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On the cd-Variation Polynomials of André and Simsun Permutations*

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Abstract. We prove a conjecture of Stanley on the cd-index of the semisuspension of the face poset of a simplicial shelling component. We give a new signed generalization of André permutations, together with a new notion of cd-variation for signed permutations. This generalization not only allows us to compute the cd-index of the face poset of a cube, but also occurs as a natural set of orbit representatives for a signed generalization of the Foata–Strehl commutative group action on the symmetric group. From the induction techniques used, it becomes clear that there is more than one way to define classes of permutations and cd-variation such that they allow us to compute the cd-index of the same poset.

Introduction

In this paper we investigate the relationship between the cd-variation monomials of certain classes of André and simsun permutations, and the cd-indices of various Eulerian posets. Our most important result is the proof of a conjecture of Stanley, stated in [11] as Conjecture 3.1, about the cd-index of the semisuspended shelling components of a simplicial complex. This conjecture refines Purtill's theorem on the cd-index of the face poset of an $(n-1)$-dimensional simplex [10, Theorem 6.1]. We also give a new signed generalization of André permutations which allows us to compute the cd-index of the face poset of the n-cube, and to prove a signed analogue of Foata and Strehl's result in [8] about the augmented André permutations being a natural set of orbit representatives for a commutative group action on the symmetric group.

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Most of our proofs have the following simple structure. We have a set $\{P_i\colon i \in I\}$ of Eulerian posets, where I is some well-ordered index set, and a set $\{\mathcal{C}_i\colon i \in I\}$ of sets of (signed or unsigned) permutations, with indices from the same set I. We want to show that (given a definition of cd-variation for signed permutations) the cd-index $\Phi_{P_i}(c, d)$ of P_i is equal to the cd-variation polynomial $V_{cd}(\mathcal{C}_i)$ of $\mathcal{C}_i$ for each i. We prove the equality by showing a recursion formula which is satisfied by both families of polynomials $\{\Phi_{P_i}(c, d)\colon i \in I\}$ and $\{V_{cd}(\mathcal{C}_i)\colon i \in I\}$, and then we check the equality of $\Phi_{P_i}(c, d)$ and $V_{cd}(\mathcal{C}_i)$ for the necessary initial values of i.

Using this approach, once we have a proof for one class of permutations, we need only perform a few modifications in order to obtain a similar result for a "slightly different" class. The fact that we are able to apply such induction proofs with a separate cd-index part and cd-variation part, indicates also that the currently discovered connections between cd-indices and cd-variations are not "canonical": there is no apparent reason to prefer one class of permutations together with one notion of cd-variation over another, if they allow us to compute the cd-index of the same poset with the same difficulty.

In Section 1 we introduce the notion of the cd-index of a poset, and we recall a very simple weight-calculating method, already used by Stanley in [11], which allows us to obtain the necessary recursion formulas for cd-indices. There is another well-known way to compute the cd-index of a poset, when it has an R-labeling, using a result of Björner and Stanley [3, Theorem 2.7]. Ehrenborg and Readdy apply this theorem in [4] to prove those recursion formulas which we also use in the case of Boolean algebra and the face lattice of the cube. It is still open whether an R-labeling argument could be given for Stanley's conjecture.

Section 2 is a short survey of André permutations of the first and second kind, and of simsun permutations. Keeping in mind the recursion formulas for cd-variations at which we are aiming, we focus on exposing the recursive structure of these permutations. While doing so, we point at a "hidden treasure" in Foata and Strehl's paper [8]. It is a proposition equivalent to saying that every André permutation of the second kind is a simsun permutation, almost two decades before simsun permutations were first defined. We also describe the Foata–Strehl group action on permutations introduced in [8].

In Section 3 we reprove Purtill's result using our elementary methods, and then we refine the argument to give a demonstration of Stanley's conjecture. We describe in detail a way to establish the necessary recursive formulas for the cd-indices and for the cd-variation polynomials of the respective augmented André permutations. As a remark, we indicate how to obtain the similar recursion formulas for the cd-variation polynomials of the corresponding subclasses of André permutations of the second kind and of augmented simsun permutations.

Finally, in Section 4 we give a new signed generalization of André permutations, which is different from the one introduced by Purtill in [10] and generalized by Ehrenborg and Readdy in [4]. In order still to have the cubical analogue of Purtill's theorem, we also modify the definition of the cd-variation for a signed permutation: it will be the cd-variation of the underlying signless permutation. Besides serving the same purpose as the signed André permutations of Purtill, our signed André permutations are also orbit representatives for a signed analogue of the Foata–Strehl group action defined on permutations. Thus we arrive at a setting in which André permutations were studied before their use in cd-index calculations. The natural question which arises is whether the Foata–Strehl group action also has an analogue for other reflection groups such that an appropriate set of orbit representatives could allow us to compute the cd-index of other symmetric posets.

1. Preliminaries on the cd-Index

Let P be a ranked poset of rank $n+1$ with minimum element $\hat{0}$, maximum element $\hat{1}$, and rank function ρ. Given a set $S \subseteq \{1, 2, \ldots, n\}$, we denote by P_S the S-rank selected subposet of P, i.e., the set $\{x \in P\colon\ \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. We define $\alpha(S)$ to be the number of maximal chains of P_S, and we call the vector $(\alpha(S), \ldots)$ the *flag f-vector* of P. We define the *beta-invariant* $\beta(S)$ by

$$\beta(S) \stackrel{\text{def}}{=} \sum_{T \subseteq S} (-1)^{|S \setminus T|} \cdot \alpha(T). \tag{1}$$

These equations are equivalent to setting $\beta(S) = (-1)^{|S|-1} \cdot \mu(P_S)$, where μ is the Möbius function. The vector $(\ldots, \beta(S), \ldots)$ is also called the *flag h-vector* of P.

The *ab-index* $\Psi_P(a, b)$ of the poset P is a polynomial in noncommuting variables a and b defined by the formula

$$\Psi_P(a, b) = \sum_{S \subseteq \{1, 2, \ldots, n\}} \beta(S) \cdot u_S, \tag{2}$$

where u_S is the monomial $u_1 \cdots u_n$ satisfying

$$u_i = \begin{cases} a & \text{if } i \notin S, \\ b & \text{if } i \in S. \end{cases}$$

In [11] Stanley gives the following useful way to compute the ab-index. Introducing $\Upsilon_P(a, b) \stackrel{\text{def}}{=} \sum_{S \subseteq \{1, \ldots, n\}} \alpha(S) \cdot u_S$, where u_S is the same as before, we have the equalities

$$\Psi_P(a, b) = \Upsilon_P(a - b, b) \quad \text{and} