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ON THE YOUNG-LAPLACE RELATION AND THE EVOLUTION OF A PERTURBED ELLIPSOID

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Abstract. The Young-Laplace relation states that the interface separating two fluids, develops in such a way that the difference between the outer and the inner pressure remains proportional to the mean curvature at every point of the interface. This relation guides the evolution of a free boundary. Considering the importance of the ellipsoidal surfaces as free boundaries in anisotropic evolutions, it is of great interest to have ready-to-use formulae for the mean curvature of a perturbed ellipsoidal surface. These formulae provide the basis for the stability analysis of free boundary value problems in Fluid Mechanics. The present work calculates the first order approximation of the local curvatures for a surface which is a perturbation of an ellipsoid.

1. Introduction. One of the most important questions concerning the development of a free boundary is the stability of the process in connection with the preservation of the initial shape of the boundary. That is, we would like to know whether a boundary which develops as a sphere, returns to its spherical shape if at some point suffers some small geometrical distortion. In an anisotropic environment, where every direction establishes its own standards, a realistic developing surface is that of an ellipsoid. Then, we are interested to know if a boundary perturbation of the ellipsoid will ultimately leave the ellipsoidal shape invariant or will it distort the ellipsoid to any other shape. In fluid mechanics, a smooth free boundary is guided by the Young-Laplace relation [1]

$$P_{\text{out}} - P_{\text{in}} = \alpha H \tag{1}$$

where H is the mean curvature

$$H = \frac{1}{2}(\kappa_1 + \kappa_2), \tag{2}$$

κ_1, κ_2 are the principal curvatures, and P_{out}, P_{in} are the values of the pressure field as we approach a point on the boundary from outside and inside, respectively. We see that in order to apply the Young-Laplace relation to any particular free boundary value problem, we need to calculate the mean curvature at any point on the boundary. If, in

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particular, we want to study the stability of the free boundary, we need to calculate the mean curvature of a small disturbance of the boundary surface.

In the present work, we consider the generic case of an ellipsoidal surface that corresponds to any confocal ellipsoidal system. In particular, we apply a disturbance, described by a smooth function f defined on the surface of the ellipsoid, which is controlled by a small positive parameter ε. Then, we calculate the principal curvatures, and therefore the mean curvature, of the disturbed surface in terms of the function f and its first and second derivatives. Finally, we perform an asymptotic analysis of the results in the parameter ε, as it approaches zero. The terms that are independent of ε recover the curvatures on the unperturbed ellipsoid, while the terms that are linear in ε provide the first order approximations of the curvatures on the perturbed surface. These results could be applied to any related free boundary value problem.

The paper is organized as follows. Section 2 provides the minimum information about the ellipsoidal system needed to make this work selfcontained. Section 3 contains the perturbation of the first fundamental form of the perturbed surface and Section 4 contains the corresponding perturbation of the outward unit vector. The perturbation of the second fundamental form is presented in Section 5 and finally the perturbation of the curvatures are given in Section 6.

2. The ellipsoidal system. The definition of an ellipsoidal system demands the determination of a reference ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \tag{3}$$

where $0 < a_3 < a_2 < a_1 < +\infty$, which fixes the foci of the system and establishes the standards of every spatial direction. The reference ellipsoid (1) plays the role of the unit sphere in the case of the spherical system. The six foci of the ellipsoidal system are located at the points $(\pm h_2, 0, 0)$, $(\pm h_3, 0, 0)$ and $(0, \pm h_1, 0)$, where

$$h_1^2 = a_2^2 - a_3^2 \tag{4}$$

$$h_2^2 = a_1^2 - a_3^2 \tag{5}$$

$$h_3^2 = a_1^2 - a_2^2 \tag{6}$$

and they are related by the equation

$$h_1^2 - h_2^2 + h_3^2 = 0. \tag{7}$$

The backbone of the ellipsoidal system is given by the focal ellipse

$$\frac{x_1^2}{h_2^2} + \frac{x_2^2}{h_1^2} = 1, \quad x_3 = 0 \tag{8}$$

and the focal hyperbola

$$\frac{x_1^2}{h_3^2} - \frac{x_3^2}{h_1^2} = 1, \quad x_2 = 0. \tag{9}$$

In the first octant, the ellipsoidal coordinates (ρ, μ, ν) are related to the Cartesian coordinates by

$$x_1 = \frac{\rho\mu\nu}{h_2 h_3}, h_2 < \rho < +\infty \quad (10)$$

$$x_2 = \frac{\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}, \quad h_3 < \mu < h_2 \quad (11)$$

$$x_3 = \frac{\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}, \quad 0 < \nu < h_3 \quad (12)$$

while the other seven octants are specified by considering the appropriate signs of the x_i's. The variable $\rho =$ constant defines an ellipsoid and therefore it corresponds to the radial variable of the spherical system. In particular, the focal ellipse (8) corresponds to the value $\rho = h_2$. The coordinates (μ, ν) identify a point on the ellipsoid $\rho =$ constant and therefore they can be considered as the orientation variables on any ellipsoidal surface.

By varying the variable ρ we obtain the family of ellipsoids

$$\frac{x_1^2}{\rho^2} + \frac{x_2^2}{\rho^2 - h_3^2} + \frac{x_3^2}{\rho^2 - h_2^2} = 1, \quad \rho^2 \in (h_2^2, +\infty). \quad (13)$$

Similarly, the variation of the variable μ defines the family of hyperboloids of one sheet

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - h_3^2} + \frac{x_3^2}{\mu^2 - h_2^2} = 1, \quad \mu^2 \in (h_3^2, h_2^2) \quad (14)$$

and the variation of the variable ν defines the family of hyperboloids of two sheets

$$\frac{x_1^2}{\nu^2} + \frac{x_2^2}{\nu^2 - h_3^2} + \frac{x_3^2}{\nu^2 - h_2^2} = 1, \quad \nu^2 \in (0, h_3^2). \quad (15)$$

Standard references for boundary value problems in ellipsoidal geometry are [3], [4]. For a fairly complete analysis of the ellipsoidal system and the related ellipsoidal harmonics we refer to [2].

