an eigenvalue with multiplicity n, and the eigenvectors corresponding to $\lambda = 0$ are precisely the elements of $T_{z_\mu} Z^c$. Let $u_j \in T_{z_\mu(c_j)} Z^c$ with $\sup_B |u_j| = 1$. Then, by using simple computations, one can prove that, up to a subsequence, $u_j \to v$ strongly in $H^1(B)$ for some function $v \in X$. We can assume w.l.o.g. that $v = v_0$ (the weak limit of v_j), so it follows that $(u_j, v_j) \to \|v_0\|^2 \neq 0$. But this is not possible, since v_j are eigenvectors corresponding to $\lambda_1 < 0$, while u_j are eigenvectors corresponding to $\lambda = 0$ and hence they are orthogonal. □

In conclusion, taking into account of Lemma 2, we can state:

Lemma 3. *The unperturbed functional I_0^c possesses an n-dimensional manifold Z^c of critical points, diffeomorphic to a ball of $\mathbb{R}^n$. Moreover I_0^c satisfies the following properties*

(i) $I_0''(z) = I - \mathcal{K}$, *where $\mathcal{K}$ is a compact operator for every $z \in Z^c$;*
(ii) $T_z Z^c = Ker\, D^2 I_0^c(z)$ *for all $z \in Z^c$.*

From (i)-(ii) it follows that the restriction of $D^2 I_0^c$ to $(T_z Z^c)^\perp$ is invertible. Moreover, denoting by $L_c(z)$ its inverse, for every $M > 0$ there exists $C > 0$ such that

$$\|L_c(z)\| \leq C \quad \textit{for all} \quad z \in Z^c \quad \textit{and for all } c > -M. \tag{11}$$

3. The Yamabe like problem

3.1. Preliminaries

Solution s of problem (1) can be found as critical points of the functional $I^c : H^1(B) \to \mathbb{R}$ defined in (4).

We recall some formulas from [3] which will be useful for our computations. We denote with g_{ij} the coefficients of the metric g in some local co-ordinates and with g^{ij} the elements of the inverse matrix $(g^{-1})_{ij}$.

The volume element dV_g of the metric $g \in \mathcal{G}_\varepsilon$, taking into account (2) is

$$dV_g = |g|^{\frac{1}{2}} \cdot dx = (1 + O(\varepsilon)) \cdot dx^2. \tag{12}$$

The Christoffel symbols are given by $\Gamma^l_{ij} = \frac{1}{2}[D_i g_{kj} + D_j g_{ki} - D_k g_{ij}]g^{kl}$. The components of the Riemann tensor, the Ricci tensor and the scalar curvature are, respectively

$$R^l_{kij} = D_i \Gamma^l_{jk} - D_j \Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}; \;\; R_{kj} = R^l_{klj}; \;\; R = R_g = R_{kj} g^{kj}. \tag{13}$$

For a smooth function u the components of $\nabla_g u$ are $(\nabla_g u)^i = g^{ij} D_j u$, so

$$(\nabla_g u)^i = \nabla u \cdot (1 + O(\varepsilon)). \tag{14}$$

From the preceding formulas and from the fact that $g \in \mathcal{G}_\varepsilon$ it readily follows that $I^c(u) = I^c_0(u) + O(\varepsilon)$. More precisely, the following lemma holds. The proof is rather technical and is postponed to the Appendix.

Lemma 4. *Given $M > 0$ there exists $C > 0$ such that for $c > -M$ and $g \in \mathcal{G}_\varepsilon$ there holds*

$$\|\nabla I^c(z)\| \leq C \cdot \varepsilon \cdot (1 + |c|)^{-\frac{n-2}{2}}, \quad \forall z \in Z^c; \tag{15}$$

$$\left\| D^2 I^c(z) - D^2 I^c_0(z) \right\| \leq C \cdot \varepsilon, \quad \forall z \in Z^c \tag{16}$$

$$\begin{aligned} &\left\| I^c(z + w) - I^c(z + w') \right\| \leq C \cdot (1 + |c|) \cdot \\ &(\varepsilon + \rho^{\frac{2}{n-2}}) \cdot \|w - w'\|, \quad \forall z \in Z^c, w, w' \in H^1(B), \quad \forall \|w\|, \|w'\| \leq \rho; \end{aligned} \tag{17}$$

$$\begin{aligned} &\left\| \nabla I^c(u + w) - \nabla I^c(u) \right\| \leq C \cdot \|w\| \cdot \\ &\left(1 + \|u\|^{\frac{4}{n-2}} + \|w\|^{\frac{4}{n-2}} + |c| \cdot \|u\|^{\frac{2}{n-2}} + |c| \cdot \|w\|^{\frac{2}{n-2}}\right), \quad \forall u, w \in H^1(B). \end{aligned} \tag{18}$$

Moreover, if $\|u\|$ is uniformly bounded and if $\|w\| \leq 1$ there results

$$\left\| D^2 I^c(u + w) - D^2 I^c(u) \right\| \leq C \cdot (1 + |c|) \cdot \|w\|^{\frac{2}{n-2}}. \tag{19}$$

3.2. *A finite dimensional reduction*

The aim of this section is to perform a finite dimensional reduction, using Lemma 3. Arguments of this kind has been emploied, e.g. in [1]. The first step is to construct, for $g \in \mathcal{G}_\varepsilon$, a perturbed manifold $Z^c_g \simeq Z^c$ which is a *natural constraint* for I^c, namely: if $u \in Z^c_g$ and $\nabla I^c|_{Z^c_g}(u) = 0$ then $\nabla I^c(u) = 0$.

For brevity, we denote by $\dot{z} \in H^1(B))^n$ an orthonormal n-tuple in $T_z Z^c$. Moreover, if $\alpha \in \mathbb{R}^n$ we set $\alpha\dot{z} = \sum \alpha_i \dot{z}_i$.

Proposition 1. *Given $M > 0$, there exist $\varepsilon_0, C > 0$, such that $\forall\, c > -M$, $\forall\, z \in Z^c$*

$\forall\, \varepsilon \le \varepsilon_0$ and $\forall\, g \in \mathcal{G}_\varepsilon$ there are

C^1 functions $w = w(z, g, c) \in H^1(B)$ and $\alpha = \alpha(z, g, c) \in \mathbb{R}^n$ such that the following properties hold

(i) w *is orthogonal to* $T_z Z^c \quad \forall z \in Z^c$, *i.e.* $(w, \dot{z}) = 0$;
(ii) $\nabla I^c(z + w) = \alpha\, \dot{z} \quad \forall z \in Z^c$;
(iii) $\|(w, \alpha)\| \le C \cdot \varepsilon \cdot (1 + |c|)^{-\frac{n-2}{2}} \quad \forall z \in Z^c$.

Furthermore, from (i)-(ii) it follows that

(iv) the manifold $Z^c_g = \{z + w(z, g, c) \mid z \in Z^c\}$ *is a natural constraint for* I^c.

Proof. Let us define [3] $H_g : Z^c \times H^1(B) \times \mathbb{R}^n \to H^1(B) \times \mathbb{R}^n$ by setting

$$H_g(z, w, \alpha) = \begin{pmatrix} \nabla I^c(z + w) - \alpha\dot{z} \\ (w, \dot{z}) \end{pmatrix}.$$

With this notation, the unknown (w, α) can be implicitly defined by the equation $H_g(z, w, \alpha) = (0, 0)$. Setting $R_g(z, w, \alpha) = H_g(z, w, \alpha) - \partial_{(w,\alpha)} H_g(z, 0, 0)$ $[(w, \alpha)]$ we have that

$$H_g(z, w, \alpha) = 0 \quad \Leftrightarrow \quad \partial_{(w,\alpha)} H_g(z, 0, 0)[(w, \alpha)] + R_g(z, w, \alpha) = 0.$$

Let $H_0 = H_{g_0}$. From (11) it follows easily that $\partial_{(w,\alpha)} H_0(z, 0, 0)$ is invertible uniformly w.r.t. $z \in Z^c$ and $c > -M$. Moreover using (16) it turns out that for ε_0 sufficiently small and for $\varepsilon \le \varepsilon_0$ also the operator $\partial_{(w,\alpha)} H_g(z, 0, 0)$ is invertible and has uniformly bounded inverse, provided $g \in \mathcal{G}_\varepsilon$. Hence, for such g there results

$$\begin{aligned} H_g(z, w, \alpha) &= 0 \Leftrightarrow (w, \alpha) = F_{z,g}(w, \alpha) \\ &\qquad := -\left(\partial_{(w,\alpha)} H_g(z, 0, 0)\right)^{-1} R_g(z, w, \alpha). \end{aligned}$$

We prove the Proposition by showing that the map $F_{z,g}$ is a contraction in some ball $B_\rho = \{(w, \alpha) \in H^1(B) \times \mathbb{R}^n \ : \ \|w\| + |\alpha| \le \rho\}$, with ρ of order

[3] H depends also on c, but such a dependence will be understood.

