On the zeta-function regularization of a two-dimensional series of Epstein–Hurwitz type

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As a further step in the general program of zeta-function regularization of multiseries expressions, some original formulas are provided for the analytic continuation, to any value of s, of two-dimensional series of Epstein–Hurwitz type, namely, $\sum_{n_1,n_2=0}^{\infty}[a_1(n_1+c_1)^2+a_2(n_2+c_2)^2]^{-s}$, where the a_j are positive reals and the c_j are not simultaneously nonpositive integers. They come out from a generalization to Hurwitz functions of the zeta-function regularization theorem of the author and Romeo [Phys. Rev. D **40**, 436 (1989)] for ordinary zeta functions. For $s=-k,0,2$, with $k=1,2,3,\ldots$, the final results are, in fact, expressed in terms of Hurwitz zeta functions only. For general s they also involve Bessel functions. A partial numerical investigation of the different terms of the exact, algebraic equations is also carried out. As a by-product, the series $\sum_{n=0}^{\infty}\exp[-a(n+c)^2]$, $a,c>0$, is conveniently calculated in terms of them.

I. INTRODUCTION

For a few years now, the study of quantum field theories in partially compactified space-time manifolds has acquired increasing importance in several domains of quantum physics. Let me just mention the issues of dimensional reduction and spontaneous compactification, and the multiple questions associated with the study of quantum field theories in the presence of boundaries (like the Casimir effect) and on curved space-time (manifolds with curvature and nontrivial topology), a step towards quantum gravity.

There are many interesting calculations in these theories that can be carried out exactly—and in a very elegant way from the mathematical point of view—by the zeta-function regularization method. In particular, if all the eigenvalues of the Hamiltonian are known, then, very commonly, one is led in this method to the computation of expressions of the general form

$$\sum_{n_1,\ldots,n_N=0}^{\infty}\left[\sum_{j=1}^{N}a_j(n_j+c_j)^{\alpha_j}\right]^{-s},\quad a_j,\alpha_j>0. \tag{1}$$

As such a multiseries, this expression only makes sense for $\mathrm{Re}(s)$ big enough, and an analytic (usually meromorphic) continuation to other values of s is in order. In the zeta-function method, this is provided by the Riemann and Hurwitz (also called Riemann generalized) zeta functions.

However, for an expression as general as (1) this program has proved to be extremely difficult (not to say impossible until now) to carry out. The simplest case is obtained when (1) corresponds to the Hamiltonian zeta function

$$\zeta(s)\equiv\sum_i E_i^{-s} \tag{2}$$

(E_i are the eigenvalues of H) of a system of N noninteracting harmonic oscillators. In this case, $a_j=1$, $j=1,2,\ldots,N$, and the a_j are the eigenfrequencies ω_j.[1] Another important case shows up in the partial toroidal compactification (space-time $T^p\times\mathbb{R}^{q+1}$). Then $\alpha_j=2$ and, usually, $c_j=0,\ \pm\frac{1}{2}$ (Ref. 2). This leads typically to Epstein zeta functions

$$Z_N(s)=\sum_{n_1,\ldots,n_N=-\infty}^{\infty}(n_1^2+\cdots+n_N^2)^{-s},$$
$$Y_N(s)=\sum_{n_1,\ldots,n_N=-\infty}^{\infty}{}' \times\left[\left(n_1+\frac{1}{2}\right)^2+\cdots+\left(n_N+\frac{1}{2}\right)^2\right]^{-s}, \tag{3}$$

(the prime prescribes omission of the term with $n_1=n_2=\cdots=n_N=0$). Other powers α_j appear when one deals with the spherical compactification (space-time $S^p\times\mathbb{R}^{q+1}$). Moreover, as string theory seems to indicate, nothing precludes the possibility of having to consider other compactification manifolds, leading to very general values for the α_j. In this work, however, we shall only deal with the particular case $\alpha_j=2$, $j=1,2,\ldots,N$, leaving more general situations for subsequent study.