3. Perturbation of the first fundamental form.

Let

$$\frac{x_1^2}{\rho_0^2} + \frac{x_2^2}{\rho_0^2 - h_3^2} + \frac{x_3^2}{\rho_0^2 - h_2^2} = 1 \quad (16)$$

be the unperturbed ellipsoid and consider the perturbation

$$\rho(\mu, \nu) = \rho_0 + \varepsilon f(\mu, \nu) \quad (17)$$

where ε is a small positive parameter and f defines a smooth function of the orientation variables (μ, ν). A fixed point (μ, ν) on the unperturbed ellipsoid ρ_0 specifies the coordinate curve $(\mu, \nu) =$ constant and $f(\mu, \nu)$ measures the local deviation of the surface point (μ, ν) along this curve. The extent of the perturbation on the ellipsoid $\rho = \rho_0$ is controlled by the support of the function f. A localized perturbation on the ellipsoid (16) coresponds to a support of f described by small variations of μ and ν.

The vectorial representation of the perturbed ellipsoid $\rho(\mu,\nu)$ with respect to a Cartesian system, with its origin in the center and its axes along the principal axes of the unperturbed ellipsoid, is given by

$$\begin{aligned} \boldsymbol{r}(\mu,\nu) &= \frac{1}{h_2h_3}(\rho_0+\varepsilon f(\mu,\nu))\mu\nu\hat{\boldsymbol{x}}_1 \\ &+ \frac{1}{h_1h_3}\sqrt{\rho_0^2-h_3^2+2\rho_0\varepsilon f(\mu,\nu)+\varepsilon^2 f^2(\mu,\nu)}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}\hat{\boldsymbol{x}}_2 \\ &+ \frac{1}{h_1h_2}\sqrt{\rho_0^2-h_2^2+2\rho_0\varepsilon f(\mu,\nu)+\varepsilon^2 f^2(\mu,\nu)}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}\hat{\boldsymbol{x}}_3. \end{aligned} \tag{18}$$

The first fundamental form on the perturbed ellipsoid $\rho(\mu,\nu)$ is given by

$$d\boldsymbol{r}\cdot d\boldsymbol{r} = E(d\nu)^2+2F(d\nu)(d\mu)+G(d\mu)^2 \tag{19}$$

where

$$E = \boldsymbol{r}_\nu\cdot\boldsymbol{r}_\nu \tag{20}$$

$$F = \boldsymbol{r}_\nu\cdot\boldsymbol{r}_\mu \tag{21}$$

$$G = \boldsymbol{r}_\mu\cdot\boldsymbol{r}_\mu \tag{22}$$

and the subindex represents differentiation with respect to the indicated variable. From (18) we obtain

$$\begin{aligned} \boldsymbol{r}_\nu &= x_1\left(\frac{1}{\nu}+\varepsilon f_\nu\frac{1}{\rho}\right)\hat{\boldsymbol{x}}_1+x_2\left(\frac{\nu}{\nu^2-h_3^2}+\varepsilon f_\nu\frac{\rho}{\rho^2-h_3^2}\right)\hat{\boldsymbol{x}}_2 \\ &+x_3\left(\frac{\nu}{\nu^2-h_2^2}+\varepsilon f_\nu\frac{\rho}{\rho^2-h_2^2}\right)\hat{\boldsymbol{x}}_3 \end{aligned} \tag{23}$$

and

$$\begin{aligned} \boldsymbol{r}_\mu &= x_1\left(\frac{1}{\mu}+\varepsilon f_\mu\frac{1}{\rho}\right)\hat{\boldsymbol{x}}_1+x_2\left(\frac{\mu}{\mu^2-h_3^2}+\varepsilon f_\mu\frac{\rho}{\rho^2-h_3^2}\right)\hat{\boldsymbol{x}}_2 \\ &+x_3\left(\frac{\mu}{\mu^2-h_2^2}+\varepsilon f_\mu\frac{\rho}{\rho^2-h_2^2}\right)\hat{\boldsymbol{x}}_3 \end{aligned} \tag{24}$$

where ρ is given by (17).

Let's denote by $\boldsymbol{r}_0=(x_1^0,x_2^0,x_3^0)$ the point on the (μ,ν)-coordinate curve that lies on the ellipsoid $\rho_0=$ constant. The local ellipsoidal system of the unperturbed ellipsoid at the point (x_1^0,x_2^0,x_3^0) is specified by

$$\hat{\boldsymbol{\rho}}_0 = \frac{\rho_0}{h_\rho^0}\left(\frac{x_1^0}{\rho_0^2},\frac{x_2^0}{\rho_0^2-h_3^2},\frac{x_3^0}{\rho_0^2-h_2^2}\right) \tag{25}$$

$$\hat{\boldsymbol{\nu}}_0 = \frac{\nu}{h_\nu^0}\left(\frac{x_1^0}{\nu^2},\frac{x_2^0}{\nu^2-h_3^2},\frac{x_3^0}{\nu^2-h_2^2}\right) \tag{26}$$

$$\hat{\boldsymbol{\mu}}_0 = \frac{\mu}{h_\mu^0}\left(\frac{x_1^0}{\mu^2},\frac{x_2^0}{\mu^2-h_3^2},\frac{x_3^0}{\mu^2-h_2^2}\right) \tag{27}$$

where

$$h_\rho^0 = \frac{\sqrt{\rho_0^2-\mu^2}\sqrt{\rho_0^2-\nu^2}}{\sqrt{\rho_0^2-h_3^2}\sqrt{\rho_0^2-h_2^2}} \tag{28}$$

$$h_\nu^0 = \frac{\sqrt{\mu^2-\nu^2}\sqrt{\rho_0^2-\nu^2}}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}} \tag{29}$$

$$h_\mu^0 = \frac{\sqrt{\mu^2-\nu^2}\sqrt{\rho_0^2-\mu^2}}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}} \tag{30}$$

are the metric coefficients on the unperturbed ellipsoid.

