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On the b-Boundary of the Closed Friedman-Model

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Abstract. Some points of the past Big Bang in the closed fourdimensional Friedman-model are found to be identical with points of the future collapse according to the bundle-boundary definition.

1. Introduction

Consider the closed Friedman-model (M, g) with metric

$$ds_g^2 = R^2(\psi)\{d\psi^2 - d\sigma^2 - \sin^2\sigma(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\}$$
$$\text{with } R(\psi) = 1 - \cos\psi\,,$$

with singularities at $\psi = 0$ and $\psi = 2\pi$. We shall investigate the structure of the b-boundary for this space-time by working with, rather than the ten-dimensional orthonormal bundle $O(M)$ (see [1, 2]), a certain three-dimensional subbundle. The construction is as follows. Consider the timelike and totally geodesic two-dimensional submanifolds $N c M$ with induced metric γ, given by

$$\vartheta = \text{const} \quad \text{and} \quad \varphi = \text{const}\,.$$

Moreover, there exists an orthonormal dyad field

$$W_\alpha, \quad \alpha = 2, 3$$

which is parallel along and orthogonal to N. Therefore we can construct a three-dimensional submanifold $\tilde{N} c O(M)$, consisting of every orthonormal tetrad Y_i, $i = 0, \ldots 3$ with

$$Y_A \in T(N) \qquad A = 0, 1$$
$$Y_\alpha = W_\alpha \qquad \alpha = 2, 3$$

at every point of N. $\tilde{N}$ is isomorphic to $O(N)$. Furthermore the induced metric in $\tilde{N}$ is equal to the bundle metric $\tilde{\gamma}$ in $O(N)$, because any curve in N, which is horizontal with respect to γ is horizontal with respect to g as well. The metric $\tilde{\gamma}$ can be easily computed. This reduction method can be applied also to other space-times, e.g. the Schwarzschild and Reissner-Nordström space-times. If we now find curves, which connect two points in the fibres of the two singularities with arbitrarily small length in $\overline{O(N)'}$ [1], the Cauchy completion of $O(N)'$ [1] [1],

[1] The prime denotes the connected component, i.e. here the manifolds consisting of every positively oriented orthonormal dyad resp. tetrad in every point of N res. M.

we have the two projected points identified. The construction of these curves is based on the two following facts:

1) For the two-dimensional submanifolds N with induced metric γ any fibre of the orthonormal bundle at $\psi=0$ and $\psi=2\pi$ is degenerated to a point, i.e. all positively oriented orthonormal dyads at a point of such a singularity are identified. This surprising and interesting fact is crucial for the identification.

2) The bundle length of a horizontal lift of a curve $C \in N$ is the "Euclidean length", measured with the aid of the components $\vartheta^i(\dot{C})$ of the tangent vector $\dot{C}$ with respect to the choosen parallely propagated dyad [1]:

$$L=\int d\lambda \sqrt{\sum_i \vartheta^i(\dot{C}(\lambda))\vartheta^i(\dot{C}(\lambda))}$$

L depends on the dyad chosen and is called generalised affine length of C. It follows clearly, that the generalised affine length of a null geodesic can be made arbitrarily small by chosing appropriately the dyad.

Now, our curve connects an orthonormal dyad X_{Ap} at a point p of the first with an orthonormal dyad X_{Aq} at a point q of the second singularity. We first boost the dyad X_{Ap}, so that its vectors approach a null direction. Then we parallely propagate this dyad along the null geodesic defined by this null direction and obtain some dyad at q. We then boost it to get the dyad X_{Aq}. The bundle length of this curve can be made as small as we want. Hence, the points p and q are identified.

2. The Submanifolds $N \subset M$

The timelike two-dimensional submanifolds $N \subset (M, g)$, defined by $\vartheta=$ const and $\varphi=$ const have an induced metric

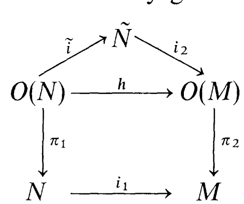
$$ds_\gamma^2=R^2(\psi)(d\psi^2-d\sigma^2)$$

with $0 \leqq \sigma < 2\pi$, $\psi \neq 2n\pi$ for $n \in \mathbb{N}$, $N=\mathbb{R}^1 \times S^1$.