The aim of the paper is to derive some new and useful expressions for the analytic continuation of two-dimensional sums of the types just mentioned. My results will come from a rigorous generalization of the zeta-function regularization theorem,[1,3,4] which is carried out in Sec. II, Eq. (7), by obtaining the appropriate counterterm (9). From it, basic expressions for zeta-functions regularization—Eqs. (22), (30), and (32) of Secs. III, IV, and V, respectively—will follow. They will give rise to the general equation (34) of Sec. V, which provides the analytical extension to any complex value of s of two-dimensional sums of the type mentioned in the Abstract, and also to the interesting particular formulas (35)–(38). Finally, in Sec. VI a recurrent procedure to extend these expressions to arbitrary-N multiseries as (1) will be sketched [Eq. (39)].

II. THE CASE $\alpha_j=2$: STATEMENT OF THE MATHEMATICAL PROBLEM

The apparently simple case $\alpha_j=2$ carries enough complication that it deserves a complete study on its own. On the

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other hand, at least formally, the general case is actually very similar to this one (the main difference being the transition from the cases $\alpha_j < 2$ to the cases $\alpha_j \geqslant 2$, as will be explained later. Thus I shall restrict myself to the expressions

$$E_N(s;a_1,...,a_N;c_1,...,c_N) \equiv \sum_{n_1,...,n_N=0}^{\infty} \left[\sum_{j=1}^{N} a_j(n_j+c_j)^2\right]^{-s}, \quad (4)$$

where it is understood that all $a_j > 0$ and that not all of the c_j are nonpositive integers. Actually, only the particular situation with $N=2$ will be worked out in detail. Let me emphasize the fact of the presence in (4) of general nonzero a_j's and c_j's. The only precedents in the literature (to my knowledge) of this kind of evaluations are restricted to very few special cases other than $a_1 = a_2 = \cdots = a_N$ and $c_1 = c_2 = \cdots = c_N = 1$.[2,5] Maybe the most famous expression in this context is the celebrated result of Hardy,[6] which can be obtained as a particular case of our final formulas in Ref. 4.

In Ref. 4, together with Romeo we began an investigation of the general expression (4), limiting ourselves to the simplest case $c_j = 1, j = 1,2,...,N$. It is not that immediate to extend the results there to the present situation, as we shall see.

A basic point in the zeta-function regularization procedure is the interchange of the order of the summations of infinite series in expressions like

$$S_c^{(\alpha)}(s) \equiv \sum_{m=0}^{\infty} (m+c)^{-s-1} \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} (m+c)^{\alpha a}. \quad (5)$$

In the case $c=1$ and $\alpha < 2$, the correct additional contribution coming from this commutation of sums was obtained by Weldon.[3] Actually, he claimed that his result was valid for any $\alpha \in \mathrm{N}$. This has turned out to be not right, as rigorously shown in Ref. 4 where the correct supplementary contributions for $\alpha \geqslant 2$ (always with $c=1$) have been obtained.

III. THE FUNDAMENTAL FORMULA FOR ZETA-FUNCTION REGULARIZATION WHEN $\alpha_j = 2$

I shall now proceed with the calculation of (5). It can be written as

$$S_c^{(\alpha)}(s) = \sum_{m=0}^{\infty} (m+c)^{-s-1} \oint_C \frac{da}{2\pi i} (m+c)^{-\alpha a}\Gamma(a), \quad (6)$$

where C is the contour ($C = L + K$) consisting of the straight line (L), $\mathrm{Re}(a) = a_0$, $0 < a_0 < 1$, and of a curved part (K), which is the semicircumference at infinity on the left of this line. For $\mathrm{Re}(s)$ big enough, we obtain