Then, if we expand formulae (18), (23) and (24) in powers of ε and keep only the linear approximation, we arrive at the asymptotic expressions

$$\begin{aligned} \boldsymbol{r} &= x_1^0\left(1+\varepsilon f\frac{1}{\rho_0}\right)\hat{\boldsymbol{x}}_1 + x_2^0(1+\varepsilon f\frac{\rho_0}{\rho_0^2-h_3^2})\hat{\boldsymbol{x}}_2 + x_3^0(1+\varepsilon f\frac{\rho_0}{\rho_0^2-h_2^2})\hat{\boldsymbol{x}}_3 + O(\varepsilon^2) \\ &= \boldsymbol{r}_0 + \varepsilon f(\mu,\nu)h_\rho^0\widehat{\boldsymbol{\rho}}_0 + O(\varepsilon^2) \end{aligned} \tag{31}$$

$$\begin{aligned} \boldsymbol{r}_\nu &= x_1^0\left(\frac{1}{\nu}+\varepsilon f\frac{1}{\nu}\frac{1}{\rho_0}+\varepsilon f_\nu\frac{1}{\rho_0}\right)\hat{\boldsymbol{x}}_1 \\ &+ x_2^0\left(\frac{\nu}{\nu^2-h_3^2}+\varepsilon f\frac{\nu}{\nu^2-h_3^2}\frac{\rho_0}{\rho_0^2-h_3^2}+\varepsilon f_\nu\frac{\rho_0}{\rho_0^2-h_3^2}\right)\hat{\boldsymbol{x}}_2 \\ &+ x_3^0\left(\frac{\nu}{\nu^2-h_2^2}+\varepsilon f\frac{\nu}{\nu^2-h_2^2}\frac{\rho_0}{\rho_0^2-h_2^2}+\varepsilon f_\nu\frac{\rho_0}{\rho_0^2-h_2^2}\right)\hat{\boldsymbol{x}}_3 + O(\varepsilon^2) \\ &= h_\nu^0\hat{\boldsymbol{\nu}}_0 - \frac{\varepsilon}{\rho_0^2-\nu^2}f[\nu h_\rho^0\hat{\boldsymbol{\rho}}_0 - \rho_0 h_\nu^0\hat{\boldsymbol{\nu}}_0] + \varepsilon h_\rho^0 f_\nu\hat{\boldsymbol{\rho}}_0 + +O(\varepsilon^2) \end{aligned} \tag{32}$$

$$\begin{aligned} \boldsymbol{r}_\mu &= x_1^0\left(\frac{1}{\mu}+\varepsilon f\frac{1}{\mu}\frac{1}{\rho_0}+\varepsilon f_\mu\frac{1}{\rho_0}\right)\widehat{\boldsymbol{x}}_1 \\ &+ x_2^0\left(\frac{\mu}{\mu^2-h_3^2}+\varepsilon f\frac{\mu}{\mu^2-h_3^2}\frac{\rho_0}{\rho_0^2-h_3^2}+\varepsilon f_\mu\frac{\rho_0}{\rho_0^2-h_3^2}\right)\hat{\boldsymbol{x}}_2 \\ &+ x_3^0\left(\frac{\mu}{\mu^2-h_2^2}+\varepsilon f\frac{\mu}{\mu^2-h_2^2}\frac{\rho_0}{\rho_0^2-h_2^2}+\varepsilon f_\mu\frac{\rho_0}{\rho_0^2-h_2^2}\right)\hat{\boldsymbol{x}}_3 + O(\varepsilon^2) \\ &= h_\mu^0\hat{\boldsymbol{\mu}}_0 - \frac{\varepsilon}{\rho_0^2-\mu^2}f[\mu h_\rho^0\hat{\boldsymbol{\rho}}_0 - \rho_0 h_\mu^0\hat{\boldsymbol{\mu}}_0] + \varepsilon h_\rho^0 f_\mu\hat{\boldsymbol{\rho}}_0 + +O(\varepsilon^2). \end{aligned} \tag{33}$$

Note that as $\varepsilon \to 0$, we recover the expected limits $\boldsymbol{r} \to \boldsymbol{r}_0$, $\boldsymbol{r}_\nu \to h_\nu^0\hat{\boldsymbol{\nu}}_0$ and $\boldsymbol{r}_\mu \to h_\mu^0\hat{\boldsymbol{\mu}}_0$.

From the asymptotic relation (31) we see that the leading perturbation of the position vector is described along the normal to the ellipsoid at the particular point. Indeed, the term of order ε is directed along $\hat{\boldsymbol{\rho}}_0$, while the actual local displacement occurs along the curve (μ, ν).