$\rho \sim \varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}}$. We first show that there exists $C > 0$ such that for all $(w, \alpha), (w', \alpha') \in B_\rho$, all $z \in Z^c$ and all $g \in \mathcal{G}_\varepsilon$, there holds

$$\begin{cases} \|F_{z,g}(w,\alpha)\| \leq C \cdot \left(\varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}} + (1+|c|) \cdot \rho^{\frac{n}{n-2}}\right), \\ \|F_{z,g}(w',\alpha') - F_{z,g}(w,\alpha)\| \leq C \cdot (1+|c|) \cdot \rho^{\frac{2}{n-2}} \cdot \|(w,\alpha)-(w',\alpha')\|. \end{cases} \tag{20}$$

Condition (20) is equivalent to the following two inequalities

$$\|\nabla I^c(z+w) - D^2 I^c(z)[w]\| \leq C \cdot \left(\varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}} + (1+|c|) \cdot \rho^{\frac{2}{n-2}}\right); \tag{21}$$

$$\begin{aligned} \|(\nabla I^c(z+w) - D^2 I^c(z)[w]) - (\nabla I^c(z+w') - D^2 I^c(z)[w'])\| \leq \\ C \cdot (1+|c|) \cdot \rho^{\frac{2}{n-2}} \cdot \|(w,\alpha)-(w',\alpha')\|. \end{aligned} \tag{22}$$

Let us first prove (21). There holds

$$\begin{aligned} \nabla I^c(z+w) - D^2 I^c(z)[w] &= \nabla I^c(z+w) - \nabla I^c(z) + \nabla I^c(z) - D^2 I^c(z)[w] \\ &= \nabla I^c(z) + \int_0^1 \left(D^2 I^c(z+sw) - D^2 I^c(z)\right)[w] ds. \end{aligned}$$

Hence it turns out that

$$\|\nabla I^c(z+w) - D^2 I^c(z)[w]\| \leq \nabla I^c(z) + \|w\| \cdot \sup_{s\in[0,1]} \|D^2 I^c(z+sw) - D^2 I^c(z)\|.$$

Using (19) we have

$$\|\nabla I^c(z+w) - D^2 I^c(z)[w]\| \leq \nabla I^c(z) + C \cdot (1+|c|) \cdot \rho^{\frac{n}{n-2}}.$$

Hence from (15) we deduce that

$$\|\nabla I^c(z+w) - D^2 I^c(z)[w]\| \leq C \cdot \left(\varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}} + (1+|c|) \cdot \rho^{\frac{n}{n-2}}\right),$$

and (21) follows. We turn now to (22). There holds

$$\begin{aligned} \|\nabla I^c(z+w) &- \nabla I^c(z+w') - D^2 I^c(z)[w-w']\| \\ &= \left\| \int_0^1 \left(D^2 I^c(z+w+s(w'-w)) - D^2 I^c(z)\right)[w'-w] ds \right\| \\ &\leq \sup_{s\in[0,1]} \|D^2 I^c(z+w+s(w'-w)) - D^2 I^c(z)\| \cdot \|w'-w\|. \end{aligned}$$

Using again (19), and taking $\|w\|, \|w'\| \leq \rho$ we have that

$$||D^2 I^c(z+w'+s(w-w')) - D^2 I^c(z)|| \leq C \cdot (1+|c|) \cdot \rho^{\frac{2}{n-2}},$$

proving (22). Taking $\rho = 2C \cdot \varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}}$ and $\varepsilon \le \varepsilon_0$, with ε_0 sufficiently small, there results

$$\begin{cases} C \cdot \left(\varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}} + (1+|c|) \cdot \rho^{\frac{n}{n-2}} \right) < \rho, \\ C \cdot (1+|c|) \cdot \rho^{\frac{2}{n-2}} < 1. \end{cases}$$

Then $F_{z,g}$ is a contraction in B_ρ and hence $H_g = 0$ has a unique solution $w = w(z, g, c)$, $\alpha = \alpha(z, g, c)$ with $\|(w, \alpha)\| \le 2C \cdot \varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}}$. □

Remark 1. In general, the preceding arguments give rise to the following result, see [1]. Let $I_\varepsilon(u) = I_0(u) + O(\varepsilon)$ denote a C^2 functional and suppose that I_0 has an n-dimensional manifold Z of critical points satisfying $(i) - (ii)$ of Lemma 3. Then for $|\varepsilon|$ small there exists a unique $w = w_\varepsilon(z)$ satisfying $(i) - (ii) - (iii)$ of Proposition 1. Furthermore, the manifold $Z_\varepsilon = \{z + w_\varepsilon(z) : z \in Z\}$ is a natural constraint for I_ε. Hence any critical point of $I_\varepsilon(z + w_\varepsilon(z))$, $z \in Z$ is a critical point of I_ε.

3.3. *Proof of Theorem 1*

Throughout this section we will take ε and c is such a way that Proposition 1 applies. The main tool to prove Theorem 1 is the following Proposition

Proposition 2. *There results*

$$\lim_{\mu \to 0} I^c(z_{\mu,\xi} + w_g(z_{\mu,\xi})) = b_c, \qquad \textit{uniformly for } \xi \textit{ satisfying } (5). \tag{23}$$

Hence $I^c|_{Z_g^c}$ can be continuously extended to ∂Z_g^c by setting

$$I^c|_{\partial Z_g^c} = b_c. \tag{24}$$

Postponing the proof of Proposition 2, it is immediate to deduce Theorem 1.

Proof of Theorem 1. The extended functional I^c has a critical point on the compact manifold $Z_g^c \cup \partial Z_g^c$. From (24) it follows that either I^c is identically constant or it achieves the maximum or the minimum in Z_g^c. In any case I^c has a critical point on Z_g^c. According to Proposition 1, such a critical point gives rise to a solution of (Y). □

In order to prove Proposition 2 we prefer to reformulate (Y) in a more convenient form using the stereographic projection σ_p, trough an appropriate point $p \in \partial S^n_+$, see Remark 3. In this way the problem reduces to study an elliptic equation in $\mathbb{R}^n_+$, where calculation are easier. More precisely, let $\tilde{g}_{ij} : \mathbb{R}^n_+ \to \mathbb{R}$ be the components of the metric g in σ_p-stereographic co-ordinates, and let

$$\overline{g}_{ij} = \left(\frac{1+|x|^2}{2} \right)^2 \tilde{g}_{ij}. \tag{$\overline{g}$}$$

Then problem (Y) is equivalent to find solutions of

$$\begin{cases} -4\dfrac{(n-1)}{(n-2)}\Delta_{\overline{g}}u + R_{\overline{g}}u = u^{\frac{n+2}{n-2}}, & \text{in } \mathbb{R}^n_+; \\ \dfrac{2}{(n-2)}\partial_{\nu_{\overline{g}}}u + h_{\overline{g}}u = cu^{\frac{n}{n-2}}, & \text{on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}, \\ u > 0, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n_+), \end{cases} \tag{$\overline{Y}$}$$

where the symbols have obvious meaning. Solutions of problem $(\overline{Y})$ can be found as critical points of the functional $f_{\overline{g}} : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \to \mathbb{R}$ defined in the following way

$$f_{\overline{g}}(u) = 2\frac{(n-1)}{(n-2)}\int_{\mathbb{R}^n_+} |\nabla_{\overline{g}}u|^2 \, dV_{\overline{g}} + \frac{1}{2}\int_{\mathbb{R}^n_+} R_{\overline{g}}\, u^2 \, dV_{\overline{g}} - \frac{1}{2^*}\int_{\mathbb{R}^n_+} u^{2^*} dV_{\overline{g}}$$
$$+ (n-1)\int_{\partial\mathbb{R}^n_+} h_{\overline{g}}\, u^2 \, d\sigma_{\overline{g}} - c(n-2)\int_{\partial\mathbb{R}^n_+} |u|^{2\frac{n-1}{n-2}} \, d\sigma_{\overline{g}}.$$

In general the transformation $(\overline{g})$ induces an isometry between $H^1(B)$ and $\mathcal{D}^{1,2}(\mathbb{R}^n_+)$ given by

$$u(x) \mapsto \overline{u}(x) := \left(\frac{2}{(x')^2 + (x_n+1)^2}\right)^{\frac{n-2}{2}}$$
$$\times \; u\left(\frac{2x'}{(x')^2 + (x_n+1)^2}, \frac{(x')^2 + x_n^2 - 1}{(x')^2 + (x_n+1)^2}\right),$$

where $x' = (x_1, \dots, x_{n-1})$.
It turns out that

$$f_{\overline{g}}(\overline{u}) = I^c(u) \tag{25}$$

as well as

$$\nabla f_{\overline{g}}(\overline{u}) = \nabla I^c(u).$$

In particular this implies that u solves (Y) if and only if $\overline{u}$ is a solution of $(\overline{Y})$.

Furthermore, there results

- g_0 corresponds to the trivial metric δ_{ij} on $\mathbb{R}^n_+$;
- z_0 corresponds to $\overline{z}_0 \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ given by

$$\overline{z}_0(x) = z_0(x - (0, a_0c)), \qquad x \in \mathbb{R}^n_+; \qquad a_0 = \frac{\kappa}{2};$$

- Z^c corresponds to $\overline{Z}^c$ given by

$$\overline{Z}^c = \left\{\overline{z}_{\mu,\xi'} := \mu^{-\frac{n-2}{2}} z_0\left(\frac{x - (\xi', a_0c\mu)}{\mu}\right), \mu > 0, \xi' \in \mathbb{R}^{n-1}\right\}.$$

Let us point out that the manifold $\overline{Z}^c$ is nothing but $\tau_p \circ \tau_S^{-1} Z^c$ (see Notations).