The orthonormal dyad field W_α, $\alpha=2, 3$

$$W_2=(R \sin\sigma)^{-1}\, \partial/\partial\vartheta, \qquad W_3=(R \sin\vartheta \sin\sigma)^{-1}\, \partial/\partial\varphi$$

is parallel along and orthogonal to N. The existence of such vectorfields implies that N is totally geodesic. We get the following maps and bundles:



i_1 is the embedding of N in M. $\tilde{N}$ is the submanifold of $O(M)$ consisting of every orthonormal tetrad Y_i, $i=0,\ldots,3$ with

$$\begin{aligned} Y_A &= i_{1*}X_A \qquad A=0, 1 \\ Y_\alpha &= W_\alpha \qquad \alpha=2, 3 \end{aligned} \tag{1}$$

at every point of $i_1 N$, where X_A is an arbitrary orthonormal dyad at the corresponding point in N.

i_2 is the embedding of N in $O(M)$,

π_1, π_2 are the bundle projections.

$\tilde{i}$ maps an orthonormal dyad X_A at the point $p \in N$ (i.e. $\tilde{p} \in O(N)$) to the tetrad Y_i at the point $i_1 p$ with (1), $\tilde{i}O(N) = \tilde{N}$.

Lemma 1. *If $\tilde{p}(s)$ is a horizontal curve in $O(N)$, then $\tilde{x}(s) = h\tilde{p}(s)$ is also horizontal in $O(M)$.*

Lemma 2. $\pi_2 \circ h = i_1 \circ \pi_1$.

3. The Metric in $O(N)'$

One can easily calculate the metric $\tilde{\gamma}$ in $O(N)'$. If χ is a canonical parameter of the one-parameter subgroup L of the Lorentzgroup Λ which acts as structure group in $O(N)'$ and if the section $\chi = 0$ is chosen to consist of the dyads $(R^{-1}\partial/\partial\psi, R^{-1}\partial/\partial\vartheta)$ one gets

$$ds^2_{\tilde{\gamma}} = \frac{e^{2\chi}}{2} R^2(\psi)(d\psi - d\sigma)^2 + \frac{e^{-2\chi}}{2} R^2(d\psi - d\sigma)^2 + \left(\frac{\dot{R}(\psi)}{R(\psi)} d\sigma + d_\chi\right)^2$$

Proposition. $\tilde{\gamma} = h^* \tilde{g}$ *or for* $U, V \in T(O(N)')$

$$\tilde{\gamma}(U, V) = \tilde{g}(h_* U, h_* V).$$

Proof. The standard horizontal and vertical vector fields

$C_A, {}^2\overset{*}{E}{}^0_1 \in T(O(N)')$ $\quad A = 0, 1$ resp.

$B_i, {}^4\overset{*}{E}{}^i_k \in T(O(M)')$ $\quad i, k = 0, \ldots, 3$

are orthonormal with respect to $\tilde{\gamma}$ resp. $\tilde{g}$. But the horizontal subspace $H_{\tilde{p}}(N) \subset T(O(N)')$ at the point $\tilde{p} \in O(N)'$ is maped into the horizontal subspace $H_{h\tilde{p}}(M) \subset T(O(M)')$ by h_* (Lemma 1). Furthermore by Lemma 2

$$\pi_{2*} \circ h_* C_{A\tilde{p}} = i_{1*} \circ \pi_{1*} C_{A\tilde{p}} = i_{1*} X_{Ap} = Y_{Ai,p} = \pi_{2*} B_{Ah\tilde{p}}.$$

Therefore

$$B_A = h_* C_A \quad A = 0, 1.$$

For the vertical vector fields let E^0_1 be the element of the Liealgebra of the Lorentzgroup Λ, which generates the one parameter structure group $L(\chi)$ of the bundle $O(N)'$. If $R_{L(\chi)}$ resp. $R'_{L(\chi)}$ denote the action of $L(\chi)$ at the points $\tilde{p} \in O(N)'$ resp. $\tilde{\chi} \in O(M)'$

$${}^2\overset{*}{E}{}^0_{1\tilde{p}} = d/d\chi (R_{L(\chi)} \tilde{p})_{\chi=0},$$

$${}^4\overset{*}{E}{}^0_{1h\tilde{p}} = d/d\chi (R'_{L(\chi)} h\tilde{p})_{\chi=0}.$$

But $R_{L(\chi)}$ transforms the dyad X_A in the same way as the two vectors $Y_A = i_{1*} X_A$ of the tetrad Y_i are transformed by $R'_{L(\chi)}$. Therefore from the definition of h we have

$${}^4\overset{*}{E}{}^0_1 = h_* {}^2\overset{*}{E}{}^0_1$$

which completes the proof.

