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On the Yang-Baxter-like matrix equation for rank-two matrices

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Abstract: Let $A = PQ^T$, where P and Q are two $n \times 2$ complex matrices of full column rank such that $Q^T P$ is singular. We solve the quadratic matrix equation $AXA = XAX$. Together with a previous paper devoted to the case that $Q^T P$ is nonsingular, we have completely solved the matrix equation with any given matrix A of rank-two.

Keywords: Rank-two matrix, Matrix equation, Jordan form

MSC: 15A18, 15A24, 65F15

1 Introduction

Recently [1], the authors have found all the solutions of the Yang-Baxter-like matrix equation

$$AXA = XAX, \tag{1}$$

where the given $n \times n$ complex matrix $A = PQ^T$, with two $n \times 2$ matrices P and Q, satisfies the assumption that the 2×2 matrix $Q^T P$ is nonsingular. In the above situation, the eigenvalue 0 of A is semi-simple with multiplicity $n - 2$. This leaves the case unsolved that $Q^T P$ is singular, which makes it much more challenging to solve the corresponding matrix equation. The purpose of this paper is to find all the solutions of (1) for the remaining case that the algebraic multiplicity of the eigenvalue 0 of A is more than $n - 2$.

The equation (1) has a similar format to the classical Yang-Baxter equation [2]. Yang [3] in 1967 first considered a one dimensional quantum mechanical many body problem with a combination of delta functions as the potential and found a factorization of the scattering matrix, and the Yang-Baxter equation was obtained as a consistence property for the factorization. Then Baxter in 1972 solved an eight-vertex model in statistical mechanics, resulting in a similar matrix equation, which, together with that from [3], was first called the Yang-Baxter equation by Russian researchers at the end of the 1970s. Since then the Yang-Baxter equation has been extensively investigated by mathematicians and physicists in knot theory, braid group theory, and quantum group theory in the past thirty years; see, e.g., [2,4-8] and the references therein. In the past several years, the quadratic matrix equation (1) has been studied with linear algebra techniques; see, for example, the references [9-14].

Solving (1) is a tough job in general since we need to solve a large system of n^2 quadratic equations with n^2 variables if we multiply out its both sides. By restricting the task to only finding the solutions that commute with A, several solution results have been obtained in [10] for matrices A of special Jordan forms, and a more general result was proved in [15] for the class of diagonalizable matrices. However, no general result has been found so far for non-

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commuting solutions for arbitrary matrices. Thus, it is our hope to find all the solutions of (1) for general matrices A. When A is a rank one matrix, all the solutions of (1) have been found in [13]. The case of A being of rank-two turns out to be much more tedious to analyze, and some special cases have been solved in our previous paper [1]. In the current paper we continue to study the rank two case to find all the solutions for the remaining Jordan form structure of A.

Matrices A of rank at most two in the equation (1) have appeared in the classic Yang-Baxter equation (see, e.g., the references in [2, 5]). For example, the two classes of 4×4 matrices

$$\begin{bmatrix} a & 0 & 0 & c \\ 0 & b & d & 0 \\ 0 & e & b & 0 \\ f & 0 & 0 & a \end{bmatrix} \text{ and } \begin{bmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{bmatrix} \tag{2}$$

with certain values of the given parameters have been studied in [16-18] for completely integrable systems and inverse scattering problems. Each matrix in the second class is actually the tensor product $T \otimes I_2$ of the 2×2 matrix $T = [t_{ij}]$ with the 2×2 identity matrix I_2, which constitutes a basic operation in the context of the classical Yang-Baxter equation. Our complete solutions to the quadratic matrix equation with a given rank-two matrix are expected to find applications in such physical applications. We shall pick up one matrix from (2) to apply our result in Section 5.

It is well known [10] that solving the Yang-Baxter-like matrix equation for any given matrix A is equivalent to solving the same equation with A replaced by a matrix that is similar to A, and all the solutions of the first equation are similar to those of the second one with the same similarity matrix. Thus, solving (1) for the given A can be reduced to solving the same equation with the Jordan form of A. Our approach in this paper will follow the above principle. That is, since any matrix is similar to its simplest possible canonical form J, the Jordan form of the matrix, which is called a Jordan matrix here for simplicity, is to be found. For the purpose of the present paper that will supplement the work of [1], we shall solve the following simpler Yang-Baxter-like matrix equation

$$JYJ = YJY \tag{3}$$

with J satisfying the conditions:
(i) J is a Jordan matrix of rank-2.
(ii) The eigenvalue 0 of J has algebraic multiplicity at least $n-1$.

If we can find the solutions of (3) for such Jordan matrices, then the solutions of (1) are available immediately for any A that is similar to J.

It turns out that the Jordan matrices J that satisfy the conditions (i) and (ii) above are

$$J = \mathrm{diag}(0, \Lambda) \tag{4}$$

such that Λ is one of the following three matrices

$$\Lambda_1 \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ \Lambda_2 \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \ \lambda \neq 0; \ \Lambda_3 \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

and the diagonal block 0 in (4) is either $(n-3) \times (n-3)$ or $(n-4) \times (n-4)$ accordingly. With respect to each of the three cases for J, there exists a corresponding nonsingular similarity matrix $W = [w_1, \ldots, w_n]$ connecting A to J, so that

$$A = PQ^T = WJW^{-1}. \tag{6}$$

More specifically, in the case that $\Lambda = \Lambda_1$, the columns $w_1, \ldots, w_{n-2}$ of W are eigenvectors of A and the columns w_{n-1} and w_n of W are generalized eigenvectors of A with degrees 2 and 3 respectively, namely

$$Aw_{n-1} \neq 0, \ A^2 w_{n-1} = 0, \ A^2 w_n \neq 0, \ A^3 w_n = 0,$$

all associated with eigenvalue 0. When $\Lambda = \Lambda_2$, the first $n-2$ columns of W are eigenvectors of A associated with eigenvalue 0, the column w_{n-1} is a generalized eigenvector of degree 2 with respect to eigenvalue 0, and the column w_n is an eigenvector of A associated with eigenvalue $\lambda \neq 0$. If $\Lambda = \Lambda_3$, then the columns $w_1, \ldots, w_{n-3}$, and w_{n-1} of W are eigenvectors of A, and the columns w_{n-2} and w_n of W are generalized eigenvectors of degrees 2, all with respect to eigenvalue 0.

In the next three sections we look for the solutions of the Yang-Baxter-like matrix equation (1) when the corresponding Jordan matrix J of A is given by (4) with $\Lambda = \Lambda_1, \Lambda_2, \Lambda_3$, respectively; such cases will be referred to as type I, type II, and type III for the given matrix A. We present two examples of our solution result in Section 5 and conclude with Section 6.

2 Solutions when A is of type I

In this and subsequent sections we let $A = PQ^T$ in (1) with $P = [p_1, p_2]$ and $Q = [q_1, q_2]$ of rank-2 such that $\det Q^T P = 0$. Let J be the Jordan form of A given by (4), where the diagonal zero sub-matrix 0 is either $(n-3) \times (n-3)$ and the sub-matrix $\Lambda = \Lambda_1$ or Λ_2 defined by (5), or the zero sub-matrix 0 is $(n-4) \times (n-4)$ and $\Lambda = \Lambda_3$ in (5). As pointed out in Section 1, it is enough to find all the solutions of the equation (3) with J the Jordan form of A, so we just focus on solving (3).

Let Y be partitioned the same way as J into the 2×2 block matrix

$$Y = \begin{bmatrix} M & U \\ V^T & K \end{bmatrix}, \tag{7}$$

where M is $(n-3) \times (n-3)$ or $(n-4) \times (n-4)$, $U = [u_1, u_2, u_3]$ or $[u_1, u_2, u_3, u_4]$, $V = [v_1, v_2, v_3]$ or $[v_1, v_2, v_3, v_4]$, and K is 3×3 or 4×4, depending on the size of Λ. Then (3) with J partitioned by (4) is

$$\begin{bmatrix} 0 & 0 \\ 0^T & \Lambda \end{bmatrix}\begin{bmatrix} M & U \\ V^T & K \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0^T & \Lambda \end{bmatrix} = \begin{bmatrix} M & U \\ V^T & K \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0^T & \Lambda \end{bmatrix}\begin{bmatrix} M & U \\ V^T & K \end{bmatrix},$$

which is equivalent to the system

$$\begin{cases} U\Lambda V^T &= 0, \\ U\Lambda K &= 0, \\ K\Lambda V^T &= 0, \\ \Lambda K \Lambda &= K\Lambda K. \end{cases} \tag{8}$$

In the current section we assume that $J = \operatorname{diag}(0, \Lambda_1)$, and the other two cases that $J = \operatorname{diag}(0, \Lambda_2)$ and $J = \operatorname{diag}(0, \Lambda_3)$ will be investigated in Sections 3 and 4, respectively. So we solve (8) with $\Lambda = \Lambda_1$. Because of the special zero structure of Λ_1, the two unknown vectors u_3 and v_1 actually do not appear at all in the above system, so they always appear as free variables in the solutions. In addition, the first equation is independent of K and is in fact

$$u_1 v_2^T + u_2 v_3^T = 0.$$

The last equation of (8),

$$\Lambda K \Lambda = K \Lambda K, \tag{9}$$

is itself a Yang-Baxter-like matrix equation of small size when $\Lambda = \Lambda_j$ with $j = 1, 2, 3$, and finding all of its solutions is the first step for solving (8).

Lemma 2.1. *The solutions K of the equation (9) with $\Lambda = \Lambda_1$ are*

$$K_1 = \begin{bmatrix} x & y & z \\ 0 & 0 & -\frac{cy}{x} \\ 0 & 0 & c \end{bmatrix}, \ x \neq 0; \ K_2 = \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & g \\ 0 & 0 & c \end{bmatrix}, c \neq 0; \ K_3 = \begin{bmatrix} 0 & y & z \\ 0 & 0 & g \\ 0 & 0 & 0 \end{bmatrix}.$$

Proof. Write

$$K = \begin{bmatrix} x & y & z \\ e & f & g \\ a & b & c \end{bmatrix}.$$

Then, the matrix equation $\Lambda_1 K \Lambda_1 = K \Lambda_1 K$ becomes

$$\begin{bmatrix} 0 & e & f \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} ex+ay & fx+by & gx+cy \\ e^2+af & ef+bf & eg+cf \\ ae+ab & af+b^2 & ag+bc \end{bmatrix}.$$

Note that there is no z in the above equation, so all the solutions have z as an arbitrary parameter. The two equations $ef+bf=a$ and $ae+ab=0$ imply $a=0$, so $b=0$ from $0=af+b^2=b^2$ and $e=0$ since $0=e^2+af=e^2$. Thus we have the remaining three equations

$$\begin{cases} fx = 0, \\ gx + cy = f, \\ cf = 0 \end{cases}$$

from which $f=0$. Hence $gx+cy=0$. If $x \neq 0$, then $g=-cy/x$, which gives the first matrix K_1 in the lemma. When $x=0$, we have $cy=0$. So $c \neq 0$ or $c=0$, resulting in the other two matrices K_2 and K_3, respectively. □

By Lemma 2.1, all the solutions of the last equation of (8) are K_1, K_2, and K_3. Substituting such matrices into the first three equations of (8) and solving them respectively, we obtain the following result.

Theorem 2.2. *Suppose $A = PQ^T$ is such that $\Lambda = \Lambda_1$ in its Jordan form (4). Then all the solutions of (1) are $X = WYW^{-1}$, where W is as given by (6) and Y is partitioned as (7) in which M is an arbitrary $(n-3)\times(n-3)$ matrix such that $Y =$*

$$\begin{bmatrix} M & u_1 & \frac{y}{x}u_1 & u_3 \\ v_1^T & x & y & z \\ -\frac{y}{x}v_3^T & 0 & 0 & -\frac{y}{x}c \\ v_3^T & 0 & 0 & c \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 \\ v_1^T & x & y & z \\ 0^T & 0 & 0 & 0 \\ 0^T & 0 & 0 & 0 \end{bmatrix}, \; x \neq 0, u_2 \neq \frac{y}{x}u_1, \tag{10}$$