From the preceding items it follows that the equation

$$\nabla f_{\overline{g}}(\overline{z} + \overline{w}) \in T_{\overline{z}} \overline{Z}^c,$$

have a unique solution $\overline{w} \perp T_{\overline{z}} \overline{Z}^c$ and there results

$$\overline{w}_{\overline{g}}(\overline{z}) = \overline{w_g(z)}.$$

From this and (25) it follows

$$I^c(z + w_g(z)) = f_{\overline{g}}(\overline{z} + \overline{w}_{\overline{g}}(\overline{z})). \tag{26}$$

Let us now introduce the metric $\overline{g}^\delta(x) := \overline{g}(\delta x),\ \delta > 0$ and let $f_{\overline{g}^\delta} : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \to \mathbb{R}$ be the corresponding Euler functional. For all $u \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ there results

$$f_{\overline{g}^\delta}(u) = f_{\overline{g}}\left(\delta^{\frac{2-n}{2}} u(\delta^{-1}x)\right).$$

Introducing the linear isometry $T_\delta : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \to \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ defined by $T_\delta(u) := \delta^{-\frac{n-2}{2}} u(x/\delta)$ this becomes

$$f_{\overline{g}^\delta}(u) = f_{\overline{g}}\left(T_\delta u\right), \tag{27}$$

Furthermore, for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ one has

$$\nabla f_{\overline{g}}(u) = T_\delta \nabla f_{\overline{g}^\delta}(T_\delta^{-1} u) \tag{28}$$

$$D^2 f_{\overline{g}}(u)[v, w] = D^2 f_{\overline{g}^\delta}(T_\delta^{-1} u)[T_\delta^{-1} v, T_\delta^{-1} w]. \tag{29}$$

Arguing as above, there exists $\overline{w}_{\overline{g}^\delta}(\overline{z}_0) \in (T_{\overline{z}_0} \overline{Z}^c)^\perp$ such that

$$\nabla f_{\overline{g}^\delta}(\overline{z}_0 + \overline{w}_{\overline{g}^\delta}) \in T_{\overline{z}_0} \overline{Z}^c.$$

and there results

$$\overline{w}_{\overline{g}^\delta}(\overline{z}_0)(x) = \delta^{\frac{n-2}{2}} \overline{w}_{\overline{g}}(\overline{z}_\delta)(\delta x),$$

namely

$$\overline{w}_{\overline{g}}(\overline{z}_\delta) = T_\delta \overline{w}_{\overline{g}^\delta}(\overline{z}_0). \tag{30}$$

Remark 2. From (27), (28), (29) and using the relations between $f_{\overline{g}}$ and I^c discussed above, it is easy to check that the estimates listed in Lemma 4 hold true, substituting I^c with $f_{\overline{g}^\delta}$ and z with $\overline{z}$. A similar remark holds for Proposition 1.

We are interested to the behaviour of $f_{\overline{g}^\delta}$ as $\delta \to 0$. To this purpose, we set

$$f_{\overline{g}(0)}(u) = \int_{\mathbb{R}^n_+} \left(2\frac{(n-1)}{(n-2)} \sum_{i,j} \overline{g}^{ij}(0) D_i u D_j u - \frac{1}{2^*}|u|^{2^*} \right) dV_{\overline{g}(0)}$$
$$-c(n-2) \int_{\partial\mathbb{R}^n_+} |u|^{2\frac{n-1}{n-2}} d\sigma_{\overline{g}(0)},$$

which is the Euler functional corresponding to the constant metric $\overline{g}(0)$.

Remark 3. Unlike the $\overline{g}^\delta$, the metric $\overline{g}(0)$ does not come from a smooth metric on B. This is the main reason why it is easier to deal with $(\overline{Y})$ instead of (Y).

Lemma 5. *For all $u \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ there results*

$$\lim_{\delta\to 0} ||\nabla f_{\overline{g}^\delta}(u) - \nabla f_{\overline{g}(0)}(u)|| = 0; \tag{31}$$

$$\lim_{\delta\to 0} f_{\overline{g}^\delta}(u) = f_{\overline{g}(0)}(u). \tag{32}$$

Proof. For any $v \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ there holds

$$\left(\nabla f_{\overline{g}^\delta}(u) - \nabla f_{\overline{g}(0)}(u), v\right) = \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5,$$

where

$$\theta_1 = 4\frac{n-1}{n-2}\left(\int_{\mathbb{R}^n_+} \nabla_{\overline{g}^\delta} u \cdot \nabla_{\overline{g}^\delta} v \, dV_{\overline{g}^\delta} - \int_{\mathbb{R}^n_+} \nabla_{\overline{g}(0)} u \cdot \nabla_{\overline{g}(0)} v \, dV_{\overline{g}(0)} \right);$$
$$\theta_2 = \int_{\mathbb{R}^n_+} R_{\overline{g}^\delta} u \; v \; dV_{\overline{g}^\delta};$$
$$\theta_3 = \int_{\mathbb{R}^n_+} |u|^{\frac{4}{n-2}} u \; v \; (dV_{\overline{g}^\delta} - dV_{\overline{g}(0)}); \; \theta_4 = 2(n-1)\int_{\partial\mathbb{R}^{n-1}} h_{\overline{g}^\delta} u \; v \; d\sigma_{\overline{g}^\delta};$$
$$\theta_5 = 2c(n-1)\left(\int_{\partial\mathbb{R}^n_+} |u|^{\frac{2}{n-2}} u \; v \; d\sigma_{\overline{g}^\delta} - \int_{\partial\mathbb{R}^n_+} |u|^{\frac{2}{n-2}} u \; v \; d\sigma_{\overline{g}(0)} \right).$$

Using the Dominated Convergence Theorem and the integrability of $|\nabla u|^2$ and of $|u|^{2^*}$, it is easy to show that θ_1, θ_3 and θ_5 converge to zero. As far as θ_2 is concerned, we first note that the bilinear form $(u, v) \to \int_{\mathbb{R}^n_+} R_{\overline{g}} \; u \; v \; dV_{\overline{g}}$ is uniformly bounded for $\overline{g} \in \overline{\mathcal{G}}_\varepsilon$, so it turns out that given $\eta > 0$ there exists $u_\eta \in C_c^\infty(\overline{\mathbb{R}}^n_+)$ such that

$$\left| \int_{\mathbb{R}^n_+} R_{\overline{g}^\delta} u \; v \; dV_{\overline{g}^\delta} - \int_{\mathbb{R}^n_+} R_{\overline{g}^\delta} u_\eta \; v \; dV_{\overline{g}^\delta} \right| \le \eta \cdot \|v\|; \qquad \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^n_+). \tag{33}$$

Hence, since it is $R_{\overline{g}^\delta}(\delta^{-1}x) = \delta^2 R_{\overline{g}}(x)$ (see (13)), it follows that for δ sufficiently small

$$\left| \int_{\mathbb{R}^n_+} R_{\overline{g}^\delta} u_\eta \, v \, dV_{\overline{g}^\delta} \right| \le \delta^2 \|R_{\overline{g}}\|_{L^\infty(B)} \|u_\eta\|_\infty \int_{supp(u_\eta)} |v| = o(1) \cdot \|v\|.$$

So, using (33) and the arbitrarity of η, one deduces that $\theta_2 = o(1) \cdot \|v\|$. Similar computations hold for the term θ_4. In the same way one can prove also (32). $\square$

We need a more complete description of $\overline{w}^0(\overline{z})$. For this, according to Remark 3, we shall study the functional $f_{\overline{g}(0)}$ in a direct fashion. If $g \in \mathcal{G}_\varepsilon$ then the constant metric $\overline{g}(0)$ on $\mathbb{R}^n_+$ satisfies $\|\overline{g}(0) - Id\|_\infty = O(\varepsilon)$ and thus $f_{\overline{g}(0)}$ can be seen as a perturbation of the functional

$$f_0(u) = 2\frac{(n-1)}{(n-2)} \int_{\mathbb{R}^n_+} |\nabla u|^2 dV_0 - \frac{1}{2^*} \int_{\mathbb{R}^n_+} u^{2^*} dV_0 - c(n-2) \int_{\partial\mathbb{R}^n_+} |u|^{2\frac{n-1}{n-2}} d\sigma_0,$$

corresponding to the trivial metric δ_{ij}.

Then the procedure used in Sect. 3.2 yields to find $\overline{w}^0(\overline{z})$ such that

(j) $\overline{w}^0(\overline{z})$ is orthogonal to $T_{\overline{z}}\overline{Z}^c$;
(jj) $\nabla f_{\overline{g}(0)}(\overline{z} + \overline{w}^0(\overline{z})) \in T_{\overline{z}}\overline{Z}^c$;
(jjj) $\|\overline{w}^0(\overline{z})\| \le C \cdot \varepsilon \cdot (1 + |c|)^{-\frac{n-2}{2}} \quad \forall \overline{z} \in \overline{Z}^c$.

The following Lemma proves that a property stronger than (jj) holds.