From (32) and (33) we can easily evaluate the coefficients (20)-(22) of the first fundamental form, which have the following asymptotic expansions

$$E = (h_\nu^0)^2 \left(1 + \varepsilon f \frac{2\rho_0}{\rho_0^2 - \nu^2}\right) + O(\varepsilon^2) \tag{34}$$

$$F = O(\varepsilon^2) \tag{35}$$

$$G = (h_\mu^0)^2 \left(1 + \varepsilon f \frac{2\rho_0}{\rho_0^2 - \mu^2}\right) + O(\varepsilon^2). \tag{36}$$

As the perturbation parameter ε tends to zero the first fundamental form on the ellipsoid ρ_0 reduces to

$$(ds)^2 = (h_\nu^0)^2 (d\nu)^2 + (h_\mu^0)^2 (d\mu)^2. \tag{37}$$

4. Perturbation of the unit normal. The unit normal on the unperturbed ellipsoid ρ_0 is given by $\hat{\boldsymbol{\rho}}_0$, and the corresponding unit normal on the perturbed surface is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{r}_\nu \times \boldsymbol{r}_\mu}{|\boldsymbol{r}_\nu \times \boldsymbol{r}_\mu|} \tag{38}$$

and since

$$\begin{aligned} |\boldsymbol{r}_\nu \times \boldsymbol{r}_\mu|^2 &= (\boldsymbol{r}_\nu \times \boldsymbol{r}_\mu) \cdot (\boldsymbol{r}_\nu \times \boldsymbol{r}_\mu) \\ &= (\boldsymbol{r}_\nu \cdot \boldsymbol{r}_\nu)(\boldsymbol{r}_\mu \cdot \boldsymbol{r}_\mu) - (\boldsymbol{r}_\nu \cdot \boldsymbol{r}_\mu)^2 = EG - F^2 \end{aligned} \tag{39}$$

we obtain

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{r}_\nu \times \boldsymbol{r}_\mu}{\sqrt{EG - F^2}}. \tag{40}$$

Using the relations (32), (33) we obtain

$$\begin{aligned} \boldsymbol{r}_\nu \times \boldsymbol{r}_\mu &= \left\{ h_\nu^0 \hat{\boldsymbol{\nu}}_0 + \varepsilon h_\rho^0 \left[\left(f_\nu - \frac{\nu}{\rho_0^2 - \nu^2} f \right) \hat{\boldsymbol{\rho}}_0 + \frac{\rho_0}{\rho_0^2 - \nu^2} \frac{h_\nu^0}{h_\rho^0} f \hat{\boldsymbol{\nu}}_0 \right] + O(\varepsilon^2) \right\} \\ &\quad \times \left\{ h_\mu^0 \hat{\boldsymbol{\mu}}_0 + \varepsilon h_\rho^0 \left[\left(f_\mu - \frac{\mu}{\rho_0^2 - \mu^2} f \right) \hat{\boldsymbol{\rho}}_0 + \frac{\rho_0}{\rho_0^2 - \mu^2} \frac{h_\mu^0}{h_\rho^0} f \hat{\boldsymbol{\mu}}_0 \right] + O(\varepsilon^2) \right\} \\ &= h_\nu^0 h_\mu^0 \hat{\boldsymbol{\rho}}_0 + \varepsilon \left[h_\nu^0 h_\mu^0 f \left(\frac{\rho_0}{\rho_0^2 - \nu^2} + \frac{\rho_0}{\rho_0^2 - \mu^2} \right) \hat{\boldsymbol{\rho}}_0 \right. \\ &\quad \left. - h_\rho^0 h_\mu^0 \left(f_\nu - f \frac{\nu}{\rho_0^2 - \nu^2} \right) \hat{\boldsymbol{\nu}}_0 - h_\rho^0 h_\nu^0 \left(f_\mu - f \frac{\mu}{\rho_0^2 - \mu^2} \right) \hat{\boldsymbol{\mu}}_0 \right] + O(\varepsilon^2) \end{aligned} \tag{41}$$

and from relations (34)-(36) we obtain

$$\begin{aligned} \frac{1}{\sqrt{EG - F^2}} &= \frac{1}{h_\nu^0 h_\mu^0} \left(1 + \varepsilon f \frac{2\rho_0}{\rho_0^2 - \nu^2} + O(\varepsilon^2) \right)^{-1/2} \left(1 + \varepsilon f \frac{2\rho_0}{\rho_0^2 - \mu^2} + O(\varepsilon^2) \right)^{-1/2} \\ &= \frac{1}{h_\nu^0 h_\mu^0} \left[1 - \varepsilon f \left(\frac{\rho_0}{\rho_0^2 - \nu^2} + \frac{\rho_0}{\rho_0^2 - \mu^2} \right) + O(\varepsilon^2) \right]. \end{aligned} \tag{42}$$

Multiplying (41) and (42) and keeping only the linear terms we arrive at the form

$$\begin{aligned}\hat{\boldsymbol{n}} &= \hat{\boldsymbol{\rho}}_0 + \varepsilon h_\rho^0 f\left[\frac{\nu}{h_\nu^0(\rho_0^2-\nu^2)}\hat{\boldsymbol{\nu}}_0 + \frac{\mu}{h_\mu^0(\rho_0^2-\mu^2)}\hat{\boldsymbol{\mu}}_0\right] \\ &\quad -\varepsilon\frac{h_\rho^0}{h_\nu^0} f_\nu \hat{\boldsymbol{\nu}}_0 - \varepsilon\frac{h_\rho^0}{h_\mu^0} f_\mu \hat{\boldsymbol{\mu}}_0 + O(\varepsilon^2). \qquad (43)\end{aligned}$$

As $\varepsilon \to 0$, relation (43) recovers the undisturbed normal $\hat{\boldsymbol{\rho}}_0$.

