ON THE ZERO-DIVISOR GRAPH OF A RING

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Abstract. Let R be a commutative ring with identity, $Z(R)$ its set of zero-divisors, and $Nil(R)$ its ideal of nilpotent elements. The zero-divisor graph of R is $\Gamma(R) = Z(R) \setminus \{0\}$, with distinct vertices x and y adjacent if and only if $xy = 0$. In this paper, we study $\Gamma(R)$ for rings R with nonzero zero-divisors which satisfy certain divisibility conditions between elements of R or comparability conditions between ideals or prime ideals of R. These rings include chained rings, rings R whose prime ideals contained in $Z(R)$ are linearly ordered, and rings R such that $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R) \setminus Nil(R)$.

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1. Introduction

Let R be a commutative ring with 1, and let $Z(R)$ be its set of zero-divisors. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Note that $\Gamma(R)$ is the empty graph if and only if R is an integral domain and that a nonempty $\Gamma(R)$ is finite if and only if R is finite and not a field [9, Theorem 2.2]. This concept is due to Beck [24], who let all the elements of R be vertices and was mainly interested in colorings. Our present definition and emphasis on the interplay between ring-theoretic properties of R and graph-theoretic properties of $\Gamma(R)$ are from [9].

In this paper, we study $\Gamma(R)$ for several classes of rings which generalize valuation domains to the context of rings with zero-divisors. These are rings with nonzero zero-divisors that satisfy certain divisibility conditions between elements or comparability conditions between ideals or prime ideals. In Sections 2 and 3, we consider rings R such that the prime ideals of R contained in $Z(R)$ are linearly ordered. In particular, we compute the diameter and girth for $\Gamma(R)$ and $\Gamma(R[X])$. In Section 4, we specialize to the case where R is a chained ring. In the final section, we investigate $\Gamma(R)$ for rings R such that $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R) \setminus Nil(R)$.

We assume throughout that all rings are commutative with $1 \neq 0$. If R is a ring, then $\dim(R)$ denotes its (Krull) dimension, $T(R)$ its total quotient ring, $U(R)$ its group of units, $Z(R)$ its set of zero-divisors, $Nil(R)$ its ideal of nilpotent elements, $N(R) = \{x \in R \mid x^2 = 0\} \subseteq Nil(R)$, and $Rad(I) = \{x \in R \mid x^n \in I$ for some integer $n \geq 1\}$ for I an ideal of R. We say that R is *reduced* if $Nil(R) = \{0\}$. For $A, B \subseteq R$, let $A^* = A \setminus \{0\}$ and $(A : B) = \{x \in R \mid xB \subseteq A\}$. We let $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Z}_{(p)}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{F}_q$ denote the rings of integers, integers modulo n, integers localized at the prime ideal $p\mathbb{Z}$, rational numbers, real numbers, and the finite field with q elements, respectively. In the next six paragraphs, we recall some background material. To avoid any trivialities when $\Gamma(R)$ is the empty graph, we implicitly assume when necessary that R is not an integral domain. For any undefined ring-theoretic concepts or terminology, see [30] or [31].

Let G be a graph. We say that G is *connected* if there is path between any two distinct vertices of G. At the other extreme, we say that G is *totally disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y in G ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\mathrm{diam}(G) = sup\{\, d(x, y) \mid x$ and y are vertices of $G\,\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). Then $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ [9, Theorem 2.3] and $\mathrm{gr}(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle [33, (1.4)]. Thus $\mathrm{diam}(\Gamma(R)) = 0, 1, 2,$ or 3, and $\mathrm{gr}(\Gamma(R)) = 3, 4,$ or ∞. For other papers on zero-divisor graphs, see [1], [2], [7], [8], [10], [11], [12], [26], [32], [33], [34], and [35]. In particular, a list of all the zero-divisor graphs with up to 14 vertices is given in [34]. A general reference for graph theory is [25].

Recall from [29] that an integral domain R with quotient field K is called a *pseudo-valuation domain (PVD)* if every prime ideal P of R is *strongly prime*, in the sense that whenever $x, y \in K$ and $xy \in P$, then $x \in P$ or $y \in P$. This concept was extended to rings with zero-divisors in [21], where R is called a *pseudo-valuation ring (PVR)* if every prime ideal P of R is *strongly prime*, in the sense that xP and yR are comparable (under inclusion) for all $x, y \in R$. Any valuation domain is a PVD, and it was shown in [21] that an integral domain is a PVD if and only if it is a PVR. It is known that a ring R is a PVR if and only if for all $x, y \in R$, we have either $x|y$ or $y|xz$ for every nonunit $z \in R$ [21, Theorem 5]. We say that a ring R is a *chained ring* if the (principal) ideals of R are linearly ordered (by inclusion), equivalently, if either $x|y$ or $y|x$ for all $x, y \in R$. By our earlier comments, a chained ring is a PVR.

Another generalization of pseudo-valuation rings is given in [15]. Recall from [27] and [14] that a prime ideal of a ring R is called a *divided prime ideal* of R if $P \subseteq xR$ for all $x \in R \backslash P$. Thus a divided prime ideal of R is comparable with every ideal of R. We say that a ring R is a *divided ring* if every prime ideal of R is divided; so the prime ideals in a divided ring are linearly ordered. Let $\mathcal{H} = \{R \mid R$ is a ring and $Nil(R)$ is a divided prime ideal of $R\,\}$. Note that an integral domain or a PVR is in $\mathcal{H}$. For any ring $R \in \mathcal{H}$, the ring homomorphism $\phi = \phi_R : T(R) \longrightarrow R_{Nil(R)}$, given by $\phi(x/y) = x/y$ for all $x \in R$ and $y \in R\backslash Z(R)$, was introduced in [15]. Then $\phi|_R : R \longrightarrow R_{Nil(R)}$ is a ring homomorphism satisfying $\phi(x) = x/1$ for all $x \in R$ and $T(\phi(R)) = R_{Nil(R)}$.

Let $R \in \mathcal{H}$, and put $K = R_{Nil(R)}$. As in [15], a prime ideal Q of $\phi(R)$ is said to be *K-strongly prime* if whenever $x, y \in K$ and $xy \in Q$, then either $x \in Q$ or $y \in Q$. A prime ideal P of R is said to be a *ϕ-strongly prime ideal* of R if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. It is known that the prime ideals of $\phi(R)$ are the sets that are (uniquely) expressible as $\phi(P)$ for some prime ideal P of R (cf. [15, Lemma 2.5]), the key fact being that $Ker(\phi) \subseteq Nil(R)$. If every prime ideal of R is a ϕ-strongly prime ideal, then R is called a *ϕ-pseudo-valuation ring (ϕ-PVR)*. It was shown in [18, Proposition 2.9] that a ring $R \in \mathcal{H}$ is a ϕ-PVR if and only if $R/Nil(R)$ is a PVD. A PVR is a ϕ-PVR, but an example of a ϕ-PVR which is not a PVR was given in [16]. Also, a ϕ-PVR is a divided ring [15, Proposition 4], and thus the prime ideals in a ϕ-PVR (or a PVR) are linearly ordered. In particular, a ϕ-PVR, and hence a PVR or a chained ring, is quasilocal.

Observe that if $Nil(R)$ is a divided prime ideal of R, then $Nil(R)$ is also the nilradical of $T(R)$ and $Ker(\phi)$ is a common ideal of R and $T(R)$. Other useful features of each ring $R \in \mathcal{H}$ include the following: (i) $\phi(R) \in \mathcal{H}$; (ii) $T(\phi(R)) = R_{Nil(R)}$ has only one prime ideal, namely, $Nil(\phi(R))$; (iii) $\phi(R)$ is naturally isomorphic to $R/Ker(\phi)$; (iv) $Z(\phi(R)) = Nil(\phi(R)) = \phi(Nil(R)) = Nil(R_{Nil(R)})$; and (v) $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$. For further studies on rings in the class $\mathcal{H}$, see [4], [5], [15], [16], [17], [18], [22], and [23].

Throughout this paper, we will use the technique of idealization of a module to construct examples. Recall that for an R-module B, the *idealization of B over R* is the ring formed from $R\times B$ by defining addition and multiplication as $(r, a)+(s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A standard notation for this "idealized ring" is $R(+)B$; see [30] for basic properties of rings resulting from the idealization construction. In particular, note that the ideal $I = \{0\}(+)B$ of $T = R(+)B$ satisfies $I^2 = \{0\}$; so $I \subseteq Nil(T)$. The zero-divisor graph $\Gamma(R(+)B)$ has recently been studied in [10] and [12].

2. LINEARLY ORDERED PRIMES

In this section, we investigate the zero-divisor graph of a ring R such that the prime ideals of R contained in $Z(R)$ are linearly ordered. These are precisely the rings R such that the prime ideals of $T(R)$ are linearly ordered, and include chained rings, divided rings, PVRs, ϕ-PVRs, rings with $Z(R) = Nil(R)$, and zero-dimensional quasilocal rings. For these rings, we show that $\mathrm{diam}(\Gamma(R)) \leq 2$ and $\mathrm{gr}(\Gamma(R)) = 3$ or ∞. We start with the following lemma (cf. [32, Lemma 2.3] and [2, Lemma 3.1]).