$$\begin{bmatrix} M & 0 & 0 & u_3 \\ v_1^T & 0 & 0 & z \\ v_2^T & 0 & 0 & g \\ v_3^T & 0 & 0 & c \end{bmatrix}, \begin{bmatrix} M & u_1 & -\frac{g}{c}u_1 & u_3 \\ v_1^T & 0 & 0 & z \\ \frac{g}{c}v_3^T & 0 & 0 & g \\ v_3^T & 0 & 0 & c \end{bmatrix}, \; c \neq 0, u_1 \neq 0, \tag{11}$$

$$\begin{bmatrix} M & 0 & 0 & u_3 \\ v_1^T & 0 & 0 & z \\ v_2^T & 0 & 0 & g \\ v_3^T & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & 0 & u_2 & u_3 \\ v_1^T & 0 & y & z \\ v_2^T & 0 & 0 & g \\ 0^T & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} M & u_1 & u_2 & u_3 \\ v_1^T & 0 & 0 & z \\ -\frac{u_1^H u_2}{\|u_1\|^2}v_3^T & 0 & 0 & 0 \\ v_3^T & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 \\ v_1^T & 0 & y & z \\ 0^T & 0 & 0 & 0 \\ 0^T & 0 & 0 & 0 \end{bmatrix}, \; y \neq 0, u_1 \neq 0. \tag{13}$$

Here u^H is the conjugate transpose of u.

Proof. We just solve the first three equations of (8) with $K = K_1$, K_2, and K_3 in succession. When $K = K_1$, those equations of (8) are

$$\begin{cases} (u_2 - \frac{y}{x}u_1)v_3^T &= 0, \\ (u_2 - \frac{y}{x}u_1)c &= 0, \\ v_2 &= -\frac{y}{x}v_3 \end{cases}$$

after simplification. From the first equation, $u_2 - yu_1/x = 0$ or $v_3 = 0$. If the former is satisfied, then $u_2 = yu_1/x$ and the second equation is satisfied. This gives the first solution matrix of (10). In the case that $v_3 = 0$, either $u_2 = yu_1/x$ or $c = 0$ from the second equation. The former case still leads to the first matrix of (10), while the latter gives the second matrix of (10).

Suppose $K = K_2$. Then $K_2\Lambda_1 V^T = 0$, so (8) is reduced to

$$\begin{cases} u_1 v_2^T + u_2 v_3^T = 0, \\ gu_1 + cu_2 = 0. \end{cases} \tag{14}$$

Thus z is a free variable in the solutions. Since $c \neq 0$, (14) is equivalent to $u_2 = -gu_1/c$ and $u_1(v_2 - gv_3/c)^T = 0$. Letting $u_1 = 0$ gives the first matrix of (11), the second matrix of (11) is the consequence of $u_1 \neq 0$, and $v_2 - gv_3/c = 0$.

If $K = K_3$, then (8) becomes

$$\begin{cases} u_1 v_2^T + u_2 v_3^T = 0, \\ gu_1 = 0, \\ yv_3^T = 0. \end{cases} \tag{15}$$

Letting $u_1 = 0, u_2 = 0$, and $y = 0$ gives the first matrix of (12), and the choice of $u_1 = 0$ and $v_3 = 0$ produces the second matrix of (12). If $u_1 \neq 0$, then the first equation of (15) implies that $v_2 = -(u_1^H u_2)v_3/\|u_1\|^2$, and the other two equations give that either $g = 0$ and $y = 0$, from which we get the first matrix of (13), or $g = 0$ and $y \neq 0$, which gives the second one. □

3 Solutions when A is of type II

We now consider the second case that the Jordan form of the matrix A is $J = \operatorname{diag}(0, \Lambda_2)$, so we solve (8) with $\Lambda = \Lambda_2$. Clearly, the structure of Λ makes u_2 and v_1 free vectors in all the solutions of (8), and the first equation of (8) is now

$$u_1 v_2^T + \lambda u_3 v_3^T = 0.$$

Lemma 3.1. *The solutions K of the equation (9) with $\Lambda = \Lambda_2$ are*

$$K_4 \equiv \begin{bmatrix} 0 & y & 0 \\ 0 & f & 0 \\ 0 & b & 0 \end{bmatrix}; \; K_5 \equiv \begin{bmatrix} 0 & y & z \\ 0 & f & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; z \neq 0;$$

$$K_6 \equiv \begin{bmatrix} x & y & z \\ 0 & -\frac{\lambda bz}{x} & 0 \\ 0 & b & 0 \end{bmatrix}, \; x \neq 0; \; K_7 \equiv \begin{bmatrix} \lambda & y & z \\ 0 & 0 & g \\ 0 & 0 & 0 \end{bmatrix}, \; g \neq 0;$$

$$K_8 \equiv \begin{bmatrix} 0 & y & 0 \\ 0 & f & 0 \\ 0 & 0 & \lambda \end{bmatrix}; \; K_9 \equiv \begin{bmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \; x \neq 0; \; K_{10} \equiv \begin{bmatrix} 0 & y & 0 \\ 0 & \lambda & 0 \\ a & b & 0 \end{bmatrix}, \; a \neq 0.$$

Proof. The equation $\Lambda_2 K \Lambda_2 = K\Lambda_2 K$ now becomes

$$\begin{bmatrix} 0 & e & \lambda g \\ 0 & 0 & 0 \\ 0 & \lambda a & \lambda^2 c \end{bmatrix} = \begin{bmatrix} ex + \lambda az & fx + \lambda bz & gx + \lambda cz \\ e^2 + \lambda ag & ef + \lambda bg & eg + \lambda cg \\ ae + \lambda ac & af + \lambda bc & ag + \lambda c^2 \end{bmatrix}. \tag{16}$$

Since the unknown y does not appear in the above, it is a free variable in all the solutions. To solve for all the other unknowns, let $a = 0$ first. Then $e = 0$ from $e^2 + \lambda ag = 0$. Because of the equation $\lambda^2 c = ag + \lambda c^2$, there are two possibilities that $c = 0$ or $c = \lambda$. Assume first that $c = 0$. Since $a = 0$ and $e = 0$, the system (16) is reduced to

$$\begin{cases} fx = -\lambda bz, \\ bg = 0, \\ g(x-\lambda) = 0. \end{cases}$$

If $g = 0$, then the above is just $fx = -\lambda bz$. When $x = 0$, we get K_4 and K_5, and $x \neq 0$ implies K_6. In the case that $g \neq 0$, we obtain K_7. The other possibility of $c = \lambda$ implies K_8 and K_9.