5. Perturbation of the second fundamental form. The second fundamental form on the perturbed ellipsoid $\rho(\mu,\nu)$ is defined as

$$-d\boldsymbol{r}\cdot d\hat{\boldsymbol{n}} = L(d\nu)^2 + 2M(d\nu)(d\mu) + N(d\mu)^2 \qquad (44)$$

where the coefficients are given by

$$L = \hat{\boldsymbol{n}}\cdot\boldsymbol{r}_{\nu\nu} \qquad (45)$$

$$M = \hat{\boldsymbol{n}}\cdot\boldsymbol{r}_{\nu\mu} \qquad (46)$$

$$N = \hat{\boldsymbol{n}}\cdot\boldsymbol{r}_{\mu\mu} \qquad (47)$$

and the minus sign represents the outward direction of the normal $\hat{\boldsymbol{n}}$.

Writing $\boldsymbol{r} = (x_1, x_2, x_3)$ and performing the differentiations component-wise we can show that

$$\frac{\partial}{\partial\nu}x_n = x_n\left[\frac{\nu}{\nu^2-a_1^2+a_n^2} + \frac{\varepsilon f_\nu(\rho_0+\varepsilon f)}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right], \quad n=1,2,3 \qquad (48)$$

$$\frac{\partial}{\partial\mu}x_n = x_n\left[\frac{\mu}{\mu^2-a_1^2+a_n^2} + \frac{\varepsilon f_\mu(\rho_0+\varepsilon f)}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right], \quad n=1,2,3 \qquad (49)$$

and after some calculations we arrive at the following expressions for the second derivatives

$$\begin{aligned}\frac{\partial^2}{\partial\nu^2}x_n &= x_n\left\{\frac{1}{\nu^2-a_1^2+a_n^2} + \frac{\varepsilon f_{\nu\nu}(\rho_0+\varepsilon f)+\varepsilon^2 f_\nu^2}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right. \\ &\quad \left. -\left[\frac{\nu}{\nu^2-a_1^2+a_n^2} - \frac{\varepsilon f_\nu(\rho_0+\varepsilon f)}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right]^2\right\} \qquad (50)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\nu\partial\mu}x_n &= x_n\left\{\left[\frac{\nu}{\nu^2-a_1^2+a_n^2} + \frac{\varepsilon f_\nu(\rho_0+\varepsilon f)}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right]\right. \\ &\quad \cdot\left[\frac{\mu}{\mu^2-a_1^2+a_n^2} + \frac{\varepsilon f_\mu(\rho_0+\varepsilon f)}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right] \\ &\quad \left. +\frac{\varepsilon f_{\nu\mu}(\rho_0+\varepsilon f)+\varepsilon^2 f_\nu f_\mu}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2} - \frac{2\varepsilon^2 f_\nu f_\mu(\rho_0+\varepsilon f)^2}{[(\rho_0+\varepsilon f)^2-a_1^2+a_n^2]^2}\right\} \qquad (51)\end{aligned}$$

$$\frac{\partial^2}{\partial\mu^2}x_n = x_n\left\{\frac{1}{\mu^2-a_1^2+a_n^2}+\frac{\varepsilon f_{\mu\mu}(\rho_0+\varepsilon f)+\varepsilon^2 f_\mu^2}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right.$$
$$\left.-\left[\frac{\mu}{\mu^2-a_1^2+a_n^2}-\frac{\varepsilon f_\mu(\rho_0+\varepsilon f)}{(\rho_0+\varepsilon f)^2-a_1^2+a_n^2}\right]^2\right\} \qquad (52)$$

which hold for every $n=1,2,3$.

Expanding (50)-(52) in powers of ε and preserving only the linear approximation we obtain

$$\frac{\partial^2}{\partial\nu^2}x_n = x_n^0\left\{-\frac{a_1^2-a_n^2}{(\nu^2-a_1^2+a_n^2)^2}+\frac{\varepsilon\rho_0}{\rho_0^2-a_1^2+a_n^2}\right.$$
$$\left.\cdot\left[f_{\nu\nu}+\frac{2\nu f_\nu}{\nu^2-a_1^2+a_n^2}-\frac{(a_1^2-a_n^2)f}{(\nu^2-a_1^2+a_n^2)^2}\right]\right\}+O(\varepsilon^2) \qquad (53)$$

$$\frac{\partial^2}{\partial\nu\partial\mu}x_n = x_n^0\left\{\frac{\nu\mu}{(\nu^2-a_1^2+a_n^2)(\mu^2-a_1^2+a_n^2)}+\frac{\varepsilon\rho_0}{\rho_0^2-a_1^2+a_n^2}\left[f_{\nu\mu}+\frac{\nu f_\mu}{\nu^2-a_1^2+a_n^2}\right.\right.$$
$$\left.\left.+\frac{\mu f_\nu}{\mu^2-a_1^2+a_n^2}+\frac{\nu\mu f}{(\nu^2-a_1^2+a_n^2)(\mu^2-a_1^2+a_n^2)}\right]\right\}+O(\varepsilon^2) \qquad (54)$$

$$\frac{\partial^2}{\partial\mu^2}x_n = x_n^0\left\{-\frac{a_1^2-a_n^2}{(\mu^2-a_1^2+a_n^2)^2}+\frac{\varepsilon\rho_0}{\rho_0^2-a_1^2+a_n^2}\right.$$
$$\left.\cdot\left[f_{\mu\mu}+\frac{2\mu f_\mu}{\mu^2-a_1^2+a_n^2}-\frac{(a_1^2-a_n^2)f}{(\mu^2-a_1^2+a_n^2)^2}\right]\right\}+O(\varepsilon^2) \qquad (55)$$

for $n=1,2,3$.