Corollary. *Let* $\tilde{p}_1, \tilde{p}_2 \in O(N)'$. *Then* $d_{\tilde{\gamma}}(\tilde{p}_1, \tilde{p}_2) \geqq d_{\tilde{g}}(h\tilde{p}_1, h\tilde{p}_2)$ *if* $d_{\tilde{\gamma}}$ *is the distance function in* $O(N)'$ *as given by* $\tilde{\gamma}$ *and* $d_{\tilde{g}}$, *similarly for* $O(M)'$.

In the following chapter we consider the sequences $\{\tilde{p}_{1n}\}: \{(\psi_n, \sigma_0, 0)\}$ and

$$\{\tilde{p}_{4n}\}: \{(2\pi - \psi_n, \sigma_0 - 2\chi_n R(\psi_n)/\dot{R}(\psi_n) + 2\pi - 2\psi_n, 0)\}$$

with $\lim_{n\to\infty} \psi_n = 0$. $\{\tilde{p}_{1n}\}$ and $\{\tilde{p}_{4n}\}$ (we anticipate here a result of Chapter 4) are Cauchy sequences without limit in $O(N)'$ and determine therefore points $\tilde{p}_1$ and $\tilde{p}_4$ of the boundary $\dot{O}(N)'$.

4. The Identification Curves

We construct a curve λ_n, consisting of three horizontal parts:

Part 1 connects the points $\tilde{p}_{1n}: (\psi_n, \sigma_0, 0)$ and $\tilde{p}_{2n}: (\psi_n, \sigma_0 - \chi_n R(\psi_n)/\dot{R}(\psi_n), \chi_n)$ and is represented by the two functions

$$\psi = \psi_n = \text{const},$$

$$\sigma(\chi) = \sigma_0 - \chi_n R(\psi_n)/R(\psi_n),$$

with $\sigma_0 = \text{const}$, $\chi_n = -\alpha \ln R(\psi_n)$ $\alpha > 1$.

The length of this part is

$$L_{1n} = \left| \frac{R^2(\psi_n)}{\dot{R}(\psi_n)} \int_0^{\chi_n} d\chi \sqrt{\operatorname{ch} 2\chi} \right| < \frac{\sqrt{2}}{2} |R(\psi_n)^{2-\alpha} + R(\psi_n)^{2+\alpha})/\dot{R}(\psi_n)| .$$

Now, if $R(\psi) = R(2\pi - \psi) \sim \psi^\beta$ for $\psi \to 0$ then

$$L_{1n} \sim \psi_n^{1+\beta-\alpha\beta} + \psi_n^{1+\beta+\alpha\beta},$$

and for arbitrary $\beta > 0$ there exists α with $1 < \alpha < (\beta+1)/\beta$. Therefore

$$\lim_{n\to\infty} L_{1n} = 0 \quad \text{for} \quad \psi_n \to 0 .$$

With this part one can show the interesting fact, that the fibre of $\dot{O}(N)'$ through the boundary point $\tilde{p}_1 \in \dot{O}(N)'$ is degenerated, i.e. that any two points in this fibre are identical. We shall give only the idea for the proof:

Consider the sequences $\{\tilde{p}_{1n}\}$ and $\{v_n\}: \{(\psi_n, \sigma_0 + \delta_n, 0)\}$. Both of them determine the same boundary point $\tilde{p}_1$. To see that we construct a curve which connects $\tilde{p}_{1n}$ and v_n and consists of three parts C_{1n}, C_{2n}, C_{3n}.

C_{1n} connects $\tilde{p}_{1n}$ with $\tilde{q}_{1n}: (\delta_n^\gamma, \sigma_0, 0)$ and is given by $d\sigma = d\chi = 0$.

C_{2n} connects $\tilde{q}_{1n}$ with $\tilde{q}'_{1n}: (\delta_n^\gamma, \sigma_0 + \delta_n, 0)$ and is given by $d\psi = d\chi = 0$.

C_{3n} connects $\tilde{q}'_{1n}$ with v_n and is given by $d\sigma = d\chi = 0$. Let $0 < \gamma < \frac{1}{2}$.