Now let $a \neq 0$. Then $e = -\lambda c$ via equating entries $(3,1)$ of the both sides of (16). Since $ag = -e^2/\lambda$ from comparing the $(2,1)$ entries, it follows from equating entries $(3,3)$ in (16) that

$$\lambda^2 c = ag + \lambda c^2 = -(-\lambda c)^2/\lambda + \lambda c^2 = 0.$$

So $c = 0$ and then $e = 0$. By equating $(3,2)$ entries, we have $f = \lambda$. Also $z = 0$ and $g = 0$ from comparing entries $(1,1)$ and $(2,1)$ of (16). Finally $x = 0$ via comparing the entries $(1,2)$ of (16), thus arriving at the corresponding solution matrix K_{10}. $\square$

Lemma 3.1 gives all the solutions $K_4, \ldots, K_{10}$ of the last equation of (8) with $\Lambda = \Lambda_2$. Substituting them for K in the system and solving the resulting three equations in succession, we are lead to the next theorem.

Theorem 3.2. *Suppose $A = PQ^T$ is such that its Jordan form is given by (4) with $\Lambda = \Lambda_2$. Then all the solutions of (1) are $X = WYW^{-1}$, where W is given by (6) and Y is partitioned as (7) in which M is an arbitrary $(n-3)\times(n-3)$ matrix such that $Y =$*

$$\begin{bmatrix} M & 0 & u_2 & 0 \\ v_1^T & 0 & y & 0 \\ v_2^T & 0 & f & 0 \\ v_3^T & 0 & b & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & 0 \\ v_1^T & 0 & y & 0 \\ 0^T & 0 & 0 & 0 \\ v_3^T & 0 & b & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 \\ v_1^T & 0 & y & 0 \\ v_2^T & 0 & f & 0 \\ -\frac{u_3^H u_1}{\lambda\|u_3\|^2} v_2^T & 0 & -\frac{f u_3^H u_1}{\lambda\|u_3\|^2} & 0 \end{bmatrix} \tag{17}$$

with $u_3 \neq 0$,

$$\begin{bmatrix} M & 0 & u_2 & u_3 \\ v_1^T & 0 & y & z \\ v_2^T & 0 & f & 0 \\ 0^T & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 \\ v_1^T & 0 & y & z \\ 0^T & 0 & 0 & 0 \\ 0^T & 0 & 0 & 0 \end{bmatrix}, \; z \neq 0, u_1 \neq 0, \tag{18}$$

$$\begin{bmatrix} M & u_1 & u_2 & u_3 \\ v_1^T & x & y & z \\ 0^T & 0 & 0 & 0 \\ 0^T & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & \frac{z}{x}u_1 \\ v_1^T & x & y & z \\ -\frac{\lambda z}{x} v_3^T & 0 & -\frac{\lambda bz}{x} & 0 \\ v_3^T & 0 & b & 0 \end{bmatrix}, \begin{bmatrix} M & 0 & u_2 & u_3 \\ v_1^T & \lambda & y & z \\ 0^T & 0 & 0 & g \\ 0^T & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

with $x \neq 0$ and $g \neq 0$,

$$\begin{bmatrix} M & 0 & u_2 & 0 \\ v_1^T & 0 & y & 0 \\ v_2^T & 0 & f & 0 \\ 0^T & 0 & 0 & \lambda \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & 0 \\ v_1^T & 0 & y & 0 \\ 0^T & 0 & 0 & 0 \\ 0^T & 0 & 0 & \lambda \end{bmatrix}, \; u_1 \neq 0, \tag{20}$$

$$\begin{bmatrix} M & u_1 & u_2 & 0 \\ v_1^T & x & y & 0 \\ 0^T & 0 & 0 & 0 \\ 0^T & 0 & 0 & \lambda \end{bmatrix}, \begin{bmatrix} M & 0 & u_2 & 0 \\ v_1^T & 0 & y & 0 \\ 0^T & 0 & \lambda & 0 \\ v_3^T & a & b & 0 \end{bmatrix}, \; x \neq 0, a \neq 0. \tag{21}$$

Proof. When $K = K_4$, the system (8) is reduced to

$$\begin{cases} u_1 v_2^T + \lambda u_3 v_3^T &= 0, \\ f u_1 + \lambda b u_3 &= 0, \end{cases}$$

which does not contain y so it will appear in all the solutions. Letting $u_3 = 0$. Then v_3 and b can be arbitrary, and the above system becomes $u_1 v_2^T = 0$ and $f u_1 = 0$. If $u_1 = 0$, then v_2 and f are arbitrary, giving the first solution matrix of (17). If $v_2 = 0$, then u_1 is arbitrary and $f = 0$, which results in the second matrix of (17). Now let $u_3 \neq 0$. Then from the above system,

$$v_3^T = -\frac{u_3^H u_1}{\lambda \|u_3\|^2} v_2^T, \quad b = -\frac{f u_3^H u_1}{\lambda \|u_3\|^2}.$$

This gives the third matrix of (17).

Next suppose $K = K_5$. Then (8) is simply $u_1 v_2^T = 0, f u_1 = 0, v_3 = 0$ since $\lambda \neq 0$ and $z \neq 0$. All the solutions are those in (18). Assume $K = K_6$. Then (8) can be written as

$$\begin{cases} u_1 v_2^T + \lambda u_3 v_3^T &= 0, \\ b u_3 - \frac{bz}{x} u_1 &= 0, \\ x v_2 + \lambda z v_3 &= 0. \end{cases}$$

Since $x \neq 0$, we can solve v_2 out from the last equation and substitute it into the first one in the above system, getting

$$\begin{cases} \left(u_3 - \frac{z}{x} u_1\right) v_3^T &= 0, \\ \left(u_3 - \frac{z}{x} u_1\right) b &= 0, \\ v_2 &= -\frac{\lambda z}{x} v_3. \end{cases}$$

Letting $b = 0$ and $v_3 = 0$ above gives the first matrix of (19), and if $u_3 = z u_1 / x$, then b and v_3 are arbitrary, giving the second one of (19).