$$S_c^{(\alpha)}(s) = \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \zeta(s+1-\alpha a,c) + \begin{cases} \dfrac{1}{\alpha}\Gamma\left(-\dfrac{s}{\alpha}\right) + \Delta_c^{(\alpha)}(s), & \dfrac{s}{\alpha} \notin \mathrm{N}, \\ (-1)^{s/\alpha}\left[\dfrac{\gamma}{\Gamma(s/\alpha+1)} - \dfrac{1}{\gamma\Gamma'(s/\alpha+1)}\right] + \Delta_c^{(\alpha)}(s), & \dfrac{s}{\alpha} \in \mathrm{N}, \end{cases} \quad (7)$$

where $\zeta(z,c)$ is Hurwitz's (or Riemann's generalized) zeta function

$$\zeta(z,c) = \sum_{n=0}^{\infty} (n+c)^{-s} \quad (8)$$

and $\Delta_c^{(\alpha)}(s)$ is the following integral over the curved part K of the contour C:

$$\Delta_c^{(\alpha)}(s) \equiv \int_K \frac{da}{2\pi i} \zeta(s+1+\alpha a,c)\Gamma(a). \quad (9)$$

The preceding expressions, Eqs. (7) and (9), constitute the more basic result in this paper. They can be viewed as a generalization of the zeta-function regularization theorem obtained in Ref. 4. There the case of the ordinary Riemann zeta function (i.e., $c=1$) was studied and a detailed discussion on the nature of the term (9) for $c=1$ (including numerical computations for different values of s) was provided. It turns out that, for arbitrary positive c, the present term (9) can be related to the one in Ref. 4; in fact, it is numerically comprised between two expressions both obtained from the case $c=1$ by suppressing a finite number of contributions, namely, the first $[c-1]$ and $[c]$, respectively (here square brackets mean integer part). As the notation (i.e., the *delta*) already suggests, this term (9) always turns out to be a correction to the first, leading terms. It is also clear from the above discussion that the most interesting new case with respect to the one dealt with in Ref. 4 appears now when $0 < c < 1$, and this is precisely the specific situation that I will consider below.

In order to be able to provide an expression for the integral (9) in terms of more elementary functions, I shall restrict myself to the case $\alpha = 2$. Use will be made of the well-known Hurwitz formula,[7] valid (in particular) for $\mathrm{Re}\, z < 0$ and $0 < c \leqslant 1$,

$$\zeta(z,c) = 2(2\pi)^{z-1}\Gamma(z-1) \sum_{n=1}^{\infty} n^{z-1} \sin\left(2\pi nc + \frac{\pi z}{2}\right). \quad (10)$$

The behavior of the lhs for $|z| \to \infty$ with $\mathrm{Re}\, z < 0$ is

$$\zeta(z,c) \sim 2(2\pi)^{z-1}\Gamma(z-1)\sin(2\pi c + \pi z/2), \quad (11)$$

while, for $c=1$, we obtain

$$\zeta(z) = \zeta(z,1) \sim 2(2\pi)^{z-1}\Gamma(z-1)\sin(\pi z/2). \quad (12)$$

From the last two expressions, we get, for $0 < c \leqslant 1$,

$$\lim_{\substack{|z|\to\infty \\ \mathrm{Re}(z)<0}} \frac{\zeta(z,c)}{\zeta(z)} = \sin(2\pi c)\cot\left(\frac{\pi z}{2}\right) + \cos(2\pi c). \quad (13)$$

Now, making use of the identity, valid also for $\mathrm{Re}(z)<0$ (Ref. 7),

$$\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)=\int_0^\infty dt\, t^{-(z+1)/2}S(t), \tag{14}$$

with

$$S(t)\equiv\sum_{n=1}^{\infty}e^{-n^2t}, \tag{15}$$

we obtain for the analytic continuation of (9) (with $\alpha=2$) to $s=-1$:

$$\begin{aligned}\Delta_c^{(2)}(-1)&=\int_K\frac{da}{2\pi i}\zeta(2a,c)\Gamma(a)\\&=\int_K\frac{da}{2\pi i}[\sin(2\pi c)\cot(\pi a)\\&\quad+\cos(2\pi c)]\zeta(2a)\Gamma(a).\end{aligned} \tag{16}$$