Lemma 2.1. *Let R be a ring, and let $x, y \in Nil(R)^*$ be distinct with $xy \neq 0$. Then $(0 : (x, y)) \neq \{0\}$, and moreover, there is a path of length 2 from x to y in $Nil(R)^* \subseteq \Gamma(R)$. In particular, if $Z(R) = Nil(R)$, then diam$(\Gamma(R)) \leq 2$.*

Proof. Since $xy \neq 0$ and $x \in Nil(R)^*$, let n (≥ 2) be the least positive integer such that $x^n y = 0$. Also, since $x^{n-1}y \neq 0$ and $y \in Nil(R)^*$, let m (≥ 2) be the least positive integer such that $x^{n-1}y^m = 0$. Then $0 \neq x^{n-1}y^{m-1} \in Nil(R)$ and $x^{n-1}y^{m-1} \in (0 : (x, y))$. Thus $x - x^{n-1}y^{m-1} - y$ is a path of length 2 from x to y in $Nil(R)^*$. The "in particular" statement is clear. □

When $Z(R) = Nil(R)$, it is easy to explicitly describe the diameter of $\Gamma(R)$; and moreover, $\mathrm{diam}(\Gamma(R)) \neq 3$ in this case. We record this as our first theorem (cf. [32, Theorem 2.6]). Note that in this case, $Nil(R)$ is the unique minimal prime ideal of R and is the only prime ideal of R contained in $Z(R)$; so this is the simplest case where the prime ideals of R contained in $Z(R)$ are linearly ordered.

Theorem 2.2. *Let R be a ring with $Z(R) = Nil(R) \neq \{0\}$. Then exactly one of the following three cases must occur.*

1. $|Z(R)^*| = 1$. *In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and $diam(\Gamma(R)) = 0$.*
2. $|Z(R)^*| \geq 2$ *and* $Z(R)^2 = \{0\}$. *In this case, $\Gamma(R)$ is a complete graph, and $diam(\Gamma(R)) = 1$.*
3. $Z(R)^2 \neq \{0\}$. *In this case, $diam(\Gamma(R)) = 2$.*

Proof. (1) If $|Z(R)^*| = 1$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ [24, Proposition 2.2]. Thus $\mathrm{diam}(\Gamma(R)) = 0$.

(2) If $Z(R)^2 = \{0\}$, then $xy = 0$ for all $x, y \in Z(R)$. Thus $\Gamma(R)$ is a complete graph with $\mathrm{diam}(\Gamma(R)) = 1$ since $|Z(R)^*| \geq 2$.

(3) Suppose that $Z(R)^2 \neq \{0\}$. Then $\Gamma(R)$ is not complete [9, Theorem 2.8], and thus $\mathrm{diam}(\Gamma(R)) \geq 2$. Hence $\mathrm{diam}(\Gamma(R)) = 2$ by Lemma 2.1. $\square$

Thus when studying the diameter of the zero-divisor graph of a ring R, the interesting case is when $Nil(R) \subsetneq Z(R)$. We next give several lemmas. Note that in Lemma 2.4 we need only assume that $x \in Z(R) \setminus N(R)$, where $N(R) = \{x \in R \mid x^2 = 0\}$.

Lemma 2.3. *Let R be a ring with $x \in Nil(R)^*$ and $y \in Z(R)^*$. Then $d(x, y) \leq 2$ in $\Gamma(R)$.*

Proof. We may assume that $x \neq y$ and $xy \neq 0$. Since $y \in Z(R)^*$ and $xy \neq 0$, there is a $z \in Z(R)^* \setminus \{x\}$ such that $yz = 0$. Let n be the least positive integer such that $x^n z = 0$ (such an n exists since $x \in Nil(R)^*$). Then $x - x^{n-1}z - y$ is a path of length 2 from x to z (if $n = 1$, then $x^{n-1}z = z$). Thus $d(x, y) \leq 2$ in $\Gamma(R)$. $\square$

Lemma 2.4. *Let R be a ring with $x \in Z(R) \setminus Nil(R)$ and $y \in Z(R)^*$ such that $x | zy^n$ for some integer $n \geq 1$ and $z \in R \setminus Z(R)$. Then $d(x, y) \leq 2$ in $\Gamma(R)$.*

Proof. We may assume that $x \neq y$ and $xy \neq 0$. Since $x \in Z(R) \setminus Nil(R)$ and $xy \neq 0$, there is a $w \in Z(R)^* \setminus \{x, y\}$ such that $xw = 0$. Since $x | zy^n$ with $z \in R \setminus Z(R)$ and $xw = 0$, we conclude that $y^n w = 0$. Let k be the least positive integer such that $y^k w = 0$. Then $x - y^{k-1}w - y$ is a path of length 2 from x to y. Thus $d(x, y) \leq 2$ in $\Gamma(R)$. $\square$

By [13, Theorem 1], the prime ideals of R are linearly ordered if and only if the radical ideals of R are linearly ordered, if and only if for all $x, y \in R$, there is an integer $n = n(x, y) \geq 1$ such that either $x | y^n$ or $y | x^n$. This result easily extends to the prime ideals of R contained in $Z(R)$.

Theorem 2.5. *Let R be a ring.*

1. *The prime ideals of R contained in $Z(R)$ are linearly ordered if and only if for all $x, y \in Z(R)$, there is an integer $n = n(x, y) \geq 1$ and an element $z \in R \setminus Z(R)$ such that either $x | zy^n$ or $y | zx^n$.*
2. *The radical ideals of R contained in $Z(R)$ are linearly ordered if and only if for all $x, y \in Z(R)$, there is an integer $n = n(x, y) \geq 1$ such that either $x | y^n$ or $y | x^n$.*
3. *If the prime ideals of R contained in $Z(R)$ are linearly ordered, then $Nil(R)$ and $Z(R)$ are prime ideals of R.*

Proof. (1) Note that the prime ideals of R contained in $Z(R)$ are linearly ordered if and only if the prime ideals of $T(R)$ are linearly ordered, if and only if for all $x, y \in T(R)$, there is an integer $n = n(x, y) \geq 1$ such that either $x|y^n$ or $y|x^n$ in $T(R)$ [13, Theorem 1]. The result now easily follows.

(2) Suppose that the radical ideals of R contained in $Z(R)$ are linearly ordered. Let $x, y \in Z(R)$. Then $Rad(xR), Rad(yR) \subseteq Z(R)$; so we may assume that $Rad(xR) \subseteq Rad(yR)$. Thus $x \in Rad(yR)$; so $y|x^n$ for some integer $n \geq 1$. Conversely, let $I, J \subseteq Z(R)$ be radical ideals of R. If I and J are not comparable, pick $x \in I \setminus J$ and $y \in J \setminus I$. If $x|y^n$, then $y^n \in xR \subseteq I$, and hence $y \in I$, a contradiction.

(3) Suppose that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then $Nil(R)$ is an intersection of linearly ordered prime ideals of R since each minimal prime ideal of R is contained in $Z(R)$ [30, Theorem 2.1], and thus $Nil(R)$ is prime. Also, $Z(R)$ is the union of linearly ordered prime ideals of R [31, page 3], and hence $Z(R)$ is prime. □

Since $Z(R)$ is a union of prime ideals of R [31, page 3], $Z(R)$ is a prime ideal of R if and only if it is an ideal of R. If $\dim(R) = 0$ (e.g., R is finite) and the prime ideals of R contained in $Z(R)$ are linearly ordered, then R is quasilocal with $Z(R) = Nil(R)$ its unique prime ideal. If $Nil(R) \subsetneq Z(R)$ and $Nil(R)$ is a prime ideal of R, then $\dim(R) \geq 1$ and $\Gamma(R)$ must be infinite. For in this case, R is not an integral domain, and thus if $\Gamma(R)$ is finite, then R must also be finite [9, Theorem 2.2], contradicting $\dim(R) \geq 1$. In particular, if the prime ideals of R contained in $Z(R)$ are linearly ordered and $Nil(R) \subsetneq Z(R)$, then $\Gamma(R)$ is infinite. It is clear that if the radical ideals of R contained in $Z(R)$ are linearly ordered, then the prime ideals of R contained in $Z(R)$ are also linearly ordered. However, we next give an example where the prime ideals of R contained in $Z(R)$ are linearly ordered, but the radical ideals of R contained in $Z(R)$ are not linearly ordered, and hence the prime ideals of R are not linearly ordered.

Example 2.6. *Let $D = \mathbb{Z} + X\mathbb{Q}[[X]]$, and let $I = \mathbb{Z}_{(2)}X + X^2\mathbb{Q}[[X]]$ be an ideal of D. Set $R = D/I$. Then $Z(R) = (2\mathbb{Z} + X\mathbb{Q}[[X]])/I = 2R = ann_R(\frac{1}{2}X + I)$, $N(R) = Nil(R) = X\mathbb{Q}[[X]]/I$, and $Nil(R)^2 = \{0\}$. The prime ideals of R contained in $Z(R)$, namely $Z(R)$ and $Nil(R)$, are linearly ordered. But the radical ideals of R contained in $Z(R)$ are not linearly ordered since the two radical ideals $(6\mathbb{Z} + X\mathbb{Q}[[X]])/I$ and $(10\mathbb{Z} + X\mathbb{Q}[[X]])/I$ are not comparable. Thus the prime ideals of R are also not linearly ordered; for example, $(2\mathbb{Z}+X\mathbb{Q}[[X]])/I$ and $(3\mathbb{Z}+X\mathbb{Q}[[X]])/I$ are not comparable. We have $diam(\Gamma(R)) = 2$ by Theorem 2.7, and $gr(\Gamma(R)) = 3$ by Theorem 2.12. Also note that $R \cong \mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z}_{(2)})$.*

The prime ideals of R contained in $Z(R)$ are linearly ordered if and only if the prime ideals of $T(R)$ are linearly ordered. Moreover, $\Gamma(R) \cong \Gamma(T(R))$ ([8, Theorem 2.2]). Thus we can often reduce to the case where the prime ideals of R are linearly ordered. Note that a reduced ring R with its prime ideals contained in $Z(R)$ linearly ordered is an integral domain. Also observe that a nonreduced ring R has $\Gamma(R)$ complete if and only if $Z(R)^2 = \{0\}$ [9, Theorem 2.8], i.e., if $xy = 0$ for all $x, y \in Z(R)$ with $x \neq y$, then $x^2 = 0$ for all $x \in Z(R)$. So if R is a nonreduced ring with $Z(R)^2 = \{0\}$, then $\{0\} \neq N(R) = Nil(R) = Z(R)$ and $\operatorname{diam}(\Gamma(R)) \leq 1$, with equality when $|Z(R)^*| \geq 2$. We are now ready for the first of the two main results of this section.