The length of this curve fulfills $\lim_{n\to\infty} L(C_n) = 0$ if $\delta_n \to 0$. Now we construct a sequence $\{u_n\}: \{\psi_n, \sigma_0 + \chi' R(\psi_n)/\dot{R}(\psi_n), 0\}$. $\{u_n\}$ is a Cauchy sequence which determines the boundary point $\tilde{p}_1$. The curve K_n, given by the two functions

$$\psi = \psi_n = \text{const},$$

$$\sigma(\chi) = \sigma_0 + \chi' R(\psi_n)/\dot{R}(\psi_n) - \chi R(\psi_n)/\dot{R}(\psi_n)$$

connects u_n with $Ra(\chi)' \tilde{p}_{1n}: (\psi_n, \sigma_0, \chi')$. But we have shown that for arbitrary χ', $\lim_{n\to\infty} L(K_n) = 0$, which completes the proof.

Part 2 is a horizontal lift of a null geodesic and connects $\tilde{p}_{2n}$ with the point

$$\tilde{p}_{3n}: (2\pi - \psi_n, \sigma_0 - \chi_n R(\psi_n)/\dot{R}(\psi_n) + 2\pi - 2\psi_n, \chi_n)$$

and is represented by the two functions

$$\sigma(\psi) = \sigma_0 - \chi_n R(\psi_n)/\dot{R}(\psi_n) + \psi - \psi_n$$
$$\chi(\psi) = \chi_n - \ln(R(\psi_n)/R(\psi))$$

The length of this part is

$$L_{2n} = \left| \sqrt{2} \frac{e^{-\chi_n}}{R(\psi_n)} \int_{\psi_n}^{2\pi - \psi_n} d\psi R^2(\psi) \right| < 3\sqrt{2}\pi R(\psi_n)^{\alpha - 1}$$

since $\int_0^{2\pi} R^2(\psi) d\psi < 3\pi$.

Part 3 connects the points $\tilde{p}_{3n}$ and

$$\tilde{p}_{4n}: (2\pi - \psi_n, \sigma_0 - 2\chi_n R(\psi_n)/\dot{R}(\psi_n) + 2\pi - 2\psi_n, 0)$$

and is represented by the two functions

$$\psi = 2\pi - \psi_n = \text{const},$$
$$\sigma(\chi) = \sigma_0 - 2\chi_n R(\psi_n)/\dot{R}(\psi_n) + 2\pi - 2\psi_n + \chi R(\psi_n)/\dot{R}(\psi_n)$$

The length of this part is also L_{1n}. Hence the length of the total curve fulfills

$$\lim_{n \to \infty} L_n = 0.$$

Therefore, for $\varepsilon > 0$ there exist N with

$$L(\lambda_n) < \varepsilon/2 \quad \text{and} \quad R(\psi_n)\psi_n < \varepsilon/2 \quad \text{for} \quad n > N.$$

This means:

1. The sequences $\{\tilde{p}_{1n}\}$, $\{\tilde{p}_{4n}\}$ have null distance, i.e. for $\varepsilon > 0$ there exist N with

$$d_{\tilde{\gamma}}(\tilde{p}_{1n}, \tilde{p}_{4m}) < L(\lambda_m) + |R(\psi_n)\psi_n - R(\psi_m)\psi_m| < \varepsilon, \qquad n, m > N,$$

therefore

$$d_{\tilde{g}}(h\tilde{p}_{1n}, h\tilde{p}_{4m}) < \varepsilon.$$

2. The sequence $\{\tilde{p}_{1n}\}$ is a Cauchy sequence without limit in $O(N)'$. Therefore, and by 1. the sequence $\{\tilde{p}_{4n}\}$ is also a Cauchy sequence, also without limit in $O(N)'$. Hence

$$\{h\tilde{p}_{1n}\} \quad \text{and} \quad \{h\tilde{p}_{4n}\}$$

are Cauchy sequences in $O(M)'$.

But the coordinates of the projections p_{1n}, p_{4n} converge to

$$p_{1n} \xrightarrow{\text{coord.}} (0, \sigma_0),$$
$$p_{4n} \xrightarrow{\text{coord.}} (2\pi, \sigma_0 + 2\pi) \text{ wich is equivalent to } (2\pi, \sigma_0).$$

Hence, $\{p_{1n}\}$ and $\{p_{4n}\}$, also $\{i_1 p_{1n}\}$ and $\{i_1 p_{4n}\}$ approach the two "different" singularities at $\psi=0$ resp. $\psi=2\pi$. But they are identified according to Schmidt's definition (b-boundary) of a singularity.

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