ON THE ZEROS OF STIELTJES AND VAN VLECK POLYNOMIALS[1]

BY

G. M. SHAH

1. Introduction

The generalized Lamé differential equation with which we shall be concerned is

$$(1.1)\quad \prod_{j=1}^{p} (x - a_j)[y'' + (\sum_{j=1}^{p} \alpha_j/(x - a_j))y'] + V(x)y = 0,$$

where all α_j are positive and $a_1 < a_2 < \cdots < a_p$. Also $V(x)$ is a polynomial of degree $(p - 2)$ in x to be specified presently. It is known [5] that there exist exactly $C(n + p - 2, p - 2)$ polynomials $V(x)$ of degree $(p - 2)$ such that corresponding to each such $V(x)$ the equation (1.1) has a polynomial solution $S(x)$ of degree n. Such $S(x)$ are called *Stieltjes polynomials* and the corresponding $V(x)$ are known as *Van Vleck polynomials* in the literature (see e.g. [2]). It has been shown that the zeros of all such $S(x)$ lie in (a_1, a_p) and those of the $V(x)$ also lie in (a_1, a_p) ([1] and [7]). Given a decomposition of the positive integer n into $n_1, n_2, \cdots, n_{p-1}$ nonnegative integers with $\sum_{j=1}^{p-1} n_j = n$, it was shown by Stieltjes [5] that there exists exactly one polynomial solution $S(x)$ of degree n with n_j $(j = 1, 2, \cdots, p - 1)$ zeros in (a_j, a_{j+1}). This result gives completely the location of the zeros of $S(x)$ in various intervals (a_j, a_{j+1}) $(j = 1, 2, \cdots, p - 1)$. The object of this paper is to give such information about the zeros of Van Vleck polynomials $V(x)$. In Section 2 we prove two lemmas which, in turn, are used in Section 3 to show that each $V(x)$ can have at most two zeros in any interval (a_j, a_{j+1}), $2 \leq j \leq p - 2$. It is also shown that each of the intervals (a_1, a_2), (a_{p-1}, a_p) contains at most one zero of $V(x)$. Section 4 deals with the bounds for the zeros of $S(x)$. These bounds can be used to give some known bounds for various classical polynomials.

2. Lemmas

In this section we intend to construct a function whose only zeros are the zeros of $S(x)$ and those of the corresponding $V(x)$. This will be done by proving the following two lemmas.

LEMMA 1. *A necessary and sufficient condition that an a_j $(j = 1, 2, \cdots, p)$ be a zero of $V(x)$ is that it be a zero of the derivative $S'(x)$ of the corresponding $S(x)$.*

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Proof. (Necessity). Suppose that a_j, $1 \leq j \leq p$ is a zero of $V(x)$. Since the equation (1.1) holds for $x = a_j$, it follows that either α_j is zero or $S'(x)$ has a zero at a_j. In view of the hypothesis that all α_j are positive we have that a_j is a zero of $S'(x)$.

(Sufficiency) Suppose that a_k is a zero of $S'(x)$. Then equation (1.1) yields for $x = a_k$, $V(a_k)S(a_k) = 0$. Since $S(a_k) \neq 0$, ($S(x)$ has all its zeros simple) it follows that $V(x)$ has a zero at a_k.

For convenience we denote by P the class of Van Vleck polynomials $V(x)$ which have each no zero at any a_j, $1 \leq j \leq p$ and the class consisting of the remaining $V(x)$ will be denoted by Q. It is easy to construct examples to show that neither P nor Q need be empty. The next lemma and the results of section 3 will deal with members of class P. It will, however, be shown at the end of Section 3 how the results obtained for the members of class P can be modified for the members of class Q.

LEMMA 2. *The zeros of a Van Vleck polynomial $V(x)$ of degree $(p - 2)$ and of class P and those of the corresponding Stieltjes polynomial $S(x)$ of degree n are the zeros of the function*

$$F(x) \equiv \sum_{j=1}^{n-1} 1/(x - x'_j) + \sum_{j=1}^{p} \alpha_j/(x - a_j)$$

and conversely, where x'_j $(j = 1, 2, \cdots, n-1)$ are the zeros of the derivative of $S(x)$.