For $K = K_7$, the system is simplified to $u_1 v_2^T + \lambda u_3 v_3^T = 0, u_1 = 0, v_2 + z v_3^T = 0, v_3 = 0$ since $g \neq 0$, whose solutions are given by the third matrix of (19). When $K = K_8$, the system (8) is actually $u_3 = 0, v_3 = 0, u_1 v_2^T = 0, f u_1 = 0$ since $\lambda \neq 0$. The solutions are the two matrices of (20). With $K = K_9$, the system becomes $v_2 = 0, v_3 = 0, u_1 v_2^T + \lambda u_3 v_3^T = 0, u_3 = 0$ since $\lambda \neq 0$ and $x \neq 0$, so the first matrix of (21) is obtained. Finally, as $K = K_{10}$, we have $u_1 = 0, v_2 = 0, u_1 v_2^T + \lambda u_3 v_3^T = 0, u_3 = 0$ since $a \neq 0$, thus obtaining the second matrix of (21). □

4 Solutions when A is of type III

Unlike the previous two cases that involve only 3×3 matrices Λ_1 and Λ_2, the third case that we shall study involves the 4×4 matrix Λ_3. The zero structure of Λ_3 ensures that u_2, u_4, v_1, v_3 are not present in the equations, so they are free vectors in the solutions. The following lemma gives all the solutions of the last equation of (8).

Lemma 4.1. *The solutions K of the equation (9) with $\Lambda = \Lambda_3$ are*

$$K_{11} \equiv \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix}; \ K_{12} \equiv \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ a_{33} \neq 0;$$

$$K_{13} \equiv \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, a_{13} \neq 0; K_{14} \equiv \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, a_{13} a_{31} \neq 0;$$

$$K_{15} \equiv \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & -\frac{a_{33}a_{42}}{a_{31}} & 0 & -\frac{a_{33}a_{44}}{a_{31}} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix}, a_{31} \neq 0;$$

$$K_{16} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & -\frac{a_{13}a_{42}}{a_{11}} & 0 & -\frac{a_{13}a_{44}}{a_{11}} \\ a_{31} & a_{32} & \frac{a_{13}a_{31}}{a_{11}} & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{bmatrix}, a_{11} \neq 0;$$

$$K_{17} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \neq 0.$$

Proof. Denote

$$K = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Then the equation $K\Lambda_3 K = \Lambda_3 K \Lambda_3$ is just

$$\begin{bmatrix} a_{11}a_{21}+a_{13}a_{41} & a_{11}a_{22}+a_{13}a_{42} & a_{11}a_{23}+a_{13}a_{43} & a_{11}a_{24}+a_{13}a_{44} \\ a_{21}a_{21}+a_{23}a_{41} & a_{21}a_{22}+a_{23}a_{42} & a_{21}a_{23}+a_{23}a_{43} & a_{21}a_{24}+a_{23}a_{44} \\ a_{31}a_{21}+a_{33}a_{41} & a_{31}a_{22}+a_{33}a_{42} & a_{31}a_{23}+a_{33}a_{43} & a_{31}a_{24}+a_{33}a_{44} \\ a_{41}a_{21}+a_{43}a_{41} & a_{41}a_{22}+a_{43}a_{42} & a_{41}a_{23}+a_{43}a_{43} & a_{41}a_{24}+a_{43}a_{44} \end{bmatrix} = \begin{bmatrix} 0 & a_{21} & 0 & a_{23} \\ 0 & 0 & 0 & 0 \\ 0 & a_{41} & 0 & a_{43} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

Comparing the $(2,1), (4,3), (2,3), (4,1)$ entries above gives the four equations

$$\begin{cases} a_{21}^2 = -a_{23}a_{41}, \\ a_{43}^2 = -a_{23}a_{41}, \\ a_{23}(a_{21}+a_{43}) = 0, \\ a_{41}(a_{21}+a_{43}) = 0. \end{cases} \tag{23}$$

We discuss two cases separately. First assume $a_{41} = 0$. Then $a_{21} = a_{43} = 0$ from (23). If $a_{23} \neq 0$, then $a_{11} = a_{31} = a_{44} = 0$ from equating entries $(1,3), (3,3), (2,4)$ of both sides in (22), from which $a_{23} = a_{11}a_{24} + a_{13}a_{44} = 0$, a contradiction. Thus $a_{23} = 0$ and (22) is reduced to

$$\begin{cases} a_{11}a_{22}+a_{13}a_{42} = 0, \\ a_{31}a_{22}+a_{33}a_{42} = 0, \\ a_{11}a_{24}+a_{13}a_{44} = 0, \\ a_{31}a_{24}+a_{33}a_{44} = 0 \end{cases} \tag{24}$$

with $a_{21} = a_{23} = a_{41} = a_{43} = 0$.

Now we assume $a_{41} \neq 0$. Then the last equation of (23) implies that $a_{21} = -a_{43}$. On the other hand, the entries $(4,2)$ and $(1,1)$ equations of (22) gives that $a_{22} = -a_{43}a_{42}/a_{41}$ and $a_{13} = -a_{11}a_{21}/a_{41}$. Hence

$$a_{21} = a_{11}a_{22} + a_{13}a_{42} = -\frac{a_{11}a_{43}a_{42}}{a_{41}} - \frac{a_{11}a_{21}a_{42}}{a_{41}} = \frac{a_{11}a_{21}a_{42}}{a_{41}} - \frac{a_{11}a_{21}a_{42}}{a_{41}} = 0,$$

and so $a_{43} = 0$. On the other hand, the entries $(3,1)$ and $(4,2)$ equalities give $a_{33} = a_{22} = 0$, so $a_{41} = a_{31}a_{22} + a_{33}a_{42} = 0$ from the entry $(3,2)$ equality of (22), contradictory to the assumption that $a_{41} \neq 0$.

Therefore, (24) is the only equation for us to solve. Let $a_{11} = 0$. Then (24) becomes $a_{13}a_{42} = 0, a_{31}a_{22} + a_{33}a_{42} = 0, a_{13}a_{44} = 0, a_{31}a_{24} + a_{33}a_{44} = 0$. If $a_{13} = 0$, then the above system is $a_{31}a_{22} + a_{33}a_{42} = 0, a_{31}a_{24} + a_{33}a_{44} = 0$. The first case is $a_{31} = 0$. Then $a_{33}a_{42} = 0$ and $a_{33}a_{44} = 0$. If $a_{33} = 0$, then we have solution matrix K_{11}; otherwise K_{12} occurs. For the second case $a_{31} \neq 0$, there hold $a_{22} = -a_{33}a_{42}/a_{31}$ and $a_{24} = -a_{33}a_{44}/a_{31}$, giving K_{15}. If $a_{13} \neq 0$, then $a_{42} = 0$ and $a_{44} = 0$, so $a_{31}a_{22} = 0, a_{31}a_{24} = 0$. The case that $a_{31} = 0$ implies K_{13} and when $a_{31} \neq 0$, we obtain K_{14}.