After making use of Eq. (14), the second term on the rhs can be integrated immediately. Writing it now at the beginning of the second member, we get

$$\begin{aligned}\Delta_c^{(2)}(-1)=&-\sqrt{\pi}S(\pi^2)\cos(2\pi c)\\&+\sin(2\pi c)\int_K\frac{da}{2\pi i}\int_0^\infty dt\\&\times t^{-a-1/2}\cot(\pi a)S(\pi^2t).\end{aligned} \tag{17}$$

The last integral in (17) turns out to be zero. In fact,

$$\begin{aligned}&\int_K da\, it^{-a}\frac{e^{i\pi a}+e^{-i\pi a}}{e^{i\pi a}-e^{-i\pi a}}\\&=\int_{\pi/2}^{\pi}d\theta\, \mathrm{R}e^{i\theta}t^{-R(\cos\theta+i\sin\theta)}\\&\quad-\int_{\pi}^{3\pi/2}d\theta\, \mathrm{R}e^{i\theta}t^{-R(\cos\theta+i\sin\theta)}\\&=-i\frac{t^{iR}+t^{-iR}}{\mathrm{Ln}\, t}=-2i\frac{\cos(R\,\mathrm{Ln}\, t)}{\mathrm{Ln}\, t},\end{aligned} \tag{18}$$

from which it follows that

$$\begin{aligned}&\lim_{R\to\infty}\int_0^\infty dt\, S(\pi^2t)\,\frac{\cos(R\,\mathrm{Ln}\, t)}{\mathrm{Ln}\, t}\\&=\mathrm{Re}\left[\int_{-\infty}^{\infty}du\, S(\pi^2e^{u/R})\,\frac{e^{iu}}{u}\right]=0.\end{aligned} \tag{19}$$

We are left with just

$$\Delta_c^{(2)}(-1)=-\sqrt{\pi}\cos(2\pi c)S(\pi^2). \tag{20}$$

Summing up, I have proved that

$$S_c\equiv S_c^{(2)}(-1)=\sum_{m=0}^{\infty}e^{-(m+c)^2} \tag{21}$$

can be expressed in terms of Hurwitz zeta functions, as

$$\begin{aligned}S_c=&\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\zeta(-2m,c)\\&+\frac{\sqrt{\pi}}{2}+\sqrt{\pi}\cos(2\pi c)S(\pi^2),\end{aligned} \tag{22}$$

with (the standard, related to Jacobi's theta function) $S(t)$ being given by Eq. (15). Equation (22) is another meaningful result of this paper. It is *exact* and holds for *any* value of c.

IV. BEHAVIOR OF THE ZETA-REGULARIZED FUNDAMENTAL SERIES

Let me now investigate the behavior of the different series in (22). Depending on the value of c, the series of Hurwitz functions can be convergent (even finite) or asymptotic. The other two series, i.e., those implicit in the definition of the S functions, are quickly convergent (the one on the rhs much more quickly than the one on the lhs). In fact, to be clearer, let us check some specific cases.

(i) In the particular case $c=1$ we recover the known equality[4,7]

$$S(1)=(\sqrt{\pi}-1)/2+\sqrt{\pi}S(\pi^2). \tag{23}$$

(ii) For $c=\frac{1}{2}$ we have $\zeta(-2m,\frac{1}{2})=0$, $k=0,1,2,\ldots$, and

$$\begin{aligned}&\sum_{m=0}^{\infty}\exp\left[-\left(m+\frac{1}{2}\right)^2\right]\\&=\frac{\sqrt{\pi}}{2}-\sqrt{\pi}\sum_{m=1}^{\infty}\exp(-m^2\pi^2).\end{aligned} \tag{24}$$