Lemma 6. *For all $\overline{z} \in \overline{Z}^c$ there results*

$$\nabla f_{\overline{g}(0)}(\overline{z} + \overline{w}_{\overline{g}(0)}(\overline{z})) = 0. \tag{34}$$

Hence $\overline{z} + \overline{w}_{\overline{g}(0)}(\overline{z})$ solves

$$\begin{cases} -4\frac{(n-1)}{(n-2)} \sum_{i,j=1}^n \overline{g}^{ij}(0) D^2_{ij} u = u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+; \\ \frac{2}{(n-2)} \frac{\partial u}{\partial \overline{\nu}} = c u^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{35}$$

Here $\overline{\nu}$ is the unit normal vector to $\partial\mathbb{R}^n_+$ with respect to $\overline{g}(0)$, namely

$$\overline{g}(0)(\overline{\nu}, \overline{\nu}) = 1; \qquad \overline{g}(0)(\overline{\nu}, v) = 0, \quad \forall v \in \partial\mathbb{R}^n_+.$$

Proof. The Lemma is a simple consequence of the invariance of the functional under the transformation $T_{\mu,\xi'} : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \to \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ defined in the following way

$$T_{\mu,\xi'}(u) = \mu^{-\frac{n-2}{2}} u \left(\frac{x - (\xi', 0)}{\mu} \right).$$

This can be achieved with an elementary computation. It then follows that

$$\overline{w}_{\overline{g}(0)}(\overline{z}_{\mu,\xi'}) = T_{\mu,\xi'}(\overline{w}_{\overline{g}(0)}(\overline{z}_0)), \qquad \text{for all } \mu, \xi'.$$

Hence, from the invariance of $f_{\overline{g}(0)}$, it turns out that

$$f_{\overline{g}(0)}(\overline{z}_{\mu,\xi'}+\overline{w}_{\overline{g}(0)}(\overline{z}_{\mu,\xi'})) = f_{\overline{g}(0)}(T_{\mu,\xi'}(\overline{z}_0+\overline{w}_{\overline{g}(0)}(\overline{z}_0))) = f_{\overline{g}(0)}(\overline{z}_0+\overline{w}_{\overline{g}(0)}(\overline{z}_0)).$$

Since $f_{\overline{g}(0)}(\overline{z}_{\mu,\xi'}+\overline{w}_{\overline{g}(0)}(\overline{z}_{\mu,\xi'}))$ is a constant function then, according to $(j)-(jj)$, any $\overline{z}+\overline{w}_{\overline{g}(0)}(\overline{z})$ is a critical point of $f_{\overline{g}(0)}$, proving the lemma. □

Let us introduce some further notation: $\overline{G}$ denotes the matrix $\overline{g}_{ij}(0)$, $\nu_{\overline{g}(0)}$ is the outward unit normal to $\partial\mathbb{R}^n_+$ with respect to $\overline{g}_{ij}(0)$, and $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

Lemma 7. *The solutions u of problem (35) are, up to dilations and translations, of the form*

$$u = \overline{z}_0(Ax),$$

where A is a matrix which satisfies

$$A\overline{G}^{-1}A^T = I, \qquad \nu_{\overline{g}(0)} = \sum_j (A^{-1})_{jn} e_j. \tag{36}$$

In particular, up to dilations, one has that

$$\overline{z}_0 + \overline{w}_{\overline{g}(0)}(\overline{z}_0) = \overline{z}_0(A\,\cdot).$$

Proof. First of all we prove the existence of a matrix A satisfying (36). The first equality simply means that the bilinear form represented by the matrix $\overline{G}^{-1}$ can be diagonalized, and this is standard. The matrix A which satisfies the first equation in (36) is defined uniquely up to multiplication on the left by an orthogonal matrix. Let $(x_1, \dots, x_n)$ be the co-ordinates with respect to the standard basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$, let $(f_1, \dots, f_n)$ be the basis given by $\mathbf{f} = (\mathbf{A}^{-1})^{\mathbf{T}}\mathbf{e}$, and let $(y_1, \dots, y_n)$ be the co-ordinates with respect to this new basis. This implies the relation between the co-ordinates $x = Ay$ and the first of (36) implies that the bilinear form $\overline{g}^{ij}(0)$ is diagonal with respect to $y_1, \dots y_n$. Moreover, by the transitive action of $O(n)$ over S^{n-1} we can ask that $f_n = \nu$; this is exactly the second equation in (36). In this way the matrix A is determined up to multiplication on the left by $O(n-1)$.

We now prove that the function $\tilde{z}_0 = \overline{z}_0(Ax) = \overline{z}_0(y)$ is a solution of (35). First of all, since $\nu_{\overline{g}(0)}$ is $\overline{g}(0)$-orthogonal to $\partial\mathbb{R}^{n-1}$, the domain $x_n > 0$ coincides with $y_n > 0$ and the equation in the interior is, by formula (36)

$$-4\frac{(n-1)}{(n-2)}\sum_{i,j=1}^n D^2_{x_i x_j}\tilde{z}_0(x) = -4\frac{(n-1)}{(n-2)}\sum_{i,j}\overline{g}^{ij}A_{li}A_{kj}D^2_{y_k y_l}\overline{z}_0(Ay) = \tilde{z}_0^{\frac{n+2}{n-2}}(x).$$

Moreover, since $\nu = f_n = \sum_j (A^{-1})^T_{nj} e_j = \sum_j (A^{-1})_{jn} e_j$, it turns out that on $\partial\mathbb{R}^n_+$

$$\begin{aligned}\frac{\partial \tilde{z}_0}{\partial \overline{\nu}}(x) &= \sum_j (A^{-1})_{jn} D_{x_j} \overline{z}_0(Ay) \\ &= \sum_{j,k} (A^{-1})_{jn} A_{kj} D_{y_k} \overline{z}_0(Ay) = D_{y_n} \overline{z}_0(Ay) = c\tilde{z}_0^{\frac{n}{n-2}}(x).\end{aligned}$$

Hence also the boundary condition is satisfied. Moreover, the function $\overline{z}_0 \in \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ is the unique solution up to dilation and translation of problem $(\overline{Y})$ with $\overline{g}_{ij} = Id$, see [14]. As pointed out before, if A and A' are two matrices satisfying (36), they differ up to $O(n-1)$. Then it is easy to check that $\overline{z}_0(Ax) = \overline{z}_0(A'x)$ and hence $\tilde{z}_0$ is unique up to dilation and translation. This concludes the proof. $\square$

Corollary 1. *The quantity* $f_{\overline{g}(0)}(\overline{z}_0 + \overline{w}^0(\overline{z}_0))$ *is independent of* $\overline{g}(0)$. *Precisely one has:*

$$f_{\overline{g}(0)}(\overline{z}_0 + \overline{w}^0(\overline{z}_0)) = b_c.$$

Proof. There holds

$$\begin{aligned}&f_{\overline{g}(0)}(\overline{z}_0 + \overline{w}_{\overline{g}(0)}(\overline{z}_0)) \\ &= 2\frac{(n-1)}{(n-2)} \int_{\mathbb{R}^n_+} \sum_{i,j,k,l} \overline{g}^{ij}(0) A_{ki} A_{lj} D_k \overline{z}_0(Ay) D_l \overline{z}_0(Ay) dV_{\overline{g}(0)}(y) \\ &\quad -\frac{1}{2^*} \int_{\mathbb{R}^n_+} |\overline{z}_0(Ay)|^{2^*} dV_{\overline{g}(0)}(y) - c(n-2) \int_{\partial\mathbb{R}^n_+} |\overline{z}_0(Ay)|^{2\frac{n-1}{n-2}} d\sigma_{\overline{g}(0)}(y).\end{aligned}$$

Using the change of variables $x = Ay$, and taking into account equations (12) and (36) we obtain the claim. This concludes the proof. $\square$

Lemma 8. *There holds*

$$\overline{w}_{\overline{g}^\delta}(\overline{z}_0) \to \overline{w}_{\overline{g}(0)} \quad \textit{as } \delta \to 0. \tag{37}$$

Proof. Define $\overline{H}^\delta : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \times \mathbb{R}^n \times \overline{Z}^c \to \mathcal{D}^{1,2}(\mathbb{R}^n_+) \times \mathbb{R}^n$ by setting

$$\overline{H}^\delta(w, \alpha, \overline{z}) = \begin{pmatrix} \nabla f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)} + w) - \alpha \dot{\overline{z}} \\ (w, \dot{\overline{z}}) \end{pmatrix}.$$

One has that

$$\nabla f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)} + w) = \nabla f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)}) + D^2 f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)})[w] + \vartheta(w)$$

where

$$\vartheta(w) := \int_0^1 \left(D^2 f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)} + sw) - D^2 f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)}) \right)[w] ds.$$

Recall that $D^2 f_{\overline{g}^\delta}(\overline{z})$ is invertible on $(T_{\overline{z}}\overline{Z}^c)^\perp$. Since $\overline{w}_{\overline{g}(0)}$ satisfies (jjj), then also $D^2 f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)})$ is invertible on $(T_{\overline{z}}\overline{Z}^c)^\perp$. As a consequence, the equation $\nabla f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)} + w) = 0$, $w \in (T_{\overline{z}}\overline{Z}^c)^\perp$ is equivalent, on $(T_{\overline{z}}\overline{Z}^c)^\perp$, to

$$w = -\left(D^2 f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)})\right)^{-1}\left[\nabla f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)}) + \vartheta(w)\right]$$

In addition, by Remark 2, we can use the estimates corresponding to (19) of Lemma 4 and to (iii) of Proposition 1, to infer that

$$\vartheta(w) = \int_0^1 \left(D^2 f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)} + sw) - D^2 f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)})\right)[w]ds = o(\|w\|).$$

Then, repeating the arguments used in Sect. 3.2 with small changes, one can show that the equation $\overline{H}^\delta = 0$ has a unique solution $w = \omega$ such that

$$\|\omega\| \leq C \cdot \|\nabla f_{\overline{g}^\delta}(\overline{z} + \overline{w}_{\overline{g}(0)})\|.$$

From (34) and (31) it follows that $\|\omega\| \to 0$ as $\delta \to 0$. Since both $\overline{w}_{\overline{g}(0)} + \omega$ and $\overline{w}_{\overline{g}^\delta}$ solve (on $(T_{\overline{z}}\overline{Z}^c)^\perp$) the same equation, we infer by uniqueness that $\overline{w}_{\overline{g}^\delta} = \overline{w}_{\overline{g}(0)} + \omega$. Finally, since $\|\omega\| \to 0$ as $\delta \to 0$, then (37) follows. □

Remark 4. All the preceding discussion has been carried out by taking the stereographic projection σ_p through an arbitrary $p \in S^{n-1}$. We are interested to the limit (23). When $\mu \to 0$ then $\xi \to \overline{\xi}$ for some $\overline{\xi} \in S^{n-1}$ and it will be convenient to choose $p = -\overline{\xi}$.