Now assume that $a_{11} \neq 0$. Then $a_{22} = -a_{13}a_{42}/a_{11}$ and $a_{24} = -a_{13}a_{44}/a_{11}$, so $a_{31}a_{22} + a_{33}a_{42} = 0$ and $a_{31}a_{24} + a_{33}a_{44} = 0$, or equivalently,

$$\begin{cases} a_{42}\left(a_{33} - \frac{a_{13}a_{31}}{a_{11}}\right) = 0, \\ a_{44}\left(a_{33} - \frac{a_{13}a_{31}}{a_{11}}\right) = 0. \end{cases}$$

In the case that $a_{33} = a_{13}a_{31}/a_{11}$, the numbers a_{42} and a_{44} are arbitrary and the matrix K_{16} appears, and otherwise $a_{42} = a_{44} = 0$, so $a_{22} = a_{24} = 0$, resulting in K_{17}. This completes the proof. $\square$

Lemma 4.1 gives all the solutions $K_{11}, \ldots, K_{17}$ of the last equation of (8) with $\Lambda = \Lambda_3$, which can be used to prove the next result.

Theorem 4.2. *Suppose $\Lambda = \Lambda_3$ in the Jordan form (4) of $A = PQ^T$. Then all the solutions of (1) are $X = WYW^{-1}$, where Y is partitioned as (7) in which M is an arbitrary $(n-4) \times (n-4)$ matrix such that $Y =$*

$$\begin{bmatrix} M & 0 & u_2 & 0 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & 0 & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \begin{bmatrix} M & 0 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & 0 & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \; u_3 \neq 0, \tag{25}$$

$$\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ -\frac{u_1^H u_3}{\|u_1\|^2} v_4^T & 0 & -\frac{u_1^H u_3}{\|u_1\|^2} a_{42} & 0 & -\frac{u_1^H u_3}{\|u_1\|^2} a_{44} \\ v_3^T & 0 & a_{32} & 0 & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \; u_1 \neq 0, \tag{26}$$

$$\begin{bmatrix} M & 0 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \; u_1 \neq 0, a_{33} \neq 0, \tag{27}$$

$$\begin{bmatrix} M & 0 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & a_{13} & a_{14} \\ v_2^T & 0 & a_{22} & 0 & a_{24} \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & 0 & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \; u_1 \neq 0, a_{13} \neq 0, \tag{28}$$

$$\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & \frac{a_{33}}{a_{31}} u_1 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ -\frac{a_{33}}{a_{31}} v_4^T & 0 & -\frac{a_{33}a_{42}}{a_{31}} & 0 & -\frac{a_{33}a_{44}}{a_{31}} \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix} \tag{29}$$

with $a_{13} \neq 0, a_{31} \neq 0$,

$$\begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & 0 & a_{12} & 0 & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & a_{11} & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & \frac{a_{13}a_{31}}{a_{11}} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

with $a_{31} \neq 0, u_3 \neq a_{33}u_1/a_{31}$ *in the left matrix, and* $a_{11} \neq 0$*,* $u_3 \neq a_{13}u_1/a_{11}$ *in the second one,*

$$\begin{bmatrix} M & u_1 & u_2 & \frac{a_{13}}{a_{11}}u_1 & u_4 \\ v_1^T & a_{11} & a_{12} & a_{13} & a_{14} \\ -\frac{a_{13}}{a_{11}}v_4^T & 0 & -\frac{a_{13}a_{42}}{a_{11}} & 0 & -\frac{a_{13}a_{44}}{a_{11}} \\ v_3^T & a_{31} & a_{32} & \frac{a_{13}a_{31}}{a_{11}} & a_{34} \\ v_4^T & 0 & a_{42} & 0 & a_{44} \end{bmatrix}, \begin{bmatrix} M & u_1 & u_2 & u_3 & u_4 \\ v_1^T & a_{11} & a_{12} & a_{13} & a_{14} \\ 0^T & 0 & 0 & 0 & 0 \\ v_3^T & a_{31} & a_{32} & a_{33} & a_{34} \\ 0^T & 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

with $a_{11} \neq 0, a_{33} \neq a_{13}a_{31}/a_{11}$.

Proof. Clearly the system (8) does not involve u_2, u_4, v_1, v_3, so they are free vectors in all the solutions. The first equation of (8) is $u_1v_2^T + u_3v_4^T = 0$. Now we solve the first three equations of (8) with $K = K_{11}, \dots, K_{17}$ separately.

When $K = K_{11}$, the system (8) is reduced to

$$\begin{cases} u_1v_2^T + u_3v_4^T &= 0, \\ a_{22}u_1 + a_{42}u_3 &= 0, \\ a_{24}u_1 + a_{44}u_3 &= 0. \end{cases}$$

Assume that $u_1 = 0$. Then $u_3v_4^T = 0, a_{42}u_3 = 0, a_{44}u_3 = 0$. If we let $u_3 = 0$, then v_4, a_{42}, a_{44} can be arbitrary, getting the first matrix of (25). Otherwise $v_4 = 0, a_{42} = a_{44} = 0$, which results in the second matrix in (25). Now assume that $u_1 \neq 0$. Then $v_2 = -(u_1^H u_3)v_4/\|u_1\|^2, a_{22} = -(u_1^H u_3)a_{42}/\|u_1\|^2, a_{24} = -(u_1^H u_3)a_{44}/\|u_1\|^2$. This gives (26).

For $K = K_{12}$, (8) becomes

$$\begin{cases} u_1v_2^T + u_3v_4^T &= 0, \\ a_{22}u_1 &= 0, \\ a_{24}u_1 &= 0, \\ a_{33}v_4^T &= 0. \end{cases}$$

Since $a_{33} \neq 0$, we have $v_4 = 0$, so the above is simplified to $u_1v_2^T = 0, a_{22}u_1 = 0, a_{24}u_1 = 0$. Letting $u_1 = 0$ gives the first matrix of (27), and otherwise we have $v_2 = 0, a_{22} = a_{24} = 0$, leading to the second matrix in (27).