Proof. Let $S(x)$ be a Stieltjes polynomial of degree n such that the corresponding $V(x)$ is of class P. Let

$$S(x) = \prod_{j=1}^{n} (x - x_j), \quad S'(x) = n \prod_{j=1}^{n-1} (x - x'_j),$$
$$V(x) = A \prod_{k=1}^{p-2} (x - t_k),$$

A a constant. Since $S'(x_k) \neq 0$, for zeros of $S(x)$ are all real and distinct [4], we have from equation (1.1), for $x = x_k$,

$$S''(x_k)/S'(x_k) + \sum_{j=1}^{p} \alpha_j/(x_k - a_j) = 0 \qquad (k = 1, 2, \cdots, n)$$

or

$$\sum_{j=1}^{n-1} 1/(x_k - x_j) + \sum_{j=1}^{p} \alpha_j \ (x_k - a_j) = 0 \qquad (k = 1, 2, \cdots, n). \tag{2.1}$$

Now, consider a zero t_k of $V(x)$. In view of the fact that $V(x) \in P$, $t_k \neq a_j$, $1 \leq j \leq p$ and $S'(t_k) \neq 0$ by Lemma 1. Hence for $x = t_k$, equation (1.1) yields

$$S''(t_k)/S'(t_k) + \sum_{j=1}^{p} \alpha_j/(t_k - a_j) = 0 \qquad (k = 1, 2, \cdots, p-2)$$

or

$$\sum_{j=1}^{n-1} 1/(t_k - x'_j) + \sum_{j=1}^{p} \alpha_j/(t_k - a_j) = 0 \quad (k = 1, 2, \cdots, p-2). \tag{2.2}$$

Equations (2.1) and (2.2) show that the zeros of any $S(x)$ and those of the corresponding $V(x)$, if $V(x) \in P$, are among the zeros of the function

$$F(x) \equiv \sum_{j=1}^{n-1} 1/(x - x'_j) + \sum_{j=1}^{p} \alpha_j/(x - a_j). \tag{2.3}$$

On the other hand, it is easy to see that $F(x)$ has only $(n + p - 2)$ zeros, equal to the number of zeros of $S(x)$ and of $V(x)$.

To prove the converse, we note that if ν is a zero of $F(x)$, then for $x = \nu$, equation (1.1) becomes

$$(2.4) \quad \prod_{j=1}^{p} (\nu - a_j)[S''(\nu) + S'(\nu) \textstyle\sum_{j=1}^{p} \alpha_j/(\nu - a_j)] + V(\nu)S(\nu) = 0.$$

We assert that $S'(\nu) \neq 0$, for otherwise $V(\nu) = 0$ and $\nu = a_k$ for some k by Lemma 1, which would contradict that $V(x) \in P$. Hence equation (2.4) can be simplified to

$$\frac{S''(\nu)}{S'(\nu)} + \sum_{j=1}^{p} \frac{\alpha_j}{\nu - a_j} + \frac{V(\nu)S(\nu)}{S'(\nu) \prod_{j=1}^{p} (\nu - a_j)} = 0$$

which in turn gives $V(\nu)S(\nu) = 0$. Hence ν is either a zero of $S(x)$ or of the corresponding $V(x)$.

3. Zeros of V(x)

Strong use of Lemma 2 is made in obtaining the results of this section.

THEOREM I. *Any interval* (a_k, a_{k+1}), $1 \leq k \leq p - 1$, *which does not contain any zero of* $S'(x)$ *contains at most one zero of the corresponding* $V(x)$, *if* $V(x) \in P$.

Proof. By Lemma 2, since $V(x) \in P$, the zeros of $V(x)$ are among the zeros of the function

$$(3.1) \qquad F(x) \equiv \textstyle\sum_{j=1}^{n-1} 1/(x - x_j') + \sum_{j=1}^{p} \alpha_j(x - a_j).$$

Differentiating the identity (3.1), we have

$$F'(x) \equiv -\textstyle\sum_{j=1}^{n-1} 1/(x - x_j')^2 - \sum_{j=1}^{p} \alpha_j/(x - a_j)^2.$$

Thus $F(x)$, apart from $(n + p - 1)$ points of discontinuity, namely x_j' $(j = 1, 2, \cdots, n - 1)$ and a_s $(s = 1, 2, \cdots, p)$, is differentiable in $[a_1, a_p]$ and a decreasing function of x in each interval of continuity.

We now restrict our attention to a fixed interval (a_k, a_{k+1}). As $x \to a_k +$, $F(x) \to +\infty$ and as $x \to a_{k-1} -$, $F(x) \to -\infty$. Thus, if (a_k, a_{k+1}) does not tain any x_j', $F(x)$ is continuous in (a_k, a_{k+1}) and decreases from $+\infty$ to $-\infty$ as x varies from a_k to a_{k+1}. Hence $F(x)$ changes sign just once in (a_k, a_{k+1}). In view of Lemma 2, either $V(x)$ or $S(x)$ has one zero, namely the zero of $F(x)$ in (a_k, a_{k+1}).