We are now in position to give:

Proof of Proposition 2. As pointed out in Remark 4, we take $p = -\overline{\xi}$ and use all the preceding results proved so far in this section. With this choice, when $(\mu, \xi) \to (0, \overline{\xi})$ with $\xi = |\xi| \cdot \overline{\xi}$, $z_{\mu,\xi}$ corresponds to $\overline{z}_{\mu'} := \overline{z}_{\mu',0}$, for some $\mu' \to 0$.

Next, in view of (26), we will show that

$$\lim_{\mu' \to 0} f_{\overline{g}}(\overline{z}_{\mu'} + \overline{w}_{\overline{g}}(\overline{z}_{\mu'})) = b_c.$$

By Corollary 1, $b_c = f_{\overline{g}(0)}(\overline{z}_0 + \overline{w}_{\overline{g}(0)})$ and hence we need to prove that

$$\lim_{\mu' \to 0} \left[f_{\overline{g}}(\overline{z}_{\mu'} + \overline{w}_{\overline{g}}(\overline{z}_{\mu'})) - f_{\overline{g}(0)}(\overline{z}_0 + \overline{w}_{\overline{g}(0)})\right] = 0.$$

Using (30), we have

$$f_{\overline{g}}(\overline{z}_{\mu'} + \overline{w}_{\overline{g}}(\overline{z}_{\mu'})) = f_{\overline{g}}(\overline{z}_{\mu'} + T_{\mu'}\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0)).$$

Then we can write

$$\begin{aligned}
&f_{\overline{g}}(\overline{z}_{\mu'}+\overline{w}_{\overline{g}}(\overline{z}_{\mu u'}))-f_{\overline{g}(0)}(\overline{z}_0+\overline{w}_{\overline{g}(0)})=f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0))\\
&=f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0))-f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}(0)}(\overline{z}_0))\\
&\quad +\text{ï} f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}(0)}(\overline{z}_0))-f_{\overline{g}(0)}(\overline{z}_0+\overline{w}_{\overline{g}(0)}).
\end{aligned}$$

From (17) with I^c substituted by $f_{\overline{g}}$, we infer

$$\begin{aligned}
&\left|f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0))-f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}(0)}(\ ovz_0))\right|\\
&\le C\cdot\|T_{\mu'}\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0)-T_{\mu'}\overline{w}_{\overline{g}(0)}(\overline{z}_0)\|\\
&\le C\cdot\|\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0)-\overline{w}_{\overline{g}(0)}(\overline{z}_0)\|\\
&=o(1)\quad\text{as } \mu'\to 0.
\end{aligned}$$

Using $\overline{z}_{\mu'}=T_{\mu'}\overline{z}_0$ and (27), we deduce

$$f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0))=f_{\overline{g}}\left(T_{\mu'}(\overline{z}_0+\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0))\right)=f_{\overline{g}^{\mu'}}(\overline{z}_0+\overline{w}_{\overline{g}(0)}).$$

Finally

$$\begin{aligned}
&\left|f_{\overline{g}}(\overline{z}_{\mu'}+T_{\mu'}\overline{w}_{\overline{g}^{\mu'}}(\overline{z}_0))-f_{\overline{g}(0)}(\overline{z}_0+\overline{w}_{\overline{g}(0)})\right|\\
&=\left|f_{\overline{g}^{\mu'}}(\overline{z}_0+\overline{w}_{\overline{g}(0)})-f_{\overline{g}(0)}(\overline{z}_0+\overline{w}_{\overline{g}(0)})\right|\to 0,
\end{aligned}$$

according to Lemma 5. Since the above arguments can be carried out uniformly with respect to $\xi'\in S^{n-1}$, the proof is completed. □

4. The scalar curvature problem

In this section the value of c is fixed. Therefore its dependence will be omitted. So we will write I_ε instead of I_ε^c, I_0 instead of I_0^c, etc.

4.1. The abstract setting

Solutions of problem (P_ε) can be found as critical points of the functional $I_\varepsilon : H^1(B)\to\mathbb{R}$ defined as

$$I_\varepsilon(u)=I_0(u)-\varepsilon G(u)$$

where the unperturbed functional $I_0^c(u)$ is defined by (see Sect. 2)

$$I_0(u)=\frac{1}{2}\|u\|_1^2-\frac{1}{2^*}\int_B|u|^{2^*}-c(n-2)\int_{S^{n-1}}|u|^{2\frac{n-1}{n-2}}$$

and the perturbation G has the form

$$G(u) = \frac{1}{2^*}\int_B K(x)|u|^{2^*}dx + (n-2)\int_{S^{n-1}} h(x)|u|^{2\frac{n-1}{n-2}}d\sigma.$$

The existence of critical points of I_ε will be faced by means of the perturbation theory studied in [1]. Precisely, let us recall that I_0 possesses an n-dimensional manifold $Z = Z^c$, given by (6). Moreover, Z is non-degenerate in the sense that $(i)-(ii)$ of Lemma 3 hold true. Then the results of [1] lead to consider the finite dimensional functional $\Gamma := G|_Z$ and give rise to the following Theorem:

Theorem 5. *In the preceding setting, let us suppose that either*

(a) *Γ has a strict maximum (minimum) on Z; or*
(b) *there exists an open subset $\Omega \subset\subset Z$ such that $deg(\Gamma', \Omega, 0) \neq 0$.*

Then I_ε has a critical point close to Z, provided ε is small enough.

In our specific case, the function $\Gamma(\mu,\xi) = G(z_{\mu,\xi})$ has the expression

$$\Gamma(\mu,\xi) = \frac{1}{2^*}\int_B K(x)z_{\mu,\xi}^{2^*}(x)dx + (n-2)\int_{S^{n-1}} h(\sigma)z_{\mu,\xi}^{2\frac{(n-1)}{(n-2)}}(\sigma)d\sigma, \tag{38}$$

where $\mu > 0$ and $\xi \in \mathbb{R}^n$ are related to c by (5), namely by

$$\mu^2 + |\xi|^2 - c\kappa\mu - 1 = 0.$$

In order to apply the preceding abstract result we need to study the behaviour of Γ at the boundary of Z, which is given by

$$\partial Z = \{z_{\mu,\xi_0} : \mu = 0,\ |\xi_0| = 1\}.$$

The following lemma will be proved in the Appendix and describes the behaviour of Γ at ∂Z. Below $a_1, \ldots, a_6$ denote positive constants defined in the Appendix.

Lemma 9. *Let $|\xi_0| = 1$ and let ν denote the outher normal direction to ∂Z at $(0,\xi_0)$. Γ can be extended to ∂Z and there results:*

(a) *$\Gamma(0,\xi_0) = a_1K(\xi_0) + a_2h(\xi_0)$;*
(b) *$\partial_\nu\Gamma(0,\xi_0) = a_3K'(\xi_0)\cdot\xi_0$;*
(c) *suppose that $K'(\xi_0)\cdot\xi_0 = 0$ and let $n > 3$. Then*

$$\partial_\nu^2\Gamma(0,\xi_0) = 4\left[a_4\Delta_T K(\xi_0) + a_5D^2K(\xi_0)[\xi_0,\xi_0] + a_6\Delta_T h(\xi_0)\right].$$

Furthermore, if $n = 3$ and $\Delta_T h(\xi_0) \neq 0$, then

$$\partial_\nu^2\Gamma(0,\xi_0) = \begin{cases} +\infty & \textit{provided } \Delta_T h(\xi_0) > 0, \\ -\infty & \textit{provided } \Delta_T h(\xi_0) < 0. \end{cases}$$

The above Lemma is the counterpart of the calculation carried out in [2] for the Scalar Curvature Problem on S^n.

4.2. A general existence result

Let us consider the auxiliary function $\psi : S^{n-1} \to \mathbb{R}$ defined by

$$\psi(x) = a_1 K(x) + a_2 h(x), \qquad x \in S^{n-1}.$$

If $x \in Crit(\psi)$ we denote by $m(x, \psi)$ its Morse index.

Theorem 6. *Suppose that either*

(a) *there exists an absolute maximum (resp. minimum) $p \in S^{n-1}$ of ψ such that $K'(p) \cdot p < 0$ (resp. $K'(p) \cdot p > 0$);*
or
(b) *ψ is a Morse function satisfying*

$$K'(x) \cdot x \neq 0, \qquad \forall x \in Crit(\psi); \tag{39}$$

$$\sum_{x \in Crit(\psi),\, K'(x)\cdot x<0} (-1)^{m(x,\psi)} \neq 1. \tag{40}$$

Then for $|\varepsilon|$ sufficiently small, problem (P_ε) has a positive solution.

Proof. We look for critical points of Γ on $Z \simeq B$. Lemma 9-(a) and the notation introduced before says that $\Gamma|_{\partial Z} = \psi$

(a) Let p_0 denote the point where Γ achieves its absolute maximum on the compact set $\overline{Z} = Z \cup \partial Z$. Lemma 9-$(b)$ and the preceding assumption (a) imply that $p_0 \in Z$. Then the existence of a critical point of I_ε, for $|\varepsilon|$ small, follows from Theorem 5-(a).