Theorem 2.7. *Let R be a ring with $Z(R)^2 \neq \{0\}$ such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then diam$(\Gamma(R)) = 2$.*

Proof. By the above comments, R is not reduced. So $\Gamma(R)$ is not a complete graph and $\text{diam}(\Gamma(R)) \geq 2$. Let $x, y \in Z(R)^*$ be distinct with $xy \neq 0$. If $x, y \in Nil(R)$, then $d(x, y) = 2$ by Lemma 2.1. If $x \in Nil(R)$ and $y \in Z(R) \setminus Nil(R)$, then $d(x, y) = 2$ by Lemma 2.3. Finally, suppose that $x, y \in Z(R) \setminus Nil(R)$. Since the prime ideals of R contained in $Z(R)$ are linearly ordered, there is an integer $n \geq 1$ and an element $z \in R \setminus Z(R)$ such that either $x|zy^n$ or $y|zx^n$ by Theorem 2.5(1). We may assume that $x|zy^n$ for some integer $n \geq 1$ and $z \in R \setminus Z(R)$. Thus $d(x, y) = 2$ by Lemma 2.4. Hence $\text{diam}(\Gamma(R)) \leq 2$, and thus $\text{diam}(\Gamma(R)) = 2$ since $\text{diam}(\Gamma(R)) \geq 2$. □

Corollary 2.8. *If R is any of the following types of rings with $Z(R)^2 \neq \{0\}$, then diam$(\Gamma(R)) = 2$.*

(1) *R is a ring such that the prime ideals of R are linearly ordered.*
(2) *R is a divided ring.*
(3) *R is a PVR.*
(4) *R is a ϕ-PVR.*
(5) *R is a chained ring.*

In view of Theorem 2.7 and [32, Theorem 2.6(3)], we have the following corollary.

Corollary 2.9. *Let R be a ring with $Z(R)^2 \neq \{0\}$ such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then $Z(R)$ is an (prime) ideal of R and each pair of distinct zero-divisors of R has a nonzero annihilator.*

Our next example illustrates what can happen when the prime ideals of R contained in $Z(R)$ are not linearly ordered.

Example 2.10. *(a) Let $D = \mathbb{R}[[X, Y]] + ZK[[Z]]$, where K is the quotient field of $\mathbb{R}[[X, Y]]$, and let $I = ZD$. Set $R = D/I$. Then R is quasilocal with maximal ideal $Z(R) = ((X, Y) + ZK[[Z]])/I$, $N(R) = Nil(R) = ZK[[Z]]/I$, $Nil(R)^2 = \{0\}$, and $((X) + ZK[[Z]])/I$ and $((Y) + ZK[[Z]])/I$ are incomparable prime ideals of R contained in $Z(R)$. One can easily show that diam$(\Gamma(R)) = 3$ and gr$(\Gamma(R)) = 3$. Also see Example 5.3(b).*

(b) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $N(R) = Nil(R) = \{0\} \times \{0, 2\} \subsetneq Z(R) = P \cup Q$, where $P = \mathbb{Z}_2 \times \{0, 2\}$ and $Q = \{0\} \times \mathbb{Z}_4$ are incomparable primes ideals of R contained in $Z(R)$. One can easily show that diam$(\Gamma(R)) = 3$ and gr$(\Gamma(R)) = \infty$.

We conclude this section with a discussion of the girth of $\Gamma(R)$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. We first handle the case where $Z(R) = Nil(R)$. In this case, $\text{gr}(\Gamma(R)) \neq 4$, and we can explicitly say when the girth is either 3 or ∞. Note that in Theorem 2.11, $\text{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R)$ is a finite star graph. (Recall that a graph is a *star graph* if it has a vertex which is adjacent to every other vertex and this is the only adjacency relation. We consider a singleton graph to be a star graph.)

Theorem 2.11. *Let R be a ring with $Z(R) = Nil(R) \neq \{0\}$. Then exactly one of the following four cases must occur.*

(1) *$|Z(R)^*| = 1$. In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and gr$(\Gamma(R)) = \infty$.*

(2) $|Z(R)^*| = 2$. *In this case, R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, and $gr(\Gamma(R)) = \infty$.*
(3) $|Z(R)^*| = 3$. *If R is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2-2)$, then $gr(\Gamma(R)) = \infty$. Otherwise, R is isomorphic to $\mathbb{Z}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2,X)^2$, $\mathbb{Z}_4[X]/(X^2+X+1)$, or $\mathbb{F}_4[X]/(X^2)$; and in this case, $gr(\Gamma(R)) = 3$.*
(4) $|Z(R)^*| \geq 4$. *In this case, $gr(\Gamma(R)) = 3$.*

Proof. By [10, Theorem 2.3], $\mathrm{gr}(\Gamma(R)) \neq 4$ when $Z(R) = Nil(R)$. Thus $\mathrm{gr}(\Gamma(R)) = 3$ or ∞. The theorem then follows from [10, Theorem 2.5], [10, Remark 2.6(a)], and [7, Example 2.1]. □

We next handle the $Nil(R) \subsetneq Z(R)$ case when $Nil(R)$ a prime ideal of R (cf. Remark 2.13(b)). In this case, we have already observed that $\Gamma(R)$ is infinite. The next theorem, together with Theorem 2.11, completely characterizes $\mathrm{gr}(\Gamma(R))$ in terms of $|Nil(R)^*|$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. In particular, we have $\mathrm{gr}(\Gamma(R)) = 3$ or ∞, with $\mathrm{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R)$ is a star graph.

Theorem 2.12. *Let R be a ring such that $Nil(R)$ is a prime ideal of R and $Nil(R) \subsetneq Z(R)$. In particular, this holds when the prime ideals of R contained in $Z(R)$ are linearly ordered and $Nil(R) \subsetneq Z(R)$. Then $gr(\Gamma(R)) = 3$ or ∞. Moreover, $gr(\Gamma(R)) = \infty$ if and only if $|Nil(R)^*| = 1$; and in this case, $\Gamma(R)$ is an infinite star graph.*

Proof. Since $\Gamma(R)) \cong \Gamma(T(R))$[8, Theorem 2.2], we may assume that $R = T(R)$. Note that R is not reduced; so if $\mathrm{gr}(\Gamma(R)) = 4$, then $R \cong D \times B$, where D is an integral domain and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by [10, Theorem 2.3]. In this case, $Nil(R) \cong \{0\} \times \mathbb{Z}_2$ is not a prime ideal of R. So we must have $\mathrm{gr}(\Gamma(R)) = 3$ or ∞. The "in particular" statement follows from Theorem 2.5(3). The "moreover" statement follows from [10, Theorem 2.5 and Remark 2.6(a)]. □

Remark 2.13. *(a) $\Gamma(R)$ is a finite star graph if and only if either $R \cong \mathbb{F}_q \times \mathbb{Z}_2$ for some finite field $\mathbb{F}_q$ (when R is reduced), or R is one of the 7 rings with $gr(\Gamma(R)) = \infty$ given in Theorem 2.11 (*[9, Theorem 2.13] *and* [26, Corollary 1.11]*).*

If $\Gamma(R)$ is an infinite star graph, then either $R \cong D \times \mathbb{Z}_2$ for D an integral domain (when R is reduced), or $Nil(R)$ is a prime ideal of R with $|Nil(R)^*| = 1$ and $Z(R)$ is a prime ideal of R ([26, Theorem 1.12] *or* [33, (2.1)]*). For example, if $R = \mathbb{Z}(+)\mathbb{Z}_2$ ($\cong \mathbb{Z}[X]/(2X, X^2)$), then $\Gamma(R)$ is an infinite star graph with center $(0,1)$ and the prime ideals of R contained in $Z(R)$ are linearly ordered.*

(b) The hypothesis that $Nil(R)$ is a prime ideal of R is needed in Theorem 2.12. For example, let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $Nil(R) \subsetneq Z(R)$, $Nil(R)$ is not a prime ideal of R, and $gr(\Gamma(R)) = 4$.

(c) It is instructive to give an elementary, self-contained proof of Theorem 2.12. If $|Nil(R)^*| = 1$, then $gr(\Gamma(R)) = \infty$ since $\Gamma(R) \setminus Nil(R)$ is totally disconnected (Theorem 3.5(1)). So suppose that $|Nil(R)^*| \geq 2$, and let $z \in Z(R) \setminus Nil(R)$. Then there is a $w \in Nil(R)^*$ with $zw = 0$. First suppose that $w^2 \neq 0$, and let m (≥ 3) be the least positive integer such that $w^m = 0$. Thus $w^{m-1} \neq w$, and hence $z - w - w^{m-1} - z$ is a cycle of length 3. Now suppose that $w^2 = 0$, and let $d \in Nil(R)^* \setminus \{w\}$. Assume that $wd \neq 0$. Since wd and w are distinct and nonzero, we conclude that $z - w - wd - z$ is a cycle of length 3. Now assume that $wd = 0$

and $w^2 = 0$. *If* $zd = 0$, *then* $z - w - d - z$ *is a cycle of length* 3. *Thus we may assume that* $zd \neq 0$. *If* $zd = w$, *then* $zd^2 = wd = 0$, *and hence* $w - z^2 - d - w$ *is a cycle of length* 3. *Thus we assume that* zd *and* w *are distinct and nonzero. Let* n *be the least positive integer such that* $zd^n = 0$. *Assume* $n > 2$. *Then it is clear that* $d \neq zd^{n-1}$. *If* $zd^{n-1} \neq w$, *then* $w - zd^{n-1} - d - w$ *is a cycle of length* 3. *Assume that* $zd^{n-1} = w$. *Then* $z^2d^{n-1} = zw = 0$. *Since* $zw = 0$, d^{n-1} *and* w *are distinct and nonzero, and thus* $w - z^2 - d^{n-1} - w$ *is a cycle of length* 3. *Now assume that* $n = 2$ *and* $zd \neq w$. *Then* $zd^2 = 0$. *If* $zd \neq d$, *then* $w - zd - d - w$ *is a cycle of length* 3. *Thus assume that* $zd = d$. *Hence* $d^2 = zd^2 = 0$. *Since* $zw = 0$ *and* $zd \neq 0$, *we have* $w + d \neq 0$. *Hence* w, d, *and* $w + d$ *are all distinct. Since* $w^2 = d^2 = wd = 0$, $w - w + d - d - w$ *is a cycle of length* 3. *Thus* $gr(\Gamma(R)) = 3$.