An immediate sequence of the above result is the following:

COROLLARY 1. *Any interval* (a_k, a_{k+1}), $1 \leq k \leq p - 1$, *which does not contain any zero of* $S'(x)$ *and* $S(x)$ *contains precisely one zero of the corresponding* $V(x)$, *if* $V(x) \in P$.

It is easy to see that among $C(n + p - 2, p - 2)$ Stieltjes polynomials $S(x)$ of degree n there are $(p - 1)$ polynomials which have each all its zeros in one interval (a_j, a_{j+1}) $(j = 1, 2, \cdots, p - 1)$. It follows from Lemma 1 that all $V(x)$ corresponding to such $S(x)$ are in class P. The following result gives the distribution of the zeros of such $V(x)$.

THEOREM II. *If all the zeros of a Stieltjes polynomial $S(x)$ of degree n lie in (a_j, a_{j+1}), $1 \leq j \leq p - 1$, then no zero of the corresponding $V(x)$ lies in (a_j, a_{j+1}) and each of the remaining $(p - 2)$ intervals (a_k, a_{k+1}) $(k \neq j, 1 \leq k \leq p - 1)$ contains precisely one zero of $V(x)$.*

Proof. As all the zeros of $S(x)$ are in (a_j, a_{j+1}), all x_j', the zeros of $S'(x)$ are contained in this interval [4]. Consequently no x_j' or x_j is contained in any (a_k, a_{k+1}) $(k \neq j, k = 1, 2, \cdots, p - 1)$. It follows by Corollary 1 that $V(x)$ has one zero in each interval (a_k, a_{k+1}) $(k \neq j, k = 1, 2, \cdots, p - 1)$. Since $V(x)$ is of degree $(p - 2)$ and the number of the intervals (a_k, a_{k+1}) $(k \neq j, k = 1, 2, \cdots, p - 1)$ is also $(p - 2)$, we have that $V(x)$ has no zero in (a_j, a_{j+1}).

THEOREM III. *Any two consecutive zeros of $S(x)$ if not separated by any a_j are not separated by a zero of the corresponding $V(x)$, if $V(x) \in P$. More generally, any q $(q \leq n)$ consecutive zeros of $S(x)$ if not separated by any a_j are not separated by zeros of the corresponding $V(x)$, if $V(x) \in P$.*

Proof. Let x_k, x_{k+1} be two successive zeros of $S(x)$ with $x_k < x_{k+1}$ which are not separated by any a_j. Thus both x_k and x_{k+1} lie in the same interval, say (a_j, a_{j+1}). We have to show that $V(x)$ has no zero in (x_k, x_{k+1}).

By Rolle's theorem, $S'(x)$ vanishes once between x_k and x_{k+1}, say at x_k'. Thus $x_k < x_k' < x_{k+1}$. $F(x_k) = 0$ and as $x \to x_k' -$, $F(x) \to -\infty$. Hence $F(x)$ decreases continuously from 0 to $-\infty$ as x varies from x_k to x_k'. Consequently $F(x)$ has no zero in the open interval (x_k, x_k'). Similarly $F(x)$ decreases from $+\infty$ to 0 as x moves from x_k' to x_{k+1} and has, therefore, no zero in the open interval (x_k', x_{k+1}).

To prove the last assertion, let $x_k < x_{k+1} < \cdots < x_{k+q-1}$ be q consecutive zeros of $S(x)$ not separated by any a_j. These zeros, then, lie in the same interval, say in (a_j, a_{j+1}). In view of the simplicity and reality of the zeros of $S(x)$ the inequalities

$$a_j < x_k < x_k' < x_{k+1} < \cdots < x_{k+q-2}' < x_{k+q-1} < a_{j+1}$$

hold, where x_j' $(j = k, \cdots, k + q - 2)$ denote the zeros of $S'(x)$ in (x_k, x_{k+q-1}). By repeated application of the argument used in the proof of the first part of this theorem it follows that $V(x)$ has no zero in (x_k, x_{k+q-1}).

The following result gives information about the zeros of those $V(x) \in P$ whose corresponding $S(x)$ have their zeros in more than one interval (a_j, a_{j+1}).