(b) According to Lemma 9-(b), if (39) holds then $\partial_\nu \Gamma(p) \neq 0$ at any critical point of $\Gamma|_{\partial Z}$. Hence Γ satisfies the *general boundary conditions* on ∂Z, see [19]. Moreover, setting

$$\partial Z^- = \{(0, \xi_0) \in \partial Z : \partial_\nu \Gamma(\xi_0) < 0\},$$

there results

$$\partial Z^- = \{(0, \xi_0) : |\xi_0| = 1,\ K'(\xi_0) \cdot \xi_0 < 0\}.$$

In particular, the critical points of ψ on the *negative boundary* ∂Z^- are precisely the $x \in Crit(\psi)$ such that $K'(x) \cdot x < 0$. Then, by a well known formula, see [13], we infer that

$$deg(\Gamma', Z, 0) = 1 - \sum_{x \in Crit(\psi) : K'(x)\cdot x<0} (-1)^{m(x,\psi)}. \tag{41}$$

Hence, by (40), $deg(\Gamma', Z, 0) \neq 0$ and Theorem 5-(b) applies yielding the existence of a critical point of I_ε, for $|\varepsilon|$ small. □

Remark 5. (a) If $h \equiv 0$ then ψ equals, up to the positive constant a_1, K. Hence the assumption made in case (b) is precisely condition (K_1), while (39) and (40), are nothing but conditions (K_2) and (K_3). As a consequence, Theorem 6-(a) implies Theorem 2 and Theorem 6-(b) implies Theorem 3.

(b) Theorem 6-(b) is the counterpart of the results of [16] where it is taken $c = h = 0$ but R' is possibly not close to a constant. Conditions like (b) are reminiscent of conditions used by Bahri-Coron [8] dealing with the scalar curvature problem on S^3, see also [2,17] for results on S^n. In contrast, assumption (a) is a new feature due to the presence of the boundary and has no counterpart in the problem on all S^n.

(c) Theorem 6 can be the starting point to prove a global result, see [18]. Here we limit ourselves to point out that (41) can be used to evaluate the degree of I'_ε. Actually, since z is a Mountain Pass critical point, the multiplicative property of the degree immediately implies that

$$deg(I'_\varepsilon, B_r, 0) = (-1) \cdot deg(\Gamma', Z, 0) = \sum_{x \in Crit(\psi): K'(x)\cdot x < 0} (-1)^{m(x,\psi)} - 1. \tag{42}$$

Our second general existence result deals with the case in which

$$K'(x) \cdot x = 0, \qquad \forall\, x \in Crit(\psi). \tag{43}$$

In such a case, motivated by Lemma 9-(c), we introduce the function $\Psi : S^{n-1} \to \mathbb{R}$,

$$\Psi(x) = a_4 \Delta_T K(x) + a_5 D^2 K(x)[x, x] + a_6 \Delta_T h(x).$$

Let us note that, according to Lemma 9-(c) there results $\partial^2_\nu \Gamma(0, \xi_0) = 4\Psi(\xi_0)$.

Theorem 7. *Suppose that (43) holds and that*

$$\Psi(x) \neq 0, \qquad \forall\, x \in Crit(\psi). \tag{44}$$

Let ψ be a Morse function and assume that

$$\sum_{x \in Crit(\psi),\, \Psi(x) < 0} (-1)^{m(x,\psi)} \neq 1. \tag{45}$$

Furthermore, if $n = 3$, we also assume that $\Delta_T h(x) \neq 0$ for all $x \in Crit(\psi)$. Then for $|\varepsilon|$ sufficiently small, problem (P_ε) has a solution

Proof. The proof will make use of arguments similar to those emploied for Theorem 6-(b). But, unlike above, the theory of critical points under general boundary conditions cannot be applied directly because now (43) implies that

$\partial_\nu \Gamma = 0$ at all the critical points of ψ. In order to overcome this problem, we consider for $\delta > 0$ sufficiently small, the set $Z_\delta := \{(\mu, \xi) \in Z : \mu > \delta\}$ with boundary $\partial Z_\delta = \{(\mu, \xi) \in Z : \mu = \delta\}$. Since ψ is a Morse function, it readily follows that for any $\xi_0 \in Crit(\psi)$ there exists (for δ small enough) a unique ξ_δ such that

(i) $(\delta, \xi_\delta) \in \partial Z_\delta$ and $\xi_\delta \to \xi_0$ as $\delta \to 0$;
(ii) ξ_δ is a critical point of $\Gamma|_{\partial Z_\delta}$; moreover, $\Gamma|_{\partial Z_\delta}$ has no other critical point but ξ_δ;
(iii) the Morse index of ξ_δ is the same $m(\xi_0, \psi)$;

Furthermore, we claim that,

(iv) Γ verifies the general boundary conditions on Z_δ.

Actually, (44), or $\Delta_T h(\xi_0) \neq 0$ if $n = 3$, jointly with Lemma 9-(c), implies that $\partial_\nu \Gamma(\delta, \xi_\delta) \neq 0$ for δ small. More precisely, $\partial_\nu \Gamma(\delta, \xi_\delta) < 0$ iff $\xi_\delta \to \xi_0$ with $\Psi(\xi_0) < 0$. Therefore, the critical points of $\Gamma|_{\partial Z_\delta}$ on the *negative boundary* ∂Z_δ^- are in one-to-one correspondence with the $x \in Crit(\psi)$ such that $\Psi(x) < 0$. From the above arguments we infer that

$$deg(\Gamma', Z_\delta, 0) = 1 - \sum_{x \in Crit(\psi) : \Psi(x) < 0} (-1)^{m(x,\psi)}.$$

Then (45) implies that $deg(\Gamma', Z_\delta, 0) \neq 0$ and the result follws. □

Remark 6. (a) If $K \equiv 0$ then, up to positive constants, $\psi = h$ and $\Psi = \Delta_T h$ and thus Theorem 4 is a particular case of Theorem 7.

(b) It can be shown that our arguments can be adapted to handle an equation like (1) with $R' = \varepsilon K$ and $h' = c + \varepsilon h$, which can be seen as an extension of (3) where $R' = 0$ and $c = 1$ is taken. This would lead to improve the results of [9]. For brevity, we do not carry out the details here.

(c) In all the above results we can deal with $-\Gamma$ instead of Γ. In such a case the condition (40) or (45) become $\sum_{x \in Crit(\psi),\, \Psi(x) > 0} (-1)^{m(x,\psi)} \neq (-1)^{n-1}$, $\sum_{x \in Crit(\psi),\, K'(x) \cdot x > 0} (-1)^{m(x,\psi)} \neq (-1)^{n-1}$, respectively.

4.3. The symmetric case

When K and h inherit a symmetry one can obtain much more general results. They can be seen as the counterpart of the ones dealing with the Scalar Curvature problem on S^n discussed in [4].

Theorem 8. *Let us suppose that K and h are invariant under the action of a group of isometries $\Sigma \subset \mathbf{O}(n)$, such that $Fix(\Sigma) = 0 \in \mathbb{R}^n$. Then for $|\varepsilon|$ sufficiently small, problem (P_ε) has a solution.*

Proof. The proof relies on the arguments of [4, Sec. 4]. For the sake of brevity, we will be sketchy, referring to such a paper for more details. We use the finite dimensional reduction discussed in the Sect. 3.2, with $I^c = I_\varepsilon$ and $Z^c = Z$, see Remark 1. From those results we infer that the manifold

$$Z_\varepsilon = \{z_{\mu,\xi} + w_\varepsilon(z_{\mu,\xi}) : \mu, \xi \text{ satisfying (5)}\}$$

is a *natural constraint* for I_ε. Let us recall that here $w = w_\varepsilon(z_{\mu,\xi})$ is the solution of the equation

$$\nabla I_\varepsilon(z_{\mu,\xi} + w) \in T_{z_{\mu,\xi}} Z.$$

According to Remark 1, it suffices to find a critical point of $\Phi_\varepsilon(\mu, \xi) := I_\varepsilon(z_{\mu,\xi} + w_\varepsilon(z_{\mu,\xi}))$. It is possible to show that Φ_ε is invariant under the action τ of a group acting on Z and depending upon Σ. Moreover, from the fact that $\mathrm{Fix}(\Sigma) = \{0\}$ it follows that $(\mu, \xi) \in \mathrm{Fix}(\tau)$ iff $\xi = 0$ and (hence) $\mu = \mu_0 := \frac{1}{2}\left(c\kappa + \sqrt{c^2\kappa^2 + 4}\right)$. Plainly, Φ_ε has a critical point at $\mu = \mu_0$, $\xi = 0$, which gives rise to a solution of (P_ε).

For the reader convenience, let us give some more details in the specific case that K and h are even functions, when the arguments do not require new notation. We claim that if K and h are even then Φ_ε is invariant under the action τ given by $\tau : (\mu, \xi) \mapsto (\mu, -\xi)$. In other words, we will show that there results

$$\Phi_\varepsilon(\mu, \xi) = \Phi_\varepsilon(\mu, -\xi). \tag{46}$$

In order to prove (46), we first remark that $z_{\mu,-\xi}(x) = z_{\mu,\xi}(-x)$. From this and using the fact that K and h are even, one checks that $w = w_\varepsilon(z_{\mu,\xi})(-x)$ satisfies the equation, defining the *natural constraint* Z_ε,

$$\nabla I_\varepsilon(z_{\mu,-\xi} + w) \in T_{z_{\mu,-\xi}} Z,$$

By uniqueness, it follows that $w_\varepsilon(z_{\mu,\xi})(-x) = w_\varepsilon(z_{\mu,-\xi})(x)$. Then one infers:

$$\begin{aligned} I_\varepsilon(z_{\mu,-\xi}(x) + w_\varepsilon(z_{\mu,-\xi})(x)) &= I_\varepsilon(z_{\mu,\xi}(-x) + w_\varepsilon(z_{\mu,\xi})(-x)) \\ &= I_\varepsilon(z_{\mu,\xi} + w_\varepsilon(z_{\mu,\xi})), \end{aligned}$$

proving (46). □

Remark 7. (*a*) Coming back to the Scalar Curvature problem on the upper half sphere S^n_+, an even function K corresponds to prescribing a scalar curvature on S^n_+ which is invariant under the symmetry $(x_1, \ldots, x_n, x_{n+1}) \mapsto (-x_1, \ldots, -x_n, x_{n+1})$.