The rhs of (24) permits us to obtain the value of the series on the lhs with 10^{-10} accuracy, *with just two terms*

$$\begin{aligned}&\sum_{m=0}^{\infty}\exp\left[-\left(m+\frac{1}{2}\right)^2\right]\\&=\frac{\sqrt{\pi}}{2}-\sqrt{\pi}e^{-\pi 2}+\mathcal{O}(10^{-10}).\end{aligned} \tag{25}$$

(iii) For $c=0$ we get an equality equivalent to (23),

$$\sum_{m=0}^{\infty}e^{-m^2}=\frac{1}{2}+\frac{\sqrt{\pi}}{2}+\sqrt{\pi}S(\pi^2). \tag{26}$$

Actually, it is an immediate consequence of the properties of the series in (22) that the equalities one obtains for $c+1$ and for $c-1$ are each equivalent to the corresponding one for c. Therefore, only the equalities (22) corresponding to c, $0<c\leqslant 1$, provide interesting (independent) relations.

(iv) For $c=\frac{1}{4}$, we get

$$\begin{aligned}&\sum_{m=0}^{\infty}\exp\left[-\left(m+\frac{1}{4}\right)^2\right]\\&=\frac{\sqrt{\pi}}{2}+\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\zeta\left(-2m,\frac{1}{4}\right).\end{aligned} \tag{27}$$

The series of Hurwitz functions on the rhs is now asymptotic. It stabilizes between the eighth and the twelfth summands and it provides a best value (with $\simeq 10^{-7}$ accuracy) exactly when we add its ten first terms.

(v) For $c=\frac{1}{3}$ and $c=\frac{1}{6}$ we obtain, respectively,

$$\begin{aligned}&\sum_{m=0}^{\infty}\exp\left[-\left(m+\frac{1}{3j}\right)^2\right]\\&=\frac{\sqrt{\pi}}{2}+\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\zeta\left(-2m,\frac{1}{3j}\right)\\&\quad+(-1)^j\frac{\sqrt{\pi}}{2}\sum_{m=1}^{\infty}\exp(-m^2\pi^2),\quad j=1,2.\end{aligned} \tag{28}$$

In these cases, contributions from the two series in the rhs must be taken into account. The first of them is asymptotic [as in (iv)] and has exactly *the same* characteristics as the one in (27), both for $j=1,2$. The second series is extremely rapidly convergent (much more than the series on the lhs). These characteristics are maintained over the full range $0<c\leqslant 1$ (but for the very special values $c=\frac{1}{2},1$ considered above).

One may ask what is gained with these asymptotic expressions. The answer has already been given before, in Eqs. (7) and (9), which extend these equalities by analytic continuation to *any* value of s, and not simply to the case $s=-1$ exemplified here. I will be more precise in what follows. Before being so, let me present two more examples of interesting, original relations that come from Eq. (22):

$$\sum_{m=-\infty}^{\infty} \exp[-(m+c)^2] = \sqrt{\pi} + 2\sqrt{\pi}\cos(2\pi c) S(\pi^2), \tag{29}$$

$$\begin{aligned}\sum_{m=0}^{\infty} m \exp[-(m+c)^2] &= \frac{1}{2} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \\ &\quad \times [\zeta(-2m-1,c) - c\zeta(-2m,c)] - \frac{\sqrt{\pi}}{2} c \\ &\quad + \sqrt{\pi}[\pi \sin(2\pi c) - c\cos(2\pi c)] S(\pi^2).\end{aligned} \tag{30}$$