THEOREM IV. *Let $x_{k+1} < x_{k+2} < \cdots < x_{k+r}$ be r zeros of $S(x)$, $1 \leq r \leq n$, in (a_j, a_{j+1}), $1 \leq j \leq p - 1$, then the corresponding $V(x)$, if $V(x) \in P$, has at most one zero in (a_j, x_{k+1}), at most one zero in (x_{k+r}, a_{j+1}) and no zero in (x_{k+1}, x_{k+r}).*

Proof. That $V(x)$ has no zero in (x_{k+1}, x_{k+r}) is the content of Theorem III. It is obvious that (a_j, x_{k+1}) contains at most one zero of $S'(x)$. In case (a_j, x_{k+1}) does not contain any zero of $S'(x)$, then $F(x)$ is a continuously decreasing function of x in (a_j, x_{k+1}). Also as $x \to a_j+$, $F(x) \to +\infty$ and $F(x_{k+1}) = 0$. Thus $F(x)$ and by Lemma 2, $V(x)$ has no zero in (a_j, x_{k+1}).

In case (a_j, x_{k+1}) does contain one zero of $S'(x)$, say x_k', then $a_j < x_k' < x_{k+1}$, for $V(x) \in P$ and zeros of $S(x)$ are simple. Again, $F(x)$ is a continuously decreasing function of x in (a_j, x_k'). As $x \to a_j+$, $F(x) \to +\infty$ and as $x \to x_k'-$, $F(x) \to -\infty$. Hence $F(x)$ has precisely one zero in (a_j, x_k'). This zero of $F(x)$ cannot be a zero of $S(x)$, since the smallest zero of $S(x)$ in (a_j, a_{j+1}), by hypothesis, is x_{k+1} and $x_k' < x_{k+1}$. Therefore, this zero of $F(x)$ must be a zero of $V(x)$. It is easy to see that no zero of $V(x)$ lies in (x_k', x_{k+1}), for $F(x)$ decreases continuously from $+\infty$ to 0 in this interval. It can be shown similarly that $V(x)$ has at most one zero in (x_{k+r}, a_{j+1}).

The following corollaries follow from the proof of the above theorem.

COROLLARY 2. *Any interval (a_j, a_{j+1}), $1 \leq j \leq p - 1$, which contains $(n - 1)$ zeros of $S(x)$, contains at most one zero of the corresponding $V(x)$, if $V(x) \in P$.*

COROLLARY 3. *The intervals (a_1, a_2) and (a_{p-1}, a_p) contain each at most one zero of $V(x)$, if $V(x) \in P$.*

We take up now the class Q of Van Vleck polynomials $V(x)$ which have some of the zeros at a_j, $2 \leq j \leq p - 1$. In view of Lemma 1, the corresponding $S'(x)$ have also zeros at these a_j. We intend to show that all the results following Lemma 2 are still valid except that open intervals (a_j, a_{j+1}) are to be replaced by closed intervals $[a_j, a_{j+1}]$.

For convenience, let us suppose that $V(x)$ has a zero at a_k and that the remaining $(p - 3)$ zeros of $V(x)$ do not coincide with any a_j. By Lemma 1, the corresponding $S'(x)$ has a zero at a_k. Then, let

$$S'(x) = n(x - a_k) \sum_{j=1}^{n-2} (x - x_j') \quad \text{and} \quad V(x) = A(x - a_k) \prod_{j=1}^{p-3} (x - t_j).$$

For a zero x_i of $S(x)$, we have then, from equation (1.1)

$$S''(x_i)/S'(x_i) + \sum_{j=1}^{p} \alpha_j/(x_i - a_j) = 0 \qquad (i = 1, 2, \cdots, n)$$

or

$$\sum_{j=1}^{n-2} 1/(x_i - x_j') + 1/(x_i - a_k) + \sum_{j=1}^{p} \alpha_j/(x_i - a_j) = 0 \qquad (i = 1, 2, \cdots, n). \tag{3.2}$$

We have used the fact that no $x_i = a_j$ $(j = 1, 2, \cdots, p)$ which is well known. Also, for a zero $t_i \neq a_k$ of $V(x)$, equation (1.1) gives,

$$S''(t_i) + S'(t_i) \sum_{j=1}^{p} \alpha_j/(t_i - a_j) = 0 \qquad (i = 1, 2, \cdots, p - 3).$$

In view of Lemma 1, $S'(t_i) \neq 0$, thus

$$S''(t_i)/S'(t_i) + \sum_{j=1}^{p} \alpha_j/(t_i - a_j) = 0$$

or

$$(3.3)\quad \sum_{j=1}^{n-2} 1/(t_i - x_j') + 1/(t_i - a_k) + \sum_{j=1}^{p} \alpha_j/(t_i - a_j) = 0 \quad (i = 1, 2, \cdots, p-3).$$

Equations (3.2) and (3.3) show that the zeros of $S(x)$ and those of the corresponding $V(x)$, apart from a_k, are the zeros of the function

$$(3.4)\quad G(x) = \sum_{j=1}^{n-2} 1/(x - x_j') + 1/(x - a_k) + \sum_{j=1}^{p} \alpha_j/(x - a_j).$$

It is clear from equation (3.4) and Lemma 2 that we can get the zeros of $S(x)$ and those of $V(x)$, apart from a_k (which is a zero of $V(x)$) directly from $F(x)$ by replacing the zero of $S'(x)$ which coincides with a_k by a_k. Also $G(x)$ has $(n + p - 3)$ zeros. Among these are n zeros of $S(x)$ and $(p - 3)$ zeros of $V(x)$. We may then state the following lemma.