Utilizing formula (25) and the identity

$$(\rho^2-a_1^2+a_n^2)(\mu^2-a_1^2+a_n^2)(\nu^2-a_1^2+a_n^2)=(-1)^{n+1}h_1^2h_2^2h_3^2\frac{x_n^2}{h_n^2},\ n=1,2,3 \qquad (56)$$

we rewrite expressions (53)-(55) in the following vector form

$$\boldsymbol{r}_{\nu\nu} = -\left[\frac{h_3^2}{(\nu^2-h_3^2)^2}x_2^0\widehat{\boldsymbol{x}}_2+\frac{h_2^2}{(\nu^2-h_2^2)^2}x_3^0\widehat{\boldsymbol{x}}_3\right]$$
$$+\ \frac{\varepsilon\rho_0}{h_1^2}f\left[\frac{1}{x_2^0}\frac{\mu^2-h_3^2}{\nu^2-h_3^2}\hat{\boldsymbol{x}}_2-\frac{1}{x_3^0}\frac{\mu^2-h_2^2}{\nu^2-h_2^2}\hat{\boldsymbol{x}}_3\right]$$
$$+\ \frac{2\varepsilon\rho_0\nu}{h_1^2h_2^2h_3^2}f_\nu\left[\frac{\mu^2h_1^2}{x_1^0}\hat{\boldsymbol{x}}_1-\frac{(\mu^2-h_3^2)h_2^2}{x_2^0}\hat{\boldsymbol{x}}_2+\frac{(\mu^2-h_2^2)h_3^2}{x_3^0}\hat{\boldsymbol{x}}_3\right]$$
$$+\ \varepsilon h_\rho^0 f_{\nu\nu}\hat{\boldsymbol{\rho}}_0+O(\varepsilon^2) \qquad (57)$$

$$\begin{aligned}
\boldsymbol{r}_{\nu\mu} &= \frac{\nu\mu}{h_1^2h_2^2h_3^2}\left[\frac{\rho_0^2h_1^2}{x_1^0}\hat{\boldsymbol{x}}_1 - \frac{(\rho_0^2-h_3^2)h_2^2}{x_2^0}\hat{\boldsymbol{x}}_2 + \frac{(\rho_0^2-h_2^2)h_3^2}{x_3^0}\hat{\boldsymbol{x}}_3\right] \\
&+ \frac{\varepsilon\rho_0\nu\mu}{h_1^2h_2^2h_3^2}f\left[\frac{h_1^2}{x_1^0}\hat{\boldsymbol{x}}_1 - \frac{h_2^2}{x_2^0}\hat{\boldsymbol{x}}_2 + \frac{h_3^2}{x_3^0}\hat{\boldsymbol{x}}_3\right] \\
&+ \frac{\varepsilon\rho_0\mu}{h_1^2h_2^2h_3^2}f_\nu\left[\frac{\nu^2h_1^2}{x_1^0}\hat{\boldsymbol{x}}_1 - \frac{(\nu^2-h_3^2)h_2^2}{x_2^0}\hat{\boldsymbol{x}}_2 + \frac{(\nu^2-h_2^2)h_3^2}{x_3^0}\hat{\boldsymbol{x}}_3\right] \\
&+ \frac{\varepsilon\rho_0\nu}{h_1^2h_2^2h_3^2}f_\mu\left[\frac{\mu^2h_1^2}{x_1^0}\hat{\boldsymbol{x}}_1 - \frac{(\mu^2-h_3^2)h_2^2}{x_2^0}\hat{\boldsymbol{x}}_2 + \frac{(\mu^2-h_2^2)h_3^2}{x_3^0}\hat{\boldsymbol{x}}_3\right] \\
&+ \varepsilon h_\rho^0 f_{\nu\mu}\hat{\boldsymbol{\rho}}_0 + O(\varepsilon^2) \qquad (58)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{r}_{\mu\mu} &= -\left[\frac{h_3^2}{(\mu^2-h_3^2)^2}x_2^0\widehat{\boldsymbol{x}}_2 + \frac{h_2^2}{(\mu^2-h_2^2)^2}x_3^0\widehat{\boldsymbol{x}}_3\right] \\
&+ \frac{\varepsilon\rho_0}{h_1^2}f\left[\frac{1}{x_2^0}\frac{\nu^2-h_3^2}{\mu^2-h_3^2}\hat{\boldsymbol{x}}_2 - \frac{1}{x_3^0}\frac{\nu^2-h_2^2}{\mu^2-h_2^2}\hat{\boldsymbol{x}}_3\right] \\
&+ \frac{2\varepsilon\rho_0\mu}{h_1^2h_2^2h_3^2}f_\mu\left[\frac{\nu^2h_1^2}{x_1^0}\hat{\boldsymbol{x}}_1 - \frac{(\nu^2-h_3^2)h_2^2}{x_2^0}\hat{\boldsymbol{x}}_2 + \frac{(\nu^2-h_2^2)h_3^2}{x_3^0}\hat{\boldsymbol{x}}_3\right] \\
&+ \varepsilon h_\rho^0 f_{\mu\mu}\hat{\boldsymbol{\rho}}_0 + O(\varepsilon^2). \qquad (59)
\end{aligned}$$

Inserting expression (43) for the normal, and expressions (57)-(59) for the second derivatives in the definitions (45)-(47) of the coefficients of the second fundamental form and keeping only terms up to the first order we obtain

$$\begin{aligned}
L &= -\frac{(h_\nu^0)^2}{h_\rho^0}\frac{\rho_0}{\rho_0^2-\nu^2} \\
&- \varepsilon h_\rho^0 f\left[\frac{(h_\nu^0)^2}{(h_\rho^0)^2}\frac{\rho_0^2}{(\rho_0^2-\nu^2)^2} + \frac{(h_\nu^0)^2}{(h_\mu^0)^2}\frac{\mu^2}{(\rho_0^2-\mu^2)(\mu^2-\nu^2)}\right. \\
&- \left.\frac{\nu^2}{\rho_0^2-\nu^2}\left(\frac{1}{\nu^2} + \frac{1}{\nu^2-h_3^2} + \frac{1}{\nu^2-h_2^2} + \frac{2}{\rho_0^2-\nu^2} + \frac{1}{\mu^2-\nu^2}\right)\right] \\
&+ \varepsilon h_\rho^0 f_\nu \frac{\nu}{(h_\nu^0)^2}\left[\frac{1}{\nu^2-h_3^2} + \frac{1}{\nu^2-h_2^2} - \frac{1}{\rho_0^2-\nu^2} + \frac{1}{\mu^2-\nu^2}\right] \\
&+ \varepsilon h_\rho^0 f_\mu \frac{(h_\nu^0)^2}{(h_\mu^0)^2}\frac{\mu}{\mu^2-\nu^2} \\
&+ \varepsilon h_\rho^0 f_{\nu\nu} + O(\varepsilon^2) \qquad (60)
\end{aligned}$$