V. THE GENERAL EXPRESSION FOR $N=2$

The calculation of the general expression (4) will be now illustrated, for the sake of clarity, in the simpler case $N=2$. By using the Mellin transform, we write

$$\begin{aligned}E_2(s;a_1,a_2;c_1,c_2) &= \sum_{n_1,n_2=0}^{\infty} [a_1(n_1+c_1)^2 + a_2(n_2+c_2)^2]^{-s} \\ &= \frac{1}{\Gamma(s)} \sum_{n_1,n_2=0}^{\infty} \int_0^\infty dt\, t^{s-1} \\ &\quad \times \exp\{-t[a_1(n_1+c_1)^2 + a_2(n_2+c_2)^2]\}.\end{aligned} \tag{31}$$

We shall need the following generalization of Eq. (22)—again a basic outcome of my regularization theorem (7)–(9) and obtained in the same way—

$$\begin{aligned}\sum_{m=0}^{\infty} \exp[-a(m+c)^2] &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^m \zeta(-2m,c) \\ &\quad + \frac{1}{2}\sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{a}} \cos(2\pi c) S\left(\frac{\pi^2}{a^2}\right).\end{aligned} \tag{32}$$

Substituting (32) into (31), we get

$$\begin{aligned}E_2(s;a_1,a_2;c_1,c_2) &= \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_1^m \zeta(-2m,c_1) \sum_{n_2=0}^{\infty} \int_0^\infty dt\, t^{s+m-1} \exp[-ta_2(n_2+c_2)^2] \\ &\quad + \frac{1}{2}\sqrt{\frac{\pi}{a_1}} \frac{1}{\Gamma(s)} \sum_{n_2=0}^{\infty} \int_0^\infty dt\, t^{s-3/2} \exp[-ta_2(n_2+c_2)^2] \\ &\quad + \sqrt{\frac{\pi}{a_1}} \frac{\cos(2\pi c_1)}{\Gamma(s)} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \int_0^\infty dt\, t^{s-3/2} \exp\left[-\frac{\pi^2 n_1^2}{a_1 t} - ta_2(n_2+c_2)^2\right].\end{aligned} \tag{33}$$

This gives

$$\begin{aligned}E_2(s;a_1,a_2;c_1,c_2) &= \frac{a_2^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(s+m)}{m!} \left(\frac{a_1}{a_2}\right)^m \zeta(-2m,c_1)\zeta(2s+2m,c_2) + \frac{a_2^{-2}}{2}\left(\frac{\pi a_1}{a_2}\right)^{1/2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta(2s-1,c_2) \\ &\quad + \frac{2\pi^s}{\Gamma(s)} \cos(2\pi c_1) a_1^{-s/2-1/4} a_2^{-s/4+1/4} \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} n_1^{s-1/2} (n_2+c_2)^{-s+1/2} K_{s-1/2}\left[2\pi\sqrt{\frac{a_2}{a_1}}\, n_1(n_2+c_2)\right],\end{aligned} \tag{34}$$

where K_ν is the modified Bessel function of the second kind. Equation (34) constitutes the general analytic continuation formula for two-dimensional series I was looking for. As is apparent, it involves Bessel functions as well as Hurwitz functions. However, the following particular cases look especially simple.

For $s=-k$, $k=0,1,2,\ldots$, one obtains

$$\begin{aligned}E_2(-k;a_1,a_2;c_1,c_2) &= \frac{a_2^k}{\Gamma(-k)} \sum_{m=0}^{k} \frac{(-1)^m \Gamma(m-k)}{m!} \left(\frac{a_1}{a_2}\right)^m \zeta(-2m,c_1)\zeta(2(m-k),c_2) \\ &= a_2^k\left(\frac{1}{2} - c_1\right)\zeta(-2k,c_2) + a_2^k \sum_{m=1}^{\infty} \frac{k(k-1)\cdots(k-m+1)}{m!} \left(\frac{a_1}{a_2}\right)^m \zeta(-2m,c_1)\zeta(2(m-k),c_2).\end{aligned} \tag{35}$$