If $K = K_{13}$, then (8) is simplified to

$$\begin{cases} v_4 &= 0, \\ u_1v_2^T &= 0, \\ a_{22}u_1 &= 0, \\ a_{24}u_1 &= 0. \end{cases}$$

$u_1 = 0$ produces the left matrix of (28) and otherwise, $v_2 = 0, a_{22} = a_{24} = 0$, so the second matrix in (28).

With $K = K_{14}$, we have

$$\begin{cases} u_1v_2^T + u_3v_4^T &= 0, \\ a_{13}v_4^T &= 0, \\ a_{31}v_2^T + a_{33}v_4^T &= 0. \end{cases}$$

Since $a_{13} \neq 0$ and $a_{31} \neq 0$, we have $v_2 = v_4 = 0$, and the first matrix of (29) is obtained.

The choice of $K = K_{15}$ gives the system

$$\begin{cases} u_1v_2^T + u_3v_4^T &= 0, \\ a_{42}\left(u_3 - \frac{a_{33}}{a_{31}}u_1\right) &= 0, \\ a_{44}\left(u_3 - \frac{a_{33}}{a_{31}}u_1\right) &= 0, \\ a_{31}v_2^T + a_{33}v_4^T &= 0. \end{cases}$$

Since $a_{31} \neq 0$, from the last equation, $v_2 = -a_{33}v_4/a_{31}$. Substituting into the first equation, we obtain $(u_3 - a_{33}u_1/a_{31})v_4^T = 0$. So if $u_3 - a_{33}u_1/a_{31} = 0$, then v_4, a_{42}, a_{44} are arbitrary and the second matrix of (29) is true. Otherwise, $v_4 = 0, a_{42} = a_{44} = 0$, which gives the first matrix of (30).

In the case $K = K_{16}$, (8) can be written as

$$\begin{cases} u_1 v_2^T + u_3 v_4^T & = 0, \\ a_{42}\left(u_3 - \frac{a_{13}}{a_{11}} u_1\right) & = 0, \\ a_{44}\left(u_3 - \frac{a_{13}}{a_{11}} u_1\right) & = 0, \\ a_{11} v_2^T + a_{13} v_4^T & = 0, \\ a_{31}\left(v_2^T + \frac{a_{13}}{a_{11}} v_4^T\right) & = 0. \end{cases}$$

By the forth equation, $v_2 = -a_{13}v_4/a_{11}$. Substituting into the first one gives $(u_3 - a_{13}u_1/a_{11})v_4^T = 0$. Hence, if $u_3 \neq a_{13}u_1/a_{11}$, then $v_4 = 0, a_{42} = a_{44} = 0$ and we get the right matrix of (30). Otherwise v_4, a_{42}, a_{44} are arbitrary, so the left matrix of (31).

The last case is $K = K_{17}$. Then

$$\begin{cases} u_1 v_2^T + u_3 v_4^T & = 0, \\ a_{11} v_2^T + a_{13} v_4^T & = 0, \\ a_{31} v_2^T + a_{33} v_4^T & = 0, \end{cases}$$

the solutions of which are given by the last matrix of (31). □

5 Examples

We give two examples to illustrate our results. The first one is artificial for the use of Theorem 3.1. Let $p_1 = [0, 1, 1, 0]^T$, $p_2 = [0, 0, 0, 1]^T$, $q_1 = [1, -1, 1, 0]^T$, and $q_2 = [-1, 1, -1, 1]^T$, so that

$$A = p_1 q_1^T + p_2 q_2^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} = WJW^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

By Theorem 3.1 and multiplying WYW^{-1} out, all the solutions of (1) are

$$X = \begin{bmatrix} m + u_2 & -u_2 & u_2 & 0 \\ m + v_1 + u_2 + y & -u_2 - y & u_2 + y & 0 \\ v_1 + v_2 + y + f & -y - f & y + f & 0 \\ v_2 + v_3 + f + b & -f - b & f + b & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} m - u_1 + u_2 & u_1 - u_2 & u_2 & 0 \\ m + v_1 - u_1 + u_2 + y & u_1 - u_2 - y & u_2 + y & 0 \\ v_1 + y & -y & y & 0 \\ v_3 + b & -b & b & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} m-u_1+u_2-u_3 & u_1-u_2+u_3 & u_2-u_3 & u_3 \\ m+v_1-u_1+u_2+y-u_3 & u_1-u_2-y+u_3 & u_2+y-u_3 & u_3 \\ v_1+v_2+y+f & -y-f & y+f & 0 \\ (v_2+f)(1-u_1/u_3) & f(u_1/u_3-1) & f(1-u_1/u_3) & 0 \end{bmatrix}, u_3 \neq 0,$$

$$X = \begin{bmatrix} m+u_2-u_3 & u_3-u_2 & u_2-u_3 & u_3 \\ m+v_1+u_2+y-u_3-z & u_3+z-u_2-y & u_2+y-u_3-z & u_3+z \\ v_1+v_2+y+f-z & z-y-f & y+f-z & z \\ v_2+f & -f & f & 0 \end{bmatrix}, z \neq 0,$$

$$X = \begin{bmatrix} m-u_1+u_2-u_3 & u_1-u_2+u_3 & u_2-u_3 & u_3 \\ m+v_1-u_1+u_2+y-u_3-z & u_1-u_2-y+u_3+z & u_2+y-u_3-z & u_3+z \\ v_1+y-z & z-y & y-z & z \\ 0 & 0 & 0 & 0 \end{bmatrix}, u_1 \neq 0, z \neq 0,$$

$$X=\begin{bmatrix} m-u_1+u_2-u_3 & u_1-u_2+u_3 & u_2-u_3 & u_3\\ m+v_1-u_1-x+u_2+y-u_3-z & u_1+x-u_2-y+u_3+z & u_2+y-u_3-z & u_3+z\\ v_1-x+y-z & x+z-y & y-z & z\\ 0 & 0 & 0 & 0\end{bmatrix}, x\neq 0,$$