$$\begin{aligned}
M &= -\varepsilon h_\rho^0 f_\nu\left(\frac{\mu}{\rho_0^2-\mu^2} + \frac{\mu}{\mu^2-\nu^2}\right) \\
&- \varepsilon h_\rho^0 f_\mu\left(\frac{\nu}{\rho_0^2-\nu^2} - \frac{\nu}{\mu^2-\nu^2}\right)
\end{aligned}$$

$$+ \quad \varepsilon h_\rho^0 f_{\nu\mu} + O(\varepsilon^2) \tag{61}$$

$$\begin{aligned}
N &= -\frac{(h_\mu^0)^2}{h_\rho^0}\frac{\rho_0}{\rho_0^2-\mu^2} \\
&- \varepsilon h_\rho^0 f\left[\frac{(h_\mu^0)^2}{(h_\rho^0)^2}\frac{\rho_0^2}{(\rho_0^2-\mu^2)^2} - \frac{(h_\mu^0)^2}{(h_\nu^0)^2}\frac{\nu^2}{(\rho_0^2-\mu^2)(\mu^2-\nu^2)}\right. \\
&- \left.\frac{\mu^2}{\rho_0^2-\mu^2}\left(\frac{1}{\mu^2}+\frac{1}{\mu^2-h_3^2}+\frac{1}{\mu^2-h_2^2}+\frac{2}{\rho_0^2-\mu^2}-\frac{1}{\mu^2-\nu^2}\right)\right] \\
&+ \varepsilon h_\rho^0 f_\mu \frac{\mu}{(h_\mu^0)^2}\left[\frac{1}{\mu^2-h_3^2}+\frac{1}{\mu^2-h_2^2}-\frac{1}{\rho_0^2-\mu^2}-\frac{1}{\mu^2-\nu^2}\right] \\
&- \varepsilon h_\rho^0 f_\nu \frac{(h_\mu^0)^2}{(h_\nu^0)^2}\frac{\nu}{\mu^2-\nu^2} \\
&+ \varepsilon h_\rho^0 f_{\mu\mu} + O(\varepsilon^2).
\end{aligned} \tag{62}$$

Note that on the unperturbed ellipsoidal surface $\rho = \rho_0$, corresponding to $\varepsilon = 0$, the coefficients of the second fundamental form read

$$L_0 = -\frac{(h_\nu^0)^2}{h_\rho^0}\frac{\rho_0}{\rho_0^2-\nu^2} \tag{63}$$

$$M_0 = 0 \tag{64}$$

$$N_0 = -\frac{(h_\mu^0)^2}{h_\rho^0}\frac{\rho_0}{\rho_0^2-\mu^2}. \tag{65}$$

6. Perturbation of the curvatures. Let k_1 and k_2 be the principal curvatures on the perturbed surface, corresponding to the variables μ and ν respectively. Then the mean curvature is given by

$$H = \frac{k_1+k_2}{2} = \frac{GL-2FM+EN}{2(EG-F^2)} \tag{66}$$

and the Gaussian curvature is given by

$$K = k_1 k_2 = \frac{LN-M^2}{EG-F^2}. \tag{67}$$

Since the terms FM, F^2 and M^2 are of order higher than ε, it follows that for the linear, with respect to ε, approximation we obtain

$$k_1 = \frac{N}{G}, \quad k_2 = \frac{L}{E}. \tag{68}$$

Therefore, the orthogonality, that holds on the ellipsoid, is also preserved on the perturbed surface as long as we keep only linear terms in the perturbation parameter ε.

Using the expressions (34), (36), (60) and (62) we obtain the principal curvatures in the following form

$$k_1 = -\frac{1}{h_\rho^0}\frac{\rho_0}{\rho_0^2-\mu^2}$$

$$\begin{aligned}
&+ \quad \varepsilon h_\rho^0 f \left[\frac{1}{(h_\rho^0)^2}\frac{\rho_0^2}{(\rho_0^2-\mu^2)^2} + \frac{1}{(h_\nu^0)^2}\frac{\nu^2}{(\rho_0^2-\nu^2)(\mu^2-\nu^2)}\right.\\
&- \quad \left.\frac{1}{(h_\mu^0)^2}\frac{\mu^2}{\rho_0^2-\mu^2}\left(\frac{1}{\mu^2}+\frac{1}{\mu^2-h_3^2}+\frac{1}{\mu^2-h_2^2}+\frac{2}{\rho_0^2-\mu^2}-\frac{1}{\mu^2-\nu^2}\right)\right]\\
&+ \quad \varepsilon h_\rho^0 f_\mu \frac{\mu}{(h_\mu^0)^2}\left[\frac{1}{\mu^2-h_3^2}+\frac{1}{\mu^2-h_2^2}-\frac{1}{\rho_0^2-\mu^2}-\frac{1}{\mu^2-\nu^2}\right]\\
&- \quad \varepsilon h_\rho^0 f_\nu \frac{\nu}{(h_\nu^0)^2}\frac{1}{\mu^2-\nu^2}\\
&+ \quad \varepsilon h_\rho^0 f_{\mu\mu}\frac{1}{(h_\mu^0)^2} + O(\varepsilon^2) \qquad (69)
\end{aligned}$$