3. LINEARLY ORDERED PRIMES-II

In this section, we continue the investigation of $\Gamma(R)$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. We show that for such rings R, $\Gamma(R) \setminus Nil(R)$ is totally disconnected, every finite set of vertices of $\Gamma(R) \setminus Nil(R)$ is adjacent to a common vertex of $Nil(R)^*$, and $\Gamma(R) \setminus Nil(R)$ is infinite when $Nil(R) \subsetneq Z(R)$. We also determine $\operatorname{diam}(\Gamma(R[X]))$ and $\operatorname{gr}(\Gamma(R[X]))$. Our first goal is to show that such a ring R is a McCoy ring, where a ring R is called a *McCoy ring* if every finitely generated ideal of R contained in $Z(R)$ has a nonzero annihilator.

Lemma 3.1. *Let* R *be a ring such that the prime ideals of* R *contained in* $Z(R)$ *are linearly ordered, and let* $z_1, \ldots, z_n \in Z(R)$. *Then there is an integer* i, $1 \leq i \leq n$, *a positive integer* m, *and an* $s \in R \setminus Z(R)$ *such that* $z_i | s z_k^m$ *for every integer* k, $1 \leq k \leq n$.

Proof. Let $T = T(R)$. Then the prime ideals of T are linearly ordered. Thus $Rad(z_1T), \ldots, Rad(z_nT)$ are prime ideals of T, and hence are linearly ordered. Thus there is an integer i, $1 \leq i \leq n$, such that $Rad(z_kT) \subseteq Rad(z_iT)$ for every integer k, $1 \leq k \leq n$. Hence there are positive integers $m_1, \ldots, m_n$ and $s_1, \ldots, s_n \in R \setminus Z(R)$ such that $z_i | s_i z_k^{m_k}$ for every integer k, $1 \leq k \leq n$. Let $s = s_1 \cdots s_n \in R \setminus Z(R)$ and $m = \max\{m_1, \ldots, m_n\}$. Then $z_i | s{z_k}^m$ for every integer k, $1 \leq k \leq n$, as desired. □

Theorem 3.2. *Let* R *be a ring such that the prime ideals of* R *contained in* $Z(R)$ *are linearly ordered. Then* R *is a McCoy ring.*

Proof. Let $I = (z_1, \ldots, z_n)$ be a nonzero finitely generated ideal of R contained in $Z(R)$. By Lemma 3.1, we may assume that there is a positive integer m and an $s \in R \setminus Z(R)$ such that $z_1 | s z_k^m$ for every integer k, $2 \leq k \leq n$. Let $w \in Z(R)^*$ such that $z_1 w = 0$. Thus there is an integer $m_2 \geq 0$ such that $z_2^{m_2} w \neq 0$ and $z_2^{m_2} w z_2 = 0$. Hence $0 \neq z_2^{m_2} w \in (0 : (z_1, z_2))$. Since $z_2^{m_2} w z_1 = 0$ and $z_1 | s z_3^m$, there is an integer $m_3 \geq 0$ such that $z_3^{m_3} z_2^{m_2} w \neq 0$ and $z_3^{m_3} z_2^{m_2} w z_3 = 0$. Thus $0 \neq z_3^{m_3} z_2^{m_2} w \in (0 : (z_1, z_2, z_3))$. Continuing in this manner, we can construct a $0 \neq z_n^{m_n} z_{n-1}^{m_{n-1}} \cdots z_2^{m_2} w \in (0 : (z_1, z_2, z_3, \ldots, z_n))$. Hence R is a McCoy ring. □

Corollary 3.3. *Let* R *be a ring such that the prime ideals of* R *contained in* $Z(R)$ *are linearly ordered, and let* $x_1, \ldots, x_n \in Z(R) \setminus Nil(R)$. *Then there is a* $y \in Nil(R)^*$ *such that* $x_i y = 0$ *for every integer* $i, 1 \leq i \leq n$.

Proof. There is a $y \in Z(R)^*$ such that each $x_i y = 0$ since R is a McCoy ring and $Z(R)$ is an ideal of R. Moreover, $y \in Nil(R)$ since $x_1 \notin Nil(R)$ and $Nil(R)$ is a prime ideal of R by Theorem 2.5(3). □

Remark 3.4. *If R is a McCoy ring and $Z(R)$ is an ideal of R, then clearly $diam(\Gamma(R)) \leq 2$. This observation, together with Theorem 3.2, gives another proof of Theorem 2.7. However, note that $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ is a McCoy ring with $diam(\Gamma(R)) = 3$ (cf. Example 2.10(b)).*

We next show that the subgraph $\Gamma(R) \setminus Nil(R)$ of $\Gamma(R)$ is infinite and totally disconnected when $Nil(R)$ is a prime ideal of R and $Nil(R) \subsetneq Z(R)$ (i.e., when $\Gamma(R) \setminus Nil(R)$ is nonempty). This fact gives another proof of the "moreover" statement of Theorem 2.12, namely, that $\Gamma(R)$ is an infinite star graph when $Nil(R)$ is a prime ideal of R, $Nil(R) \subsetneq Z(R)$, and $|Nil(R)^*| = 1$.

Theorem 3.5. *Let R be a ring.*

1. *$\Gamma(R) \setminus Nil(R)$ is totally disconnected if and only if $Nil(R)$ is a prime ideal of R.*
2. *If $Nil(R)$ is a prime ideal of R and $Nil(R) \subsetneq Z(R)$, then $Z(R) \setminus Nil(R)$ is infinite.*

In particular, $\Gamma(R) \setminus Nil(R)$ is infinite and totally disconnected when the prime ideals of R contained in $Z(R)$ are linearly ordered and $Nil(R) \subsetneq Z(R)$.

Proof. (1) Suppose that $\Gamma(R) \setminus Nil(R)$ is totally disconnected. Let $xy \in Nil(R)$ with $x, y \notin Nil(R)$. Then $x^n y^n = 0$ for some positive integer n. Thus $x^n, y^n \in Z(R) \setminus Nil(R)$ and $x^n \neq y^n$ since $x, y \notin Nil(R)$. But then x^n and y^n are adjacent in $\Gamma(R) \setminus Nil(R)$, a contradiction. Hence $Nil(R)$ is a prime ideal of R. The converse is clear.

(2) Let $x \in Z(R) \setminus Nil(R)$. Suppose that $x^n = x^m$ for some integers $n > m \geq 1$. Then $x^m(1 - x^{n-m}) = 0 \in Nil(R)$ and $x \notin Nil(R)$ implies $1 - x^{n-m} \in Nil(R)$ since $Nil(R)$ is prime. Thus $x^{n-m} = 1 - (1 - x^{n-m}) \in U(R)$, and hence $x \in U(R)$, a contradiction. Thus $Z(R) \setminus Nil(R)$ is infinite.

The "in particular" statement holds since in this case $Nil(R)$ is a prime ideal of R by Theorem 2.5(3). □

Combining Lemma 2.1, Theorem 3.5, and Corollary 3.3, we have the following structure theorem for $\Gamma(R)$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. Then $Nil(R)^*$ is a subgraph of $\Gamma(R)$ of diameter at most 2, $\Gamma(R) \setminus Nil(R)$ is infinite and totally disconnected when $Nil(R) \subsetneq Z(R)$, and for each finite set of vertices $Y \subseteq \Gamma(R) \setminus Nil(R)$, there is a vertex $y \in Nil(R)^*$ which is adjacent to every element of Y.

Our next goal is to investigate $\mathrm{diam}(\Gamma(R[X]))$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. The diameter of $\Gamma(R[X])$ has recently been studied in [11], [10], and [32]. In particular, [32, Theorems 3.4 and 3.6] give nice characterizations of $\mathrm{diam}(\Gamma(R[X]))$. If $Z(R)^2 = \{0\}$ (i.e., $\Gamma(R)$ is a complete graph), then $Z(R[X])^2 = \{0\}$; so $\Gamma(R[X])$ is a complete graph with $\mathrm{diam}(\Gamma(R[X])) = 1$. McCoy's Theorem for polynomial rings states that $f(X) \in Z(R[X])$ if and only if $rf(X) = 0$ for some $0 \neq r \in R$, i.e., $Z(R[X]) \subseteq Z(R)[X]$. Thus $Z(R[X])$ is an ideal of $R[X]$ if and only if R is a McCoy ring and $Z(R)$ is an ideal of R [32, Theorem 3.3], and in this case, $Z(R[X]) = Z(R)[X]$.