LEMMA 2′. *The zeros of a $V(x)$, which has one zero at a_k, $2 \leq k \leq p - 1$, and the remaining zeros not coinciding with any a_j, and the zeros of the corresponding $S(x)$ are the zeros of the function $(x - a_k)G(x)$, where $G(x)$ is given by equation* (3.4).

In view of Lemma 2′, the modification in the proofs of earlier results in case $V(x) \in Q$ is obvious. In those results the open intervals (a_j, a_{j+1}) are to be replaced by the closed intervals $[a_j, a_{j+1}]$.

4. Bounds for the zeros of S(x)

The following theorem of Laguerre [3, p. 59] will be used to obtain some bounds for the zeros of $S(x)$.

THEOREM (Laguerre). *Let $f(x)$ be a polynomial of degree n and x_0 one of its simple zeros. Then any circle through the points x_0 and $x_0' = x_0 - 2(n-1)f'(x_0)/f''(x_0)$ separates the zeros of $f(x)$ unless all the zeros lie on the circumference of this circle. The same is true if a straight line replaces this circle.*

The following result gives the bounds for the zeros of $S(x)$.

THEOREM V. *If x_1 and x_n are the smallest and the largest zeros of any Stieltjes polynomial $S(x)$ of degree n, then*

$$(4.1)\quad \begin{aligned} &\text{(i)}\quad \sum_{j=1}^{p} \alpha_j/(x_n - a_j) < 2(n-1)/(a_1 - x_n) \\ &\text{(ii)}\quad \sum_{j=1}^{p} \alpha_j/(x_1 - a_j) > 2(n-1)/(a_p - x_1). \end{aligned}$$

Proof. We prove only (i). (ii) can be proved similarly. For $x = x_n$, equation (1.1) gives

$$(4.2)\quad S''(x_n) + \left(\sum_{j=1}^{p} \alpha_j/(x_n - a_j)\right)S'(x_n) = 0$$

or

$$2 \sum_{j=1}^{n-1} 1/(x_n - x_j) + \sum_{j=1}^{p} \alpha_j/(x_n - a_j) = 0,$$

where $S(x) = \prod_{j=1}^{n} (x - x_n)$. Thus

$$\sum_{j=1}^{p} \alpha_j/(x_n - a_j) = -2 \sum_{j=1}^{n-1} 1/(x_n - x_j) < 0. \tag{4.3}$$

Also, $S''(x_n) \neq 0$, for otherwise, equation (4.2) would give $S'(x_n) = 0$. Therefore,

$$S'(x_n)/S''(x_n) = -\left(\sum_{j=1}^{p} \alpha_j/(x_n - a_j)\right)^{-1}.$$

So

$$x_n' = x_n - 2(n-1)S'(x_n)/S''(x_n) = x_n + 2(n-1)\left(\sum_{j=1}^{p} \alpha_j/(x_n - a_j)\right)^{-1}.$$

We assert that

$$x_1 < x_n + 2(n-1)\left(\sum_{j=1}^{p} \alpha_j/(x_n - a_j)\right)^{-1},$$

for otherwise, since in view of inequality (4.3), $x_n' < x_n$, we could draw a circle through x_n and x_n' which would include all the zeros of $S(x)$ in its interior, a contradiction to the above theorem of Laguerre.

Thus

$$a_1 < x_1 < x_n + 2(n-1)\left(\sum_{j=1}^{p} \alpha_j/(x_n - a_j)\right)^{-1}$$

or

$$\sum_{j=1}^{p} \alpha_j/(x_n - a_j) < 2(n-1)/(a_1 - x_n).$$

It may be remarked that some classical orthogonal polynomials, e.g., Legendre, Jacobi, and Tchebychif polynomials are special cases of Stieltjes polynomials up to a constant factor. The bounds for their zeros given in [6, p. 118] can be obtained directly from inequalities (4.1).

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University of Wisconsin-Milwaukee.
Milwaukee, Wisconsin