(*b*) Using again the arguments of [4] one could treat the invariance under a group Σ such that $\mathrm{Fix}(\Sigma) \neq \{0\}$.

A. Appendix

A.1. Proofs of technical Lemmas

First we prove

Lemma 10. *Given $M > 0$, there exists $C > 0$ such that for all $c > -M$ there holds*

$$\|z\| \le C \cdot (1+|c|)^{-\frac{n-2}{2}} \qquad \textit{for all } z \in Z^c. \tag{47}$$

Proof. By symmetry it suffices to take By symmetry it suffices to take $\xi = 0$ and consider $z = z_\mu$. As $c \to +\infty$ one has that $\mu \sim \kappa c$ and $z_\mu \sim \mu^{(n-2)/2}$ in B. Then the lemma follows by a straight calculation. □

Now we start by proving Eq. (15). Since it is clearly $\nabla I_0^c(z) = 0$, it is sufficient to estimate the quantity $\|\nabla I^c(z) - \nabla I_0^c(z)\|$. Given $v \in H^1(B)$ and setting

$$\alpha_1 = 4\frac{(n-1)}{(n-2)}\int_B \nabla_g z \cdot \nabla_g v \, dV_g - 4\frac{(n-1)}{(n-2)}\int_B \nabla z \cdot \nabla v \, dV_0;$$
$$\alpha_2 = \int_B R_g \, z \, v \, dV_g;$$
$$\alpha_3 = \int_B z^{\frac{n+2}{n-2}} v \, dV_0 - \int_B z^{\frac{n+2}{n-2}} v \, dV_g; \qquad \alpha_4 = 2(n-1)\int_{\partial B} h_g \, z \, v \, d\sigma_g;$$
$$\alpha_5 = 2(n-1)\, c \int_{\partial B} z^{\frac{n}{n-2}} v \, d\sigma_g - 2(n-1)\, c \int_{\partial B} z^{\frac{n}{n-2}} v \, d\sigma_0,$$

there holds

$$\left(\nabla I^c(z) - \nabla I_0^c(z), v\right) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5. \tag{48}$$

As far as α_1 is concerned, taking into account of equations (12), (14) and the fact that $\|z\| \le C \cdot (1+|c|)^{-\frac{n-2}{2}}$ (see Lemma 10) one deduces that

$$\begin{aligned} |\alpha_1| &\le C \int_B \left|\nabla_g z \cdot \nabla_g v - \nabla z \cdot \nabla v\right| dx + C \int_B |\nabla z \cdot \nabla v| \, |dV_g - dV_0| \\ &\le C \cdot \varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}} \cdot \|v\|. \end{aligned} \tag{49}$$

Turning to α_2 we recall that the expression of R_g as a function of g, is of the type

$$R_g = D\Gamma + G^2; \qquad \Gamma = Dg, \qquad \Rightarrow \qquad R_g = D^2 g + (Dg)^2.$$

We start by estimating the quantity $\int_B R_g \, z \, v \, dV_0$. Integrating by parts, the term $\int_B D^2 g \, z \, v \, dV_0$ transforms into

$$\int_B D^2 g \, z \, v \, dV_0 = \int_{\partial B} Dg \, z \, v \, d\sigma_0 + \int_B Dg \, D(zv) dV_0.$$

Hence, if $g \in \mathcal{G}_\varepsilon$ (see expression (2)), from the Hölder inequality it follows that

$$\int_B R_g zv \, dV_0 \simeq \int_B (D^2 g + (Dg)^2) zv \, dV_0 \leq C \cdot \varepsilon \cdot \|z\| \cdot \|v\|,$$

and hence

$$|\alpha_2| \leq \int_B \left| R_g zv \right| dV_0 + \int_B |R_g zv| |dV_g - dV_0| \leq C \cdot \varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}} \cdot \|v\|. \tag{50}$$

With simple estimates one can also prove that

$$|\alpha_3| \leq C \cdot \varepsilon \cdot (1+|c|)^{-\frac{n+2}{2}} \cdot \|v\|. \tag{51}$$

The function h_g is of the form $h_g = Dg$ so, taking into account (2) one finds

$$|\alpha_4| \leq C \cdot \varepsilon \cdot (1+|c|)^{-\frac{n-2}{2}} \cdot \|v\|. \tag{52}$$

In order to estimate the last term α_5, using the continuous embedding $H^1(B) \hookrightarrow L^{2\frac{n-1}{n-2}}(S^{n-1})$ and the Hölder inequality one deduces that

$$|\alpha_5| \leq C \cdot \varepsilon \cdot (1+|c|) \cdot \|z\|^{\frac{n}{n-2}}_{L^{\frac{n}{n-2}}(S^{n-1})} \cdot \|v\| \leq C \cdot \varepsilon \cdot (1+|c|) \cdot (1+|c|)^{-\frac{n}{2}} \cdot \|v\|.$$

Putting together equations (49)-(52) one deduces (15).

Turning to equation (19) and given $v_1, v_2 \in H^1(B)$, there holds

$$(D^2 I^c(z+w) - D^2 I^c(z))[v_1, v_2] = \delta_1 + \delta_2,$$

where

$$\delta_1 = \frac{(n+2)}{(n-2)} \left(\int_B u^{\frac{4}{n-2}} v_1 \, v_2 \, dV_g - \int_B (u+w)^{\frac{4}{n-2}} v_1 \, v_2 \, dV_g \right)$$
$$\delta_2 = 2n \frac{(n-1)}{(n-2)} c \left(\int_{\partial B} u^{\frac{2}{n-2}} v_1 \, v_2 \, d\sigma_g - \int_{\partial B} (u+w)^{\frac{2}{n-2}} v_1 \, v_2 \, d\sigma_g \right).$$

Using standard inequalities one finds that

$$|\delta_1| \leq \begin{cases} C \cdot \|w\|^{\frac{4}{n-2}} & \text{for } n \geq 6, \\ C \cdot \|w\| \cdot \left(\|u\|^{\frac{6-n}{n-2}} + \|w\|^{\frac{6-n}{n-2}} \right) & \text{for } n < 6; \end{cases}$$

$$|\delta_2| \leq \begin{cases} C \cdot (1+|c|) \cdot \|w\|^{\frac{4}{n-2}} & \text{for } n \geq 4, \\ C \cdot (1+|c|) \cdot \|w\| \cdot \left(\|u\|^{\frac{4-n}{n-2}} + \|w\|^{\frac{4-n}{n-2}} \right) & \text{for } n < 4, \end{cases}$$

$$\|D^2 I^c(z+w) - D^2 I^c(z)\| \leq C \cdot |c| \cdot \|w\|^{\frac{2}{n-2}}. \tag{53}$$

so we obtain the estimate.

We now prove inequality (16). Given $v_1, v_2 \in H^1(B)$ and setting

$$\beta_1 = 4\frac{(n-1)}{(n-2)}\int_B \nabla_g v_1 \cdot \nabla_g v_2 \, dV_g - 4\frac{(n-1)}{(n-2)}\int_B \nabla v_1 \cdot \nabla v_2 \, dV_0;$$
$$\beta_2 = \int_B R_g \, v_1 \, v_2 \, dV_g;$$
$$\beta_3 = \frac{(n+2)}{(n-2)}\int_B z^{\frac{4}{n-2}} v_1 \, v_2 \, dV_0 - \frac{(n+2)}{(n-2)}\int_B z^{\frac{4}{n-2}} v_1 \, v_2 \, dV_g;$$
$$\beta_4 = 2(n-1)\int_{\partial B} h_g \, v_1 \, v_2 \, d\sigma_g;$$
$$\beta_5 = 2n\frac{(n-1)}{(n-2)} \, c \int_{\partial B} z^{\frac{2}{n-2}} v_1 \, v_2 \, d\sigma_g - 2n\frac{(n-1)}{(n-2)} \, c \int_{\partial B} z^{\frac{2}{n-2}} v_1 \, v_2 \, d\sigma_0,$$

there holds

$$(D^2 I^c(z) - D^2 I_0^c(z))[v_1, v_2] = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5. \tag{54}$$

For β_1, taking into account equation (14) one finds

$$\begin{aligned} |\beta_1| &\le C \int_B \left|\nabla_g v_1 \cdot \nabla_g v_2 - \nabla v_1 \cdot \nabla v_2\right| dV_0 + C \int_B |\nabla v_1 \cdot \nabla v_2| \cdot \left|dV_g - dV_0\right| \\ &\le C \cdot \varepsilon \cdot \|v_1\| \cdot \|v_2\|. \end{aligned} \tag{55}$$

Turning to β_2 reasoning as for the above term α_2 one deduces that

$$|\beta_2| \le \int_B \left|R_g \, z \, v\right| dV_g \le C \cdot \varepsilon \cdot \|v_1\| \cdot \|v_2\|. \tag{56}$$

In the same way one can prove that

$$|\beta_3| \le C \cdot \varepsilon \cdot \|z\|^{\frac{4}{n-2}} \cdot \|v_1\| \cdot \|v_2\| \le C \cdot \varepsilon \cdot (1+|c|)^{-2} \cdot \|v_1\| \cdot \|v_2\|. \tag{57}$$

For the term β_4, similarly to the expression α_4 above there holds

$$|\beta_4| \le C \cdot \varepsilon \cdot \|v_1\| \cdot \|v_2\|. \tag{58}$$

Turning to β_5, using the Hölder inequality one deduces that

$$|\beta_5| \le C \cdot c \cdot \varepsilon \cdot (1+|c|) \cdot \|z\|^{\frac{2}{n-2}}_{L^{\frac{2}{n-2}}(S^{n-1})} \cdot \|v_1\| \cdot \|v_2\| \le C \cdot \varepsilon \cdot \|v_1\| \cdot \|v_2\|. \tag{59}$$

Putting together equations (55)-(59) (59) one deduces inequality (16).