Theorem 3.6. *Let R be a ring such that the prime ideals of R contained in $Z(R)$ are linearly ordered.*

(1) *$Z(R[X])$ is an (prime) ideal of $R[X]$.*
(2) *If R is not an integral domain and $Z(R)^2 = \{0\}$, then $diam(\Gamma(R[X])) = 1$.*
(3) *If $Z(R)^2 \neq \{0\}$, then $diam(\Gamma(R[X])) = 2$.*

Proof. Part (1) follows from Theorem 3.2 and [32, Theorem 3.3]. We have already observed part (2) above. Part (3) follows from Theorem 3.2, Corollary 2.9, and [32, Theorem 3.4(3)]. □

Corollary 3.7. *If R is any of the following types of rings with $Z(R)^2 \neq \{0\}$, then $diam(\Gamma(R[X])) = 2$.*

(1) *R is a ring such that the prime ideals of R are linearly ordered.*
(2) *R is a divided ring.*
(3) *R is a PVR.*
(4) *R is a ϕ-PVR.*
(5) *R is a chained ring.*

Corollary 3.8. *Let R be a nonreduced ring such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then exactly one of the following four cases must occur.*

(1) *$|Z(R)^*| = 1$. In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[Y]/(Y^2)$, $diam(\Gamma(R)) = 0$, and $diam(\Gamma(R[X])) = 1$.*
(2) *$|Z(R)^*| \geq 2$, $Z(R) = Nil(R)$, and $Z(R)^2 = \{0\}$. In this case, $diam(\Gamma(R)) = diam(\Gamma(R[X])) = 1$.*
(3) *$Z(R) = Nil(R)$ and $Z(R)^2 \neq \{0\}$. In this case, $diam(\Gamma(R)) = diam(\Gamma(R[X])) = 2$.*
(4) *$Nil(R) \subsetneq Z(R)$. In this case, $diam(\Gamma(R)) = diam(\Gamma(R[X])) = 2$.*

Proof. This follows directly from Theorem 2.2 and Theorem 3.6. □

The following example illustrates the four cases stated in Corollary 3.8. In each case, the ring R is actually a chained ring. The routine details are left to the reader.

Example 3.9. *(a) Let $R = \mathbb{Z}_4$. Then R is a a chained ring with $|Z(R)^*| = 1$. Thus $diam(\Gamma(R)) = 0$ and $diam(\Gamma(R[X])) = 1$.*

(b) Let $R = \mathbb{Z}_9$. Then R is a chained ring with $|Z(R)^*| = 2, Z(R) = Nil(R) = N(R)$, and $Z(R)^2 = \{0\}$. Thus $diam(\Gamma(R)) = diam(\Gamma(R[X])) = 1$.

(c) Let $R = \mathbb{Z}_8$. Then R is a chained ring with $N(R) \subsetneq Nil(R) = Z(R)$ and $Z(R)^2 \neq \{0\}$. Thus $diam(\Gamma(R)) = diam(\Gamma(R[X])) = 2$.

(d) Let $D = \mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$ and $I = XD = \mathbb{Z}_{(2)}X + X^2\mathbb{Q}[[X]]$. Set $R = D/I$. Then D is a valuation domain; so R is a chained ring. Note that $Z(R) = (2\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]])/I = 2R$ and $N(R) = Nil(R) = X\mathbb{Q}[[X]]/I$; so $Nil(R) \subsetneq Z(R)$ and $Nil(R)^2 = \{0\}$. Thus $diam(\Gamma(R)) = diam(\Gamma(R[X])) = 2$.

Unlike the case for the diameter of the zero-divisor graph of a polynomial ring as in Corollary 3.8, the girth case is very easy. The girth of $\Gamma(R[X])$ and $\Gamma(R[[X]])$ has been studied in [11] and [10], and a complete characterization is given in [10, Theorem 3.2]. For any nonreduced ring R, we always have $\mathrm{gr}(\Gamma(R[X])) = \mathrm{gr}(\Gamma(R[[X]])) = 3$ by [10, Lemma 3.1] (since $aX - aX^2 - aX^3 - aX$ forms a triangle for any $a \in N(R)^*$).

4. CHAINED RINGS

In this section, we investigate $\Gamma(R)$ when R is a chained ring. This is probably the nicest case where the prime ideals of R contained in $Z(R)$ are linearly ordered since in a chained ring all the ideals are linearly ordered. A typical example of a chained ring is a homomorphic image of a valuation domain. In particular, $\mathbb{Z}_n$ is a chained ring if and only if n is a prime power. In fact, it was an open question (attributed to Kaplansky) if every chained ring is the homomorphic image of a valuation domain (cf. [30, Chapter V]). However, an example in [28] shows that this is not true in general. It will turn out that the subset $N(R) = \{x \in R \mid x^2 = 0\}$ of $Nil(R)$ will play a major role in describing $\Gamma(R)$ when R is a chained ring. Note that if R is a chained ring, then $N(R) = \{0\}$ if and only if $Z(R) = \{0\}$. Also note that for any ring R, we have $N(R) = Nil(R)$ when $Nil(R)^2 = \{0\}$, and $N(R) = \{0\}$ if and only if $Nil(R) = \{0\}$. We start with several lemmas. In some cases, these results are special cases of ones from previous sections; however, the proofs are much easier in the chained ring setting.

Lemma 4.1. *Let R be a ring, $N(R) = \{x \in R \mid x^2 = 0\}$, and $x \in Nil(R) \setminus N(R)$. Then $xy = 0$ for some $y \in N(R)^* \setminus \{x\}$.*

Proof. Let n (≥ 3) be the least positive integer such that $x^n = 0$, and let $y = x^{n-1}$. Then $xy = x^n = 0$, $y = x^{n-1} \neq 0$, and $y^2 = (x^{n-1})^2 = x^{2n-2} = 0$ because $2n - 2 \geq n$ since $n \geq 3$. Clearly $x \neq y$ since $x^2 \neq 0$. □

Thus any vertex of the subgraph $Nil(R) \setminus N(R)$ of $\Gamma(R)$ is adjacent to a vertex of $N(R)^*$. We next show, among other things, that for a chained ring R, any vertex of $\Gamma(R) \setminus N(R)$ is adjacent to a vertex of $N(R)^*$ and any two vertices of $N(R)^*$ are adjacent.

Lemma 4.2. *Let R be a chained ring, $N(R) = \{x \in R \mid x^2 = 0\}$, and $x, y \in R$.*

1. *If $xy = 0$, then either $x \in N(R)$ or $y \in N(R)$.*
2. *If $x, y \in N(R)$, then $xy = 0$.*
3. *If $x, y \in Z(R) \setminus N(R)$, then $xy \neq 0$.*
4. *If $x \in Z(R)^*$, then $xy = 0$ for some $y \in N(R)^*$.*
5. *If $x_1, \ldots, x_n \in Z(R)^*$, then there is a $y \in N(R)^*$ such that $x_i y = 0$ for every integer $i, 1 \leq i \leq n$.*
6. *$N(R)$ is an ideal of R.*
7. *$N(R)$ is a prime ideal of R if and only if $N(R) = Nil(R)$.*

Proof. (1) Suppose that $x|y$. Then $y = rx$ for some $r \in R$; so $y^2 = rxy = 0$.

(2) Suppose that $x|y$. Then $y = rx$ for some $r \in R$, and hence $xy = rx^2 = 0$.

(3) This follows from part (1).

(4) If $x \in N(R)^*$, then let $y = x$. If $x \in Z(R) \setminus N(R)$, then $xy = 0$ for some $0 \neq y \in R$. By part (3) above, we must have $y \in N(R)$.

(5) There is an integer $j, 1 \leq j \leq n$, such that $x_j | x_i$ for all $i, 1 \leq i \leq n$. By part (4) above, there is a $y \in N(R)^*$ such that $x_j y = 0$; so $x_i y = 0$ for all $i, 1 \leq i \leq n$.

(6) Clearly $xN(R) \subseteq N(R)$ for all $x \in R$; so we need only show that $N(R)$ is closed under addition. Let $x, y \in N(R)$. Then $x^2 = y^2 = 0$, and $xy = 0$ by part (2) above. Thus $(x + y)^2 = x^2 + 2xy + y^2 = 0$, and hence $x + y \in N(R)$.

(7) This is clear since $Nil(R)$ is the unique minimal prime ideal of R. □

One can ask if part(5) above extends to any subset of $Z(R)^*$. Of course, if $X \subseteq xR$ and $yx = 0$, then $yX = \{0\}$. So if $X \subseteq Z(R)^*$ and $X \subseteq xR$ for some $x \in Z(R)^*$, then $yX = \{0\}$ for some $y \in N(R)^*$. Our next remark addresses this question.

Remark 4.3. *(a) Let $D = V + XK[[X]]$, where V is a valuation domain with nonzero maximal ideal M and quotient field K; so D is also a valuation domain. Let $I = XD = VX + X^2K[[X]]$, and set $R = D/I$. Then R is a chained ring with maximal ideal $Z(R) = (M + XK[[X]])/I$ and $N(R) = Nil(R) = XK[[X]]/I$. Note that there is a $y \in N(R)^*$ such that $yZ(R) = \{0\}$ if and only if there is a $y \in M^{-1} \setminus V$. (So for dim$(V) = 1$, this happens if and only if V is a DVR.)*

(b) If R is a chained ring, then $N(R) = \{x \in R \mid x^2 = 0\}$ is an ideal of R by Lemma 4.2(6). In general, $N(R)$ need not be an ideal of R (see Examples 5.5 and 5.6). However, if char$(R) = 2$, then $N(R)$ is an ideal of R. Also note that if $2 \in U(R)$ and $N(R)$ is an ideal of R, then $xy = 0$ for all $x, y \in N(R)$.

By Theorem 3.5(1), $\Gamma(R) \setminus Nil(R)$ is totally disconnected when R is a chained ring. Lemma 4.2(3) yields the following stronger result (also see Example 5.5).

Theorem 4.4. *Let R be a chained ring and $N(R) = \{x \in R \mid x^2 = 0\}$. Then $\Gamma(R) \setminus N(R)$ is totally disconnected.*

Our next result is a special case of Theorem 2.7, but we give a proof in the spirit of this section. We can also explicitly say when diam$(\Gamma(R))$ is 0, 1, or 2.