$$X=\begin{bmatrix} m-u_1+u_2-\frac{z}{x}u_1 & u_1-u_2+\frac{z}{x}u_1 & u_2-\frac{z}{x}u_1 & \frac{z}{x}u_1\\ m+v_1-u_1-x+u_2+y-\frac{z}{x}u_1-z & u_1+x-u_2-y+\frac{z}{x}u_1+z & u_2+y-\frac{z}{x}u_1-z & \frac{z}{x}u_1+z\\ v_1-\frac{z}{x}v_3-x+y+\frac{z}{x}b-z & x+z-y+\frac{z}{x}b & y-z-\frac{z}{x}b & z\\ v_3(1-\frac{z}{x})-\frac{z}{x}b & \frac{z}{x}b & -\frac{z}{x}b & 0\end{bmatrix}, x\neq 0,$$

$$X=\begin{bmatrix} m+u_2-u_3 & u_3-u_2 & u_2-u_3 & u_3\\ m+v_1-1+u_2+y-u_3-z & 1-u_2-y+u_3+z & u_2+y-u_3-z & u_3+z\\ v_1-1+y-z-g & 1+z-y+g & y-z-g & z+g\\ -g & g & -g & g\end{bmatrix}, g\neq 0,$$

$$X=\begin{bmatrix} m+u_2 & -u_2 & u_2 & 0\\ m+v_1+u_2+y & -u_2-y & u_2+y & 0\\ v_1+v_2+y+f & -y-f & y+f & 0\\ v_2+f-1 & 1-f & f-1 & 1\end{bmatrix},$$

$$X=\begin{bmatrix} m-u_1+u_2 & u_1-u_2 & u_2 & 0\\ m+v_1-u_1+u_2+y & u_1-u_2-y & u_2+y & 0\\ v_1+y & -y & y & 0\\ -1 & 1 & -1 & 1\end{bmatrix}, u_1\neq 0,$$

$$X=\begin{bmatrix} m-u_1+u_2 & u_1-u_2 & u_2 & 0\\ m+v_1-u_1-x+u_2+y & u_1+x-u_2-y & u_2+y & 0\\ v_1-x+y & x-y & y & 0\\ -1 & 1 & -1 & 1\end{bmatrix}, x\neq 0,$$

$$X=\begin{bmatrix} m+u_2 & -u_2 & u_2 & 0\\ m+v_1+u_2+y & -u_2-y & u_2+y & 0\\ v_1+y+1 & -y-1 & y+1 & 0\\ v_3-a+b+1 & a-b-1 & b+1 & 0\end{bmatrix}, a\neq 0.$$

The matrix in our second example has appeared in the application of the classical Yang-Baxter equation to the inverse scattering theory [18]. Let $p_1=[1,0,0,0]^T$, $p_2=[0,1,0,0]^T$, $q_1=[0,0,1,0]^T$, and $q_2=[0,0,0,1]^T$, so

$$A=p_1q_1^T+p_2q_2^T=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}=WJW^{-1}=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}.$$

Note that A is a member of the second matrix class in (2) as the tensor product $J_0\otimes I_2$ of the 2×2 Jordan block J_0 with eigenvalue 0 and I_2. Since $J=\Lambda_3$ in this example, all the solutions Y of (3) are exactly $K_{11},\dots,K_{17}$ as given by Lemma 4.1. Multiplying $X=WYW^{-1}=WK_kW$ out with $k=11,\dots,17$, we find all the solutions of (1):

$$X=\begin{bmatrix}0&0&a_{12}&a_{14}\\0&0&a_{32}&a_{34}\\0&0&a_{22}&a_{24}\\0&0&a_{42}&a_{44}\end{bmatrix};\quad \begin{bmatrix}0&0&a_{12}&a_{14}\\0&a_{33}&a_{32}&a_{34}\\0&0&a_{22}&a_{24}\\0&0&0&0\end{bmatrix},\ a_{33}\neq 0;$$

$$X=\begin{bmatrix}0&a_{13}&a_{12}&a_{14}\\0&a_{33}&a_{32}&a_{34}\\0&0&a_{22}&a_{24}\\0&0&0&0\end{bmatrix},\ a_{13}\neq 0;\quad \begin{bmatrix}0&a_{13}&a_{12}&a_{14}\\a_{31}&a_{33}&a_{32}&a_{34}\\0&0&0&0\\0&0&0&0\end{bmatrix},\ a_{13}a_{31}\neq 0;$$

$$X=\begin{bmatrix}0&0&a_{12}&a_{14}\\a_{31}&a_{33}&a_{32}&a_{34}\\0&0&-\frac{a_{33}a_{42}}{a_{31}}&-\frac{a_{33}a_{44}}{a_{31}}\\0&0&a_{42}&a_{44}\end{bmatrix},\ a_{31}\neq 0;$$

$$X=\begin{bmatrix} a_{11} & a_{13} & a_{12} & a_{14} \\ a_{31} & \frac{a_{13}a_{31}}{a_{11}} & a_{32} & a_{34} \\ 0 & 0 & -\frac{a_{13}a_{42}}{a_{11}} & -\frac{a_{13}a_{44}}{a_{11}} \\ 0 & 0 & a_{42} & a_{44} \end{bmatrix},\ a_{11}\neq 0;$$

$$X=\begin{bmatrix} a_{11} & a_{13} & a_{12} & a_{14} \\ a_{31} & a_{33} & a_{32} & a_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ a_{11}\neq 0,\ \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\neq 0.$$

6 Conclusions

As continuation of our earlier paper in which all the solutions of the Yang-Baxter-like matrix equation (1) were found when $A = PQ^T$, with P and Q being $n \times 2$ and of rank-two such that $Q^T P$ is nonsingular, we have found in this paper all the solutions of (1) when $Q^T P$ is singular. For each of the resulting three kinds of Jordan forms of A we solved the corresponding simplified matrix equation, thus obtaining all the structures of the solution matrices.

The various structures of the solution matrices reflect the complicated structure of the algebraic varieties as solutions of polynomial systems of multi-variable. Although we were not able to use algebraic geometry to find all the solutions of (1) in the general case, we have been successful in finding all the solutions when the rank of A is at most 2. Our future work will be devoted to the exploration of solving the Yang-Baxter-like matrix equation with a given matrix A of rank-k.

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