and

$$\begin{aligned}
k_2 \quad &= \quad -\frac{1}{h_\rho^0}\frac{\rho_0}{\rho_0^2-\nu^2}\\
&+ \quad \varepsilon h_\rho^0 f \left[\frac{1}{(h_\rho^0)^2}\frac{\rho_0^2}{(\rho_0^2-\nu^2)^2} - \frac{1}{(h_\mu^0)^2}\frac{\mu^2}{(\rho_0^2-\mu^2)(\mu^2-\nu^2)}\right.\\
&- \quad \left.\frac{1}{(h_\nu^0)^2}\frac{\nu^2}{\rho_0^2-\nu^2}\left(\frac{1}{\nu^2}+\frac{1}{\nu^2-h_3^2}+\frac{1}{\nu^2-h_2^2}+\frac{2}{\rho_0^2-\nu^2}+\frac{1}{\mu^2-\nu^2}\right)\right]\\
&+ \quad \varepsilon h_\rho^0 f_\nu \frac{\nu}{(h_\nu^0)^2}\left[\frac{1}{\nu^2-h_3^2}+\frac{1}{\nu^2-h_2^2}-\frac{1}{\rho_0^2-\nu^2}+\frac{1}{\mu^2-\nu^2}\right]\\
&+ \quad \varepsilon h_\rho^0 f_\mu \frac{\mu}{(h_\mu^0)^2}\frac{1}{\mu^2-\nu^2}\\
&+ \quad \varepsilon h_\rho^0 f_{\nu\nu}\frac{1}{(h_\nu^0)^2} + O(\varepsilon^2). \qquad (70)
\end{aligned}$$

Then, the mean curvature is written as

$$\begin{aligned}
H \quad &= \quad -\frac{1}{2h_\rho^0}\left(\frac{\rho_0}{\rho_0^2-\mu^2}+\frac{\rho_0}{\rho_0^2-\nu^2}\right) \qquad (71)\\
&+ \quad \frac{\varepsilon h_\rho^0 f}{2}\left[\frac{1}{(h_\rho^0)^2}\frac{\rho_0^2}{(\rho_0^2-\mu^2)^2} - \frac{1}{(h_\mu^0)^2}\frac{\mu^2}{\rho_0^2-\mu^2}\left(\frac{1}{\mu^2}+\frac{1}{\mu^2-h_3^2}+\frac{1}{\mu^2-h_2^2}+\frac{2}{\rho_0^2-\mu^2}\right)\right.\\
&+ \quad \left.\frac{1}{(h_\rho^0)^2}\frac{\rho_0^2}{(\rho_0^2-\nu^2)^2} - \frac{1}{(h_\nu^0)^2}\frac{\nu^2}{\rho_0^2-\nu^2}\left(\frac{1}{\nu^2}+\frac{1}{\nu^2-h_3^2}+\frac{1}{\nu^2-h_2^2}+\frac{2}{\rho_0^2-\nu^2}\right)\right]\\
&+ \quad \frac{\varepsilon h_\rho^0 f_\mu}{2}\frac{\mu}{(h_\mu^0)^2}\left[\frac{1}{\mu^2-h_3^2}+\frac{1}{\mu^2-h_2^2}-\frac{1}{\rho_0^2-\mu^2}\right]\\
&+ \quad \frac{\varepsilon h_\rho^0 f_\nu}{2}\frac{\nu}{(h_\nu^0)^2}\left[\frac{1}{\nu^2-h_3^2}+\frac{1}{\nu^2-h_2^2}-\frac{1}{\rho_0^2-\nu^2}\right]\\
&+ \quad \frac{\varepsilon h_\rho^0 f_{\mu\mu}}{2}\frac{1}{(h_\mu^0)^2}\\
&+ \quad \frac{\varepsilon h_\rho^0 f_{\nu\nu}}{2}\frac{1}{(h_\nu^0)^2} + O(\varepsilon^2) \qquad (72)
\end{aligned}$$

and the Gaussian curvature is written as

$$\begin{aligned} K \quad = \quad & \frac{1}{(\rho_0^2-\mu^2)(\mu^2-\nu^2)}\left[\frac{\rho_0^2}{(h_\rho^0)^2}\right. \\ + \quad & \frac{2\varepsilon\rho_0^3 f}{(h_\rho^0)^2}\left(\frac{1}{\rho_0^2}+\frac{1}{\rho_0^2-h_3^2}+\frac{1}{\rho_0^2-h_2^2}-\frac{2}{\rho_0^2-\mu^2}-\frac{2}{\rho_0^2-\nu^2}\right) \\ + \quad & \frac{\varepsilon\rho_0\mu f_\mu}{\mu^2-\nu^2}\left(2\mu^2-h_3^2-h_2^2\right)-\frac{\varepsilon\rho_0\nu f_\nu}{\mu^2-\nu^2}\left(2\nu^2-h_3^2-h_2^2\right) \\ - \quad & \left.\frac{\varepsilon\rho_0 f_{\mu\mu}}{(h_\mu^0)^2}(\rho_0^2-\mu^2)-\frac{\varepsilon\rho_0 f_{\nu\nu}}{(h_\nu^0)^2}(\rho_0^2-\nu^2)\right]+O(\varepsilon^2). \end{aligned} \tag{73}$$

The corresponding formulae for the unperturbed ellipsoid are obtained in the limit as $\varepsilon \to 0$.

The above formulae can be used in the investigation of the stability of evolving boundaries in the shape of ellipsoids.

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