Theorem 4.5. *Let R be a chained ring. Then diam$(\Gamma(R)) \leq 2$.*

Proof. We may assume that $|Z(R)^*| \geq 2$. Let $N(R) = \{x \in R \mid x^2 = 0\}$, and let $x, y \in Z(R)^*$ be distinct. If $x, y \in N(R)$, then $xy = 0$ by Lemma 4.2(2), and thus $d(x, y) = 1$. If $x \in N(R)$ and $y \notin N(R)$, then $yz = 0$ for some $z \in N(R)^*$ by Lemma 4.2(4), and hence $xz = 0$ by Lemma 4.2(2). Thus $d(x, y) \leq 2$. Finally, let $x \notin N(R)$ and $y \notin N(R)$. Then $xz = yz = 0$ for some $z \in N(R)^*$ by Lemma 4.2(5). Thus $d(x, y) \leq 2$, and hence diam$(\Gamma(R)) \leq 2$. □

Theorem 4.6. *Let R be a chained ring with $Z(R) \neq \{0\}$, and let $N(R) = \{x \in R \mid x^2 = 0\}$. Then exactly one of the following three cases must occur.*

1. *$|Z(R)^*| = 1$. In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and diam$(\Gamma(R)) = 0$.*
2. *$|Z(R)^*| \geq 2$ and $N(R) = Z(R)$. In this case, diam$(\Gamma(R)) = 1$.*
3. *$N(R) \subsetneq Z(R)$. In this case, diam$(\Gamma(R)) = 2$.*

Proof. The first part follows from [24, Proposition 2.2]. The other two follow directly from Lemma 4.2 and Theorem 4.5. □

Let R be a chained ring with $N(R) = \{x \in R \mid x^2 = 0\}$. It is now easy to describe the structure of $\Gamma(R)$. First, observe that $N(R)^*$ is a complete subgraph of $\Gamma(R)$ by Lemma 4.2(2), $\Gamma(R) \setminus N(R)$ is totally disconnected by Lemma 4.2(3), and $\Gamma(R) \setminus N(R)$ is infinite if $Nil(R) \subsetneq Z(R)$. Moreover, for any finite set of vertices $Y \subseteq \Gamma(R) \setminus N(R)$, there is a vertex $z \in N(R)^*$ adjacent to every element in Y by Lemma 4.2(5). In particular, $\Gamma(R)$ is complete if and only if $Z(R) = N(R)$. Note that this description of $\Gamma(R)$ recovers Theorem 4.6. Also note that $Nil(R)^*$ need not be a complete subgraph of $\Gamma(R)$ (e.g., when R is the chained ring $\mathbb{Z}_{16}$).

The structure of $\Gamma(R)$ described in the preceding paragraph also extends to $\Gamma(R[X])$ when R is a chained ring. Note that when R is a chained ring, we have $N(R[X]) = N(R)[X], Nil(R[X]) = Nil(R)[X]$, and $Z(R[X]) = Z(R)[X]$ (of course, $Nil(R[X]) = Nil(R)[X]$ holds for any ring R). These statements are easy to verify directly, or just note that for any $0 \neq f \in R[X]$, we have $f = af^*$, where $a \in R$ and $f^* \in R[X]$ has unit content. Then $f \in N(R[X])$ (resp., $Nil(R[X]), Z(R[X])$) if and only if $a \in N(R)$ (resp., $Nil(R), Z(R)$). Thus $N(R[X])^*$ is a complete subgraph of $\Gamma(R[X]), \Gamma(R[X]) \setminus N(R[X])$ is totally disconnected, and for any finite set of vertices $Y \subseteq \Gamma(R[X]) \setminus N(R[X])$, there is a vertex $f \in N(R[X])^*$ which is adjacent to every element in Y when R is a chained ring. Moreover, $N(R[X])^*$ and $\Gamma(R[X]) \setminus N(R[X])$ are both infinite when R is a nonreduced chained ring. This observation shows that $\mathrm{diam}(\Gamma(R[X])) = 1$ when $Z(R)^2 = \{0\}$ and $\mathrm{diam}(\Gamma(R[X])) = 2$ when $Z(R)^2 \neq \{0\}$.

The above description of $\Gamma(R)$ also enables us to easily determine $\mathrm{gr}(\Gamma(R))$ when R is a chained ring (cf. Theorem 2.12). Note that $\Gamma(R)$ is a finite star graph in the first three cases of the next theorem, but it is not possible to have $\Gamma(R)$ be an infinite star graph when R is a chained ring (cf. Theorem 2.12).

Theorem 4.7. *Let R be a chained ring with $N(R) = \{x \in R \mid x^2 = 0\} \neq \{0\}$. Then exactly one of the following five cases must occur.*

(1) *$|N(R)^*| = 1$ and $N(R) = Z(R)$. In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and $gr(\Gamma(R)) = \infty$.*
(2) *$|N(R)^*| = 1$ and $N(R) \subsetneq Z(R)$. In this case, R is isomorphic to $\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, and $gr(\Gamma(R)) = \infty$.*
(3) *$|N(R)^*| = 2$ and $N(R) = Z(R)$. In this case, R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, and $gr(\Gamma(R)) = \infty$.*
(4) *$|N(R)^*| = 2$ and $N(R) \subsetneq Z(R)$. In this case, $gr(\Gamma(R)) = 3$.*
(5) *$|N(R)^*| \geq 3$. In this case, $gr(\Gamma(R)) = 3$.*

Proof. If $|N(R)^*| \geq 3$, then clearly $\mathrm{gr}(\Gamma(R)) = 3$ by Lemma 4.2(2). Suppose that $|N(R)^*| = 2$; say $N(R)^* = \{x, y\}$. If $y \neq -x$, then $x + y$ is a third nonzero element of $N(R)$, a contradiction. Thus $y = -x$; so $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$. If there is a $z \in Z(R) \setminus N(R)$, then $x - y - z - x$ is a triangle by Lemma 4.2(4); so $\mathrm{gr}(\Gamma(R)) = 3$. Otherwise, $Z(R) = N(R)$, and thus $\mathrm{gr}(\Gamma(R)) = \infty$. Finally, suppose that $|N(R)^*| = 1$, say $N(R) = \{0, x\}$. If $Z(R) = N(R)$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by [24, Proposition 2.2]. In this case, $\mathrm{gr}(\Gamma(R)) = \infty$. So suppose that $N(R) \subsetneq Z(R)$. By parts (3) and (4) of Lemma 4.2, $\Gamma(R)$ is a star graph with center x. Thus $|R| = 8, |R| = 9$, or $|R| > 9$ and $Nil(R) = \{0, x\}$ by [8, Lemma 3.7]. The $|R| > 9$ case can not happen. For in this case, $Nil(R) = N(R) = xR$ is a prime ideal of R. Let $y \in Z(R)^* \setminus \{x\}$. Then $xR \subsetneq yR$; so $x = yr$ for some $0 \neq r \in R$. Hence $r \in xR = \{0, x\}$ since xR is a prime ideal of R; so $r = x$. Thus $x = yx$, and hence $x(1 - y) = 0$. But R is quasilocal; so $1 - y \in U(R)$, and thus $x = 0$, a contradiction. If $|R| = 8$, then $R \cong \mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$; and if $|R| = 9$, then $R \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^3)$ by [8, Corollary 3.11]. As each of these rings is a chained ring, the result follows. □

We close this section with several examples.

Example 4.8. *(a) Let R be the (nonreduced) chained ring $\mathbb{Z}_{p^n}$, where p is prime and $n \geq 2$. Then $diam(\Gamma(R)) = 0$ if and only if $p = 2$ and $n = 2$, $diam(\Gamma(R)) = 1$ if and only if $p > 2$ and $n = 2$, and $diam(\Gamma(R)) = 2$ if and only if $n \geq 3$.*

We have $gr(\Gamma(R)) = \infty$ *if either* $p = 2$ *and* $2 \leq n \leq 3$ *or* $p = 3$ *and* $n = 2$*; otherwise,* $gr(\Gamma(R)) = 3$.

(b) We have $N(R) \subseteq Nil(R) \subseteq Z(R)$ *for any ring* R*. We give examples to show that all four cases for inclusion or proper inclusion are possible when* R *is a chained ring. The easy details are left to the reader. Recall that* $\{0\} \neq Nil(R) \subsetneq Z(R)$ *forces a chained ring* R *to be infinite, and thus so is* $\Gamma(R)$*. (i) Let* $R = \mathbb{Z}_4$*. Then* $N(R) = Nil(R) = Z(R)$*. (ii) Let* $R = (\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]])/(X)$*. Then* $N(R) = Nil(R) \subsetneq Z(R)$*. (iii) Let* $R = \mathbb{Z}_8$*. Then* $N(R) \subsetneq Nil(R) = Z(R)$*. (iv) Let* $R = (\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]])/(X^2)$*. Then* $N(R) \subsetneq Nil(R) \subsetneq Z(R)$*.*

(c) Let R_1 *and* R_2 *be chained rings and* $R = R_1 \times R_2$*. Then* $N(R) = N(R_1) \times N(R_2)$ *and* R *is never a chained ring since the ideals* $(1,0)R$ *and* $(0,1)R$ *are not comparable. Note that* $N(R)^*$ *is still a complete subgraph of* $\Gamma(R)$ *and any* $(x,y) \in \Gamma(R)$ *is still adjacent to some element of* $N(R)^*$*, but* $\Gamma(R) \setminus N(R)$ *is not totally disconnected since* $(0,1)$ *and* $(1,0)$ *are adjacent.*

(d) We have already observed that for a chained ring R*, its zero-divisor graph* $\Gamma(R)$ *is complete if and only if* $Z(R) = N(R)$*. However, if* R *is not a chained ring, then* $Z(R) = N(R)$ *does not imply that* $\Gamma(R)$ *is complete. For example, let* $R = \mathbb{Z}_2[X,Y]/(X^2,Y^2) = \mathbb{Z}_2[x,y]$*. Then* R *is not a chained ring since the ideals* xR *and* yR *are not comparable. However,* $N(R) = Nil(R) = Z(R) = \{0, x, y, x+y, xy, x+xy, y+xy, x+y+xy\}$*, but* $\Gamma(R)$ *is not complete since* $xy \neq 0$*. Note that the prime ideals of* R *are (trivially) linearly ordered,* $diam(\Gamma(R)) = 2$*, and* $gr(\Gamma(R)) = 3$*.*

(e) A ring R *such that* $Nil(R)^* (= N(R)^*)$ *is a complete subgraph of* $\Gamma(R)$ *and* $\Gamma(R) \setminus Nil(R)$ *is totally disconnected, but* R *is not a chained ring. Let* D *be an integral domain which is not a valuation domain, and let* K *be the quotient field of* D*. Set* $R = D(+)(K/D)$*; for example, let* $R = \mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z})$*. Note that* $N(R) = Nil(R) = \{0\}(+)(K/D) \subsetneq Z(R) = (D \setminus U(D))(+)(K/D)$ *and* $Nil(R)^2 = \{0\}$*. Thus one can easily verify that* R *satisfies the desired conditions.*

5. $\Gamma(R)$ WHEN $R \in \mathcal{H}$

In this final section, we are interested in the case where the ring R satisfies $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R) \setminus Nil(R)$. In particular, this condition holds when $R \in \mathcal{H}$ is not an integral domain (i.e., when $Nil(R)$ is a nonzero divided prime ideal of R; so $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in R \setminus Nil(R)$). We start by showing that if $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R) \setminus Nil(R)$, then $Nil(R)$ is a prime ideal of R (cf. the proof of [3, Proposition 5.1]), and that $Nil(R)$ is a divided prime ideal of R when $Nil(R) \subsetneq Z(R)$.

Theorem 5.1. *Let* R *be a ring with* $\{0\} \neq Nil(R) \subseteq zR$ *for all* $z \in Z(R) \setminus Nil(R)$*.*

1. $Nil(R)$ *is a prime ideal of* R*.*
2. $Nil(R) \subseteq \bigcap_{n \geq 1} z^n R$ *for all* $z \in Z(R) \setminus Nil(R)$*.*
3. *If* $Nil(R) \subsetneq Z(R)$*, then* $Nil(R)$ *is a divided prime ideal of* R*.*

Proof. (1) If $Nil(R) = Z(R)$, then $Nil(R)$ is a prime ideal of R. So we may assume that $Nil(R) \subsetneq Z(R)$ and $Nil(R) \subseteq zR$ for all $z \in Z(R) \setminus Nil(R)$. Suppose that $Nil(R)$ is not prime. Then there are $x, y \in Z(R) \setminus Nil(R)$ with $xy \in Nil(R)$. Thus $x^2 \in Z(R) \setminus Nil(R)$, and hence $Nil(R) \subseteq x^2R$. Thus $xy = x^2d$ for some $d \in R$, and hence $y - xd \notin Nil(R)$ since $xd \in Nil(R)$ and $y \notin Nil(R)$. Since

$(y - xd)x = 0$, we have $y - xd \in Z(R) \setminus Nil(R)$. Thus $Nil(R) \subseteq (y - xd)R$, and hence $xNil(R) \subseteq x(y - xd)R = \{0\}$. Let $0 \neq z \in Nil(R) \subseteq x^2R$. Then $z = x^2r$ for some $r \in R$, and $xr \in Nil(R)$. Thus $z = x(xr) = 0$, a contradiction. Hence $Nil(R)$ is a prime ideal of R.

(2) Let $z \in Z(R) \setminus Nil(R)$. Then $z^n \in Z(R) \setminus Nil(R)$ for all integers $n \geq 1$ since $Nil(R)$ is a prime ideal of R by part (1), and thus $Nil(R) \subseteq z^nR$ for all integers $n \geq 1$. Hence $Nil(R) \subseteq \bigcap_{n\geq 1} z^nR$.

(3) Let $z \in R \setminus Nil(R)$ and $w \in Z(R) \setminus Nil(R)$. Then $wz \in Z(R) \setminus Nil(R)$, and thus $Nil(R) \subseteq wzR \subseteq zR$. Hence $Nil(R)$ is a divided prime ideal of R. ☐

Corollary 5.2. *The following statements are equivalent for a ring R.*

1. $\{0\} \neq Nil(R) \subseteq zR$ *for all* $z \in Z(R) \setminus Nil(R)$ *and* $Nil(R) \subsetneq Z(R)$.
2. $R \in \mathcal{H}$ *and* $Nil(R) \subsetneq Z(R)$.

The simplest example of a ring R with $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Nil(R) \setminus Nil(R)$ and $Nil(R) \subsetneq Z(R)$ is a nondomain chained ring R with $\dim(R) \geq 1$. We next give two examples to show that the condition $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R) \setminus Nil(R)$ neither implies nor is implied by the condition that the prime ideals of R contained in $Z(R)$ are linearly ordered. We also show that the $Nil(R) \subsetneq Z(R)$ hypothesis is needed in part (3) of Theorem 5.1.

Example 5.3. *(a) Let* $R = \mathbb{Z}(+)\mathbb{Z}_2$. *Then* $N(R) = Nil(R) = \{0\}(+)\mathbb{Z}_2$ *and* $Z(R) = 2\mathbb{Z}(+)\mathbb{Z}_2$. *Thus the prime ideals of R contained in $Z(R)$, namely $Nil(R)$ and $Z(R)$, are linearly ordered, but* $Nil(R) \not\subseteq (2, 0)R$ *for* $(2, 0) \in Z(R) \setminus Nil(R)$.

(b) Let $R = \mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z})$. *Then* $N(R) = Nil(R) = \{0\}(+)(\mathbb{Q}/\mathbb{Z})$ *and* $Z(R) = (\mathbb{Z} \setminus \{1, -1\})(+)(\mathbb{Q}/\mathbb{Z})$. *Thus the prime ideals of R contained in $Z(R)$ are not linearly ordered, but* $Nil(R) \subseteq zR$ *for all* $z \in Z(R) \setminus Nil(R)$*; so* $R \in \mathcal{H}$. *We have* $diam(\Gamma(R)) = 3$ *since* $d((2, 0), (3, 0)) = 3$. *Also note that R is a McCoy ring,* $gr(\Gamma(R)) = 3$, *and* $R \cong (\mathbb{Z} + X\mathbb{Q}[[X]])/(X)$.

(c) Let $R = \mathbb{Z}_4[X]$ *(or* $\mathbb{Z}_4[[X]]$*). Then* $N(R) = Nil(R) = Z(R) = 2R$*; so* $\{0\} \neq Nil(R) \subseteq zR$ *for all* $z \in Z(R) \setminus Nil(R)$. *But $Nil(R)$ is not divided since* $Nil(R) = 2R \not\subseteq XR$.

Suppose that $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$. Then we have already observed that $Z(R) \setminus Nil(R)$ must be infinite (Theorem 3.5(2)). In fact, both $Nil(R)$ and $Z(R) \setminus Nil(R)$ are infinite.

Theorem 5.4. *Let* $R \in \mathcal{H}$ *with* $Nil(R) \subsetneq Z(R)$.

1. *If* $xy = 0$ *for* $x \in Z(R) \setminus Nil(R)$ *and* $y \in R$, *then* $y \in N(R) \subseteq Nil(R)$ *and* $yNil(R) = \{0\}$. *Thus* $ann_R(x) \subseteq ann_R(Nil(R))$.
2. $Nil(R)$ *is infinite.*
3. $\Gamma(R) \setminus Nil(R)$ *is infinite and totally disconnected.*

Proof. (1) Suppose that $xy = 0$ for $x \in Z(R) \setminus Nil(R)$ and $y \in R$. Then $y \in Nil(R)$ since $Nil(R)$ is a prime ideal of R. Then $Nil(R) \subseteq xR$ since $Nil(R)$ is a divided prime ideal, and thus $yNil(R) \subseteq xyR = \{0\}$. In particular, $y^2 = 0$; so $y \in N(R)$.

(2) Let $x \in Z(R) \setminus Nil(R)$. We have $xz = 0$ for some $z \in Nil(R)^*$. Then for each integer $n \geq 1$, we have $z = z_nx^n$ for some $z_n \in R$ by Theorem 5.1(2). Note that $z_n \in Nil(R)^*$ since $Nil(R)$ is a prime ideal of R and $x^n \notin Nil(R)$. If $z_n = z_m$ for some integers $n > m \geq 1$, then $z = x^nz_n = x^nz_m = x^{n-m}(x^mz_m) = x^{n-m}z = 0$, a contradiction. Thus $Nil(R)$ is infinite.

(3) Since $Nil(R)$ is a prime ideal of R, the graph $\Gamma(R) \setminus Nil(R)$ is totally disconnected by Theorem 3.5(1) and infinite by Theorem 3.5(2). $\square$

We can now describe the structure of $\Gamma(R)$ when $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$. The subgraph $\Gamma(R) \setminus Nil(R)$ is infinite and totally disconnected, $Nil(R)^*$ is infinite, and for each vertex $x \in \Gamma(R) \setminus Nil(R)$, there is a vertex $y \in Nil(R)^*$ such that y is adjacent to x and to all other elements of $Nil(R)^*$.

Since $N(R) \subseteq Nil(R)$, the graph $\Gamma(R) \setminus Nil(R)$ is totally disconnected when $\Gamma(R) \setminus N(R)$ is totally disconnected (so this happens when R is a chained ring). However, our next example shows that we may have $\Gamma(R) \setminus Nil(R)$ totally disconnected, but $\Gamma(R) \setminus N(R)$ is not totally disconnected for a ring $R \in \mathcal{H}$ with the prime ideals of R contained in $Z(R)$ linearly ordered.

Example 5.5. *Let $D = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$ and $I = X^2D = \mathbb{Z}_{(2)}X^2 + X^3\mathbb{R}[[X]]$. Set $R = D/I$. Then R is quasilocal with maximal ideal $Z(R) = (2\mathbb{Z}_{(2)} + X\mathbb{R}[[X]])/I = 2R$ and $Nil(R) = X\mathbb{R}[[X]]/I$. Note that R is not a chained ring and the prime ideals of R contained in $Z(R)$, namely $Nil(R)$ and $Z(R)$, are linearly ordered. Let $f = \pi X + I$ and $g = \pi^{-1}X + I$. Then $f, g \in Nil(R) \setminus N(R)$, but $fg = X^2 + I = 0$; so $\Gamma(R) \setminus N(R)$ is not totally disconnected. Also $N(R)$ is not an ideal of R and $N(R)^2 \neq \{0\}$ (and hence $Nil(R)^2 \neq \{0\}$) since $f = \sqrt{2}X + I, g = \sqrt{3}X + I \in N(R)$, but $f + g \notin N(R)$ and $fg = \sqrt{6}X^2 + I \neq 0$. It is easy to check that $R \in \mathcal{H}$.*

The next example shows that Theorem 5.4(1) need not hold if we only assume that the prime ideals of R contained in $Z(R)$ are linearly ordered.

Example 5.6. *Let $D = \mathbb{Q}[X, Y, Z]_{(X,Y,Z)}$ and $I = (X^2, Y^2, XZ)_{(X,Y,Z)}$. Set $R = D/I = \mathbb{Q}[x, y, z]$. Then $Nil(R) = (x, y) \subsetneq (x, y, z) = Z(R)$. The prime ideals of R contained in $Z(R)$, namely $Nil(R)$ and $Z(R)$, are linearly ordered. Then $z \in Z(R) \setminus Nil(R)$ and $xz = 0$, but $xNil(R) \neq \{0\}$ since $xy \neq 0$. Note that $N(R)$ is not an ideal of R and $Nil(R)^2 \neq \{0\}$.*

Observe that if $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$, then $Ker(\phi) = \{w \in Nil(R) \mid zw = 0 \text{ for some } z \in Z(R) \setminus Nil(R)\} \subseteq Nil(R)$. Thus $Ker(\phi)^*$ is precisely the set of vertices of $\Gamma(R)$ which are adjacent to some vertex of $\Gamma(R) \setminus Nil(R)$. Clearly $Nil(R) \subseteq Ker(\phi)$ when $\phi(R)$ is an integral domain, and thus $Ker(\phi) = Nil(R)$ when $\phi(R)$ is an integral domain.

Corollary 5.7. *Let $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$. Then $Nil(R)Ker(\phi) = \{0\}$, and thus $Ker(\phi)^2 = \{0\}$ (so $Ker(\phi) \subseteq N(R)$). In particular, when $\phi(R)$ is an integral domain, then $Nil(R)^2 = \{0\}$, and hence $Nil(R)^*$ is a complete subgraph of $\Gamma(R)$.*

Proof. Let $y \in Ker(\phi)$. Then there is a $z \in Z(R) \setminus Nil(R)$ with $zy = 0$. Thus $yNil(R) = \{0\}$ by Theorem 5.4(1), and hence $Nil(R)Ker(\phi) = \{0\}$. Thus $Ker(\phi)^2 = \{0\}$ since $Ker(\phi) \subseteq Nil(R)$. Now suppose that $\phi(R)$ is an integral domain. Then $Nil(R) = Ker(\phi)$, and hence $Nil(R)^2 = \{0\}$. Thus $Nil(R)^*$ is a complete subgraph of $\Gamma(R)$. $\square$

Remark 5.8. *The proof of Theorem 5.4(2) actually shows that $Ker(\phi)$ is infinite since z and each z_n are in $Ker(\phi)$. Thus by the above corollary, $Ker(\phi)^*$ is an infinite complete subgraph of $\Gamma(R)$ when $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$. Also $Ker(\phi) \subseteq N(R) \subseteq Nil(R)$; so all three are infinite when $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$.*

The following is an example of a ring $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$ and $Nil(R)^2 = \{0\}$, but $\phi(R)$ is not an integral domain.

Example 5.9. *Let $R = \mathbb{Z}(+)(\mathbb{R}/\mathbb{Z}_{(2)})$. Then $N(R) = Nil(R) = \{0\}(+)(\mathbb{R}/\mathbb{Z}_{(2)})$, $Nil(R)^2 = \{0\}$, $Z(R) = 2\mathbb{Z}(+)(\mathbb{R}/\mathbb{Z}_{(2)})$, and $Ker(\phi) = \{0\}(+)(\mathbb{Q}/\mathbb{Z}_{(2)})$. Thus $R \in \mathcal{H}$ and $Ker(\phi) \subsetneq Nil(R) \subsetneq Z(R)$; so $\phi(R)$ is not an integral domain. In fact, $\phi(R) \cong R/Ker(\phi) \cong \mathbb{Z}(+)(\mathbb{R}/\mathbb{Q})$. Note that $Nil(R)^*$ (and hence $Ker(\phi)^*$) is a complete subgraph of $\Gamma(R)$, and $\Gamma(R) \setminus Nil(R)$ is totally disconnected by Theorem 5.4(3). However, $\Gamma(R) \setminus Ker(\phi)$ is not totally disconnected; for example, $(0, \pi + \mathbb{Z}_{(2)})$ and $(0, \pi^{-1} + \mathbb{Z}_{(2)})$ are adjacent in $\Gamma(R) \setminus Ker(\phi)$ (cf. Theorem 5.10).*

We next give another characterization for when $\phi(R)$ is an integral domain in terms of complete and totally disconnected subgraphs of $\Gamma(R)$.

Theorem 5.10. *The following statements are equivalent for a ring $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$.*

(1) *$\phi(R)$ is an integral domain.*
(2) *$Nil(R) = Ker(\phi)$.*
(3) *$Ker(\phi)^*$ is a complete subgraph of $\Gamma(R)$ and $\Gamma(R) \setminus Ker(\phi)$ is totally disconnected.*
(4) *$\Gamma(R) \setminus Ker(\phi)$ is totally disconnected.*

Proof. (1) $\Leftrightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) This follows from Theorem 5.4(3) and Corollary 5.7.

(3) $\Rightarrow$ (4) This is also clear.

(4) $\Rightarrow$ (2) We always have $Ker(\phi) \subseteq Nil(R)$ since $R \in \mathcal{H}$. Suppose that there is a $w \in Nil(R) \setminus Ker(\phi)$, and let $z \in Z(R) \setminus Nil(R)$. Then $zw \in Nil(R) \setminus Ker(\phi)$; so $zw \neq 0$. For if $zw \in Ker(\phi)$, then $tzw = 0$ for some $t \in Z(R) \setminus Nil(R)$. Thus $w \in Ker(\phi)$ since $tz \in Z(R) \setminus Nil(R)$, a contradiction. Also $zw \neq w$. For if $zw = w$, then $(z-1)w = 0$, and hence $z - 1 \in Z(R)^*$. Also $z - 1 \notin Nil(R)$ since $z - 1 \in Nil(R)$ implies that $z = 1 + (z-1) \in U(R)$, a contradiction. But then $z - 1 \in Z(R) \setminus Nil(R)$ and $(z-1)w = 0$; so $w \in Ker(\phi)$, a contradiction. If $w^2 = 0$, then $w - zw$ is an edge in $\Gamma(R) \setminus Ker(\phi)$, a contradiction. Hence we may assume that $w^2 \neq 0$. Let m (≥ 3) be the least positive integer such that $w^m = 0$. If $w^{m-1} \notin Ker(\phi)$, then $w - w^{m-1}$ is an edge in $\Gamma(R) \setminus Ker(\phi)$, which is again a contradiction. Thus let $k, 1 \leq k \leq m-1$, be the least positive integer such that $w^k \in Ker(\phi)$, and let $d \in Z(R) \setminus Nil(R)$ such that $dw^k = 0$. Then $k \geq 2$ since $w \notin Ker(\phi)$. Also $dw^{k-1} \notin Ker(\phi)$. For if $dw^{k-1} \in Ker(\phi)$, then $tdw^{k-1} = 0$ for some $t \in Z(R) \setminus Nil(R)$. Hence $w^{k-1} \in Ker(\phi)$ since $td \in Z(R) \setminus Nil(R)$, a contradiction. Since $w \neq dw^{k-1}$ because $w^2 \neq 0$, we have that $w - dw^{k-1}$ is an edge in $\Gamma(R) \setminus Ker(\phi)$, a contradiction. Hence $Ker(\phi) = Nil(R)$. □

Example 5.3(b) shows that a ring $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$ may have $\mathrm{diam}(\Gamma(R)) = 3$. Thus any of the possible diameters, $0, 1, 2$, or 3, may be realized by a ring in $\mathcal{H}$. However, if $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$, then $\mathrm{diam}(\Gamma(R))$ is either 2 or 3. For if $\mathrm{diam}(\Gamma(R)) = 0$ or 1, then $Z(R)^2 = \{0\}$, and thus $Nil(R) = Z(R)$.

We end the paper with the analog of Theorem 2.12 for rings in $\mathcal{H}$. Note that the $\mathrm{gr}(\Gamma(R)) = \infty$ case is not possible since $\Gamma(R)$ can not be an infinite star graph.

Theorem 5.11. *Let $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$. Then $gr(\Gamma(R)) = 3$.*

Proof. The theorem follows directly from Theorem 2.12 and Theorem 5.4(2). □

As an alternate proof of the above theorem, just note that $Ker(\phi)^*$ is an infinite complete subgraph of $\Gamma(R)$ when $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$ by Remark 5.8; so $\mathrm{gr}(\Gamma(R)) = 3$.

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