In particular, for $s=0$,

$$E_2(0;a_1,a_2;c_1,c_2) = (c_1-\tfrac{1}{2})(c_2-\tfrac{1}{2}), \tag{36}$$

and, for $s = -1$,

$$E_2(-1;a_1,a_2;c_1,c_2) = a_2(\tfrac{1}{2}-c_1)\zeta(-2,c_2) + a_1(\tfrac{1}{2}-c_2)\zeta(-2,c_1) = \tfrac{1}{3}(c_1-\tfrac{1}{2})(c_2-\tfrac{1}{2})[a_1c_1(1-c_1)+a_2c_2(1-c_2)]. \quad (37)$$

For $s = 2$, we obtain

$$E_2(2;a_1,a_2;c_1,c_2) = \frac{1}{a_2^2}\sum_{m=0}^{\infty}(-1)^m(m+1)\left(\frac{a_1}{a_2}\right)^m\zeta(-2m,c_1)\zeta(2m+4,c_2) + \frac{\pi}{4a_2}\frac{1}{\sqrt{a_1a_2}}\zeta(3,c_2) + \frac{\pi^2\cos(2\pi c_1)}{a_1a_2}\sum_{m=0}^{\infty}\left\{(n+c_2)^{-2}\left[\exp\left(2\pi\sqrt{\frac{a_2}{a_1}}(n+c_2)\right)-1\right]^{-2} + \left[(n+c_2)^{-2}+\sqrt{\frac{a_1}{a_2}}\frac{(n+c_2)^{-3}}{2\pi}\right]\left[\exp\left(2\pi\sqrt{\frac{a_2}{a_1}}(n+c_2)\right)-1\right]^{-1}\right\}. \quad (38)$$

The first two terms on the rhs yield properly the result of the zeta-function regularization method (naive commutation of the series summations plus Weldon's additional contribution[3]). They produce the desired expression of (4) in terms of zeta functions. The last term in (38) generalizes to arbitrary $c_1,c_2>0$ the supplementary corrections detected in Ref. 4 for $c_1=c_2=1$ and which had been loosely forgotten in Ref. 3. In spite of the imposing aspect of this last term, its contribution is actually very small, and the series in n is very quickly convergent (only the first couple of summands need to be taken into account in practice). For an arbitrary value of s, one must use the general expression (34).

VI. A GENERAL EXPRESSION FOR ARBITRARY *N*

The preceding calculations can be generalized to multiple sums (4) with arbitary N. The fundamental formula (32) introduced into the Mellin transform [as in (31)–(33)] allows us to proceed recurrently. One obtains the (exact) equation

$$E_N(s;a_1,\ldots,a_N;c_1,\ldots,c_N) = \frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a_1^m\zeta(-2m,c_1)\Gamma(s+m)E_{N-1}(s+m;a_2,\ldots,a_N;c_2,\ldots,c_N) + \frac{1}{2}\sqrt{\frac{\pi}{a_1}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}E_{N-1}\left(s-\frac{1}{2};a_2,\ldots,a_N;c_2,\ldots,c_N\right) + \sqrt{\frac{\pi}{a_1}}\frac{\cos(2\pi c_1)}{\Gamma(s)}\sum_{n_1=1}^{\infty}\sum_{n_2,\ldots,n_N=0}^{\infty}\int_0^{\infty}dt\, t^{s-3/2}\exp\left[-\frac{\pi^2n_1^2}{a_1t}-t\sum_{j=2}^{N}a_j(n_j+c_j)^2\right]. \quad (39)$$

Notice, once more, that the last term is a small correction to the first two, so that, in practice Eq. (39) can be viewed as a recursive formula with a small correction term Δ (the last one) that can be estimated numerically. This is also discussed in Ref. 4 (for the particular case $c_1=\cdots=c_N=1$) in greater detail.

The application of the formulas derived in this paper to the direct evaluation (exact, or at worst, six to seven decimal places precise) of the Casimir effect, by just summing over modes (provided that they are known exactly) and by zeta-regularizing the resulting expressions, will be developed in a separate publication.

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