Equation (17) follows from similar computations.

A.2. Proof of Lemma 9

Given $\xi_0| = 1$, we introduce a reference frame in $\mathbb{R}^n$ such that $e_n = -\xi_0$. Let $\alpha = \alpha(\mu)$ be such that the pair (μ, ξ), with $\xi = \alpha\xi_0$, satisfies (5). Setting

$$\gamma(\mu) = \Gamma(\mu, -\alpha(\mu)e_n),$$

one has that

$$\Gamma(0, \xi_0) = \gamma(0), \quad \partial_\nu \Gamma(0, \xi_0) = -\gamma'(0), \quad \partial^2_\nu \Gamma(0, \xi_0) = \gamma''(0).$$

In order to evaluate the above quantities, it is convenient to make a change of variables. This will considerably simplify the calculation when we deal with $\gamma'(0)$ and $\gamma''(0)$.

Let $\psi : \mathbb{R}^n_+ \to B$ be the map given by

$$\begin{aligned} &(y', y_n) \in \mathbb{R}^n_+ \to (x', x_n) \in B; \\ &x' = \frac{2y'}{(y')^2 + (y_n+1)^2}, \quad x_n = \frac{(y')^2 + y_n^2 - 1}{(y')^2 + (y_n+1)^2}. \end{aligned}$$

Here and in the sequel,if $x \in \mathbb{R}^n$ we will set $x' = (x_1, \dots, x_{n-1})$ so that $x = (x', x_n)$.

By using simple computations it turns out that

$$\gamma(\mu) = \tilde{\gamma}(\tilde{\mu}),$$

where

$$\tilde{\gamma}(\tilde{\mu}) = \frac{1}{2^*}\int_{\mathbb{R}^n_+} \tilde{K}(y)(z^c_{\tilde{\mu},0})^{2^*}(y)dy + (n-2)\int_{\partial\mathbb{R}^n_+} \tilde{h}(\omega)(z^c_{\tilde{\mu},0})^{2\frac{n-1}{n-2}}(\omega)d\omega,$$

and

$$\tilde{\mu} = \frac{2\mu}{1+\mu^2+\alpha(\mu)}; \qquad \tilde{K}(y) = K(\psi(y)).$$

Let us point out that the derivatives of K and $\tilde{K}$ satisfy the following relations:

$$\begin{aligned} D_{y_n}\tilde{K}(0,0) &= 2D_{x_n}K(\xi_0); \\ D_{y'}\tilde{K}(0,0) &= 2D_{x'}(\xi_0); \\ D^2_{y_n}\tilde{K}(0,0) &= 4\left(D^2_{x_n}K - D_{x_n}K\right)(\xi_0); \\ D^2_{y'}\tilde{K}(0,0) &= 4\left(D^2_{x'}K - D_{x_n}K\right)(\xi_0); \\ D^2_{y',y_n}\tilde{K}(0,0) &= 4\left(D^2_{x',x_n}K - D_{x'}K\right)(\xi_0). \end{aligned}$$

The change of variables $y = \tilde{\mu}q$, $\omega = \tilde{\mu}\sigma$ yields

$$\tilde{\gamma}(\tilde{\mu}) = \frac{1}{2^*}\int_{\mathbb{R}^n_+} \tilde{K}(\tilde{\mu}q)(z^c_{1,0})^{2^*}(q)dq + (n-2)\int_{\partial\mathbb{R}^n_+} \tilde{h}(\tilde{\mu}\sigma)(z^c_{1,0})^{2\frac{n-1}{n-2}}(\sigma)d\sigma. \tag{60}$$

Hence, passing to the limit for $\tilde{\mu} \to 0$, it follows that

$$\gamma(0) = \tilde{\gamma}(0) = a_1 \tilde{K}(0) + a_2 \tilde{h}(0) = a_1 K(\xi_0) + a_2 h(\xi_0),$$

with

$$a_1 = \frac{1}{2^*} \int_{\mathbb{R}^n_+} z_0^{2^*}(q', q_n - \kappa c/2) dq, \qquad a_2 = (n-2) \int_{\partial \mathbb{R}^n_+} z_0^{2\frac{n-1}{n-2}}(\sigma, \kappa c/2) d\sigma.$$

Let us now evaluate the first derivative. There holds

$$\gamma'(0) = \frac{d\tilde{\gamma}}{d\tilde{\mu}}(0) \cdot \frac{d\tilde{\mu}}{d\mu}(0) = 2\tilde{\gamma}'(0).$$

Moreover from formula (60) we deduce

$$\begin{aligned} \tilde{\gamma}'(\tilde{\mu}) = \frac{1}{2^*} \int_{\mathbb{R}_n^+} \nabla \tilde{K}(\tilde{\mu} q) \cdot q \ |z_{1,0}^c(q)|^{2^*} dq + (n-2) \\ \times \int_{\partial \mathbb{R}^n_+} \nabla \tilde{h}(\tilde{\mu}\sigma) \cdot \sigma \ |z_{1,0}^c(\sigma)|^{2\frac{n-1}{n-2}}(\sigma) d\sigma. \end{aligned} \tag{61}$$

For symmetry reasons when $\tilde{\mu} \to 0$, the parallel component to $\partial \mathbb{R}^n_+$ in the first integral and the second integral vanishes, hence it follows that

$$\gamma'(0) = 2\tilde{\gamma}'(0) = \frac{2}{2^*} D_n \tilde{K}(0) \int_{\mathbb{R}^n_+} q_n |z_{1,0}^c(q)|^{2^*} dq = -a_3 K'(\xi_0) \cdot \xi_0, \tag{62}$$

where

$$a_3 = \frac{4}{2^*} \int_{\mathbb{R}^n_+} q_n z_0^{2^*}(q', q_n - \kappa c/2) dq.$$

We are interested in the study of the second derivative only in the case in which the first derivative vanishes, namely when $K'(\xi_0) \cdot \xi_0 = 0$.

As for the second derivative, there holds:

$$\begin{aligned} \tilde{\gamma}''(\tilde{\mu}) \ = \ & \frac{1}{2^*} \int_{\mathbb{R}_n^+} \sum_{i,j=1}^{n} D^2_{ij} \tilde{K}(\tilde{\mu} q) q_i q_j |z_{1,0}^c(q)|^{2^*} dq \\ & + (n-2) \int_{\partial \mathbb{R}^n_+} \sum_{i,j=1}^{n-1} D^2_{ij} \tilde{h}(\tilde{\mu}\sigma) \sigma_i \sigma_j |z_{1,0}^c(\sigma)|^{2\frac{(n-1)}{(n-2)}} d\sigma \\ := \ & \delta(\tilde{\mu}) + \rho(\tilde{\mu}). \end{aligned} \tag{63}$$

Now we have to distinguish the case $n = 3$ and the case $n > 3$. In fact the boundary integral $\rho(\tilde{\mu})$ in (63) is uniformly dominated by a function in $L^1(\partial \mathbb{R}^n_+)$

if and only if $n > 3$. However it is possible to determine the sign of this integral also for $n = 3$: it turns out that

$$\begin{aligned}\lim_{\tilde{\mu}\to 0} \delta(\tilde{\mu}) = \frac{1}{2^*(n-1)} \int_{\mathbb{R}^n_+} |q'|^2 |z^c_{1,0}(q)|^{2^*} dq \cdot \Delta_T \tilde{K}(0)\\ +\frac{1}{2^*} \int_{\mathbb{R}^n_+} q_n^2 |z^c_{1,0}(q)|^{2^*} dq \cdot D^2_{nn} \tilde{K}(0);\end{aligned}$$

and

$$\begin{cases} \lim\limits_{\tilde{\mu}\to 0} \rho(\tilde{\mu u}) = (+\infty)\cdot \Delta_T \tilde{h}(0), & \text{for } n = 3;\\ \lim\limits_{\tilde{\mu}\to 0} \rho(\tilde{\mu}) = \dfrac{(n-2)}{(n-1)} \displaystyle\int_{\partial\mathbb{R}^n_+} |\sigma|^2 |z^c_{1,0}(\sigma)|^{2\frac{(n-1)}{(n-2)}} d\sigma \cdot \Delta_T \tilde{h}(0), & \text{for } n > 3.\end{cases}$$

Hence we have that

$$\tilde{\gamma}''(0) = \begin{cases} (+\infty)\cdot \Delta_T h(\xi_0) & \text{for } n = 3;\\ \\ a_4 \Delta_T K(\xi_0) + a_5 D^2 K(\xi_0)[\xi_0, \xi_0] + a_6 \Delta_T h(\xi_0) & \text{for } n > 3,\end{cases} \tag{64}$$

where

$$\begin{aligned} a_4 &= \frac{4}{(n-1)2^*} \int_{\mathbb{R}^n_+} |q'|^2 z_0^{2^*}(q', q_n - \kappa c/2) dq,\\ a_5 &= \frac{4}{2^*} \int_{\mathbb{R}^n_+} q_n^2 z_0^{2^*}(q', q_n - \kappa c/2) dq,\\ a_6 &= 4\frac{(n-2)}{(n-1)} \int_{\partial\mathbb{R}^n_+} |\sigma|^2 z_0^{2\frac{n-1}{n-2}}(\sigma, \kappa c/2) d\sigma.\end{aligned}$$

Finally, since $\gamma''(0) = 4\tilde{\gamma}''(0)$, the lemma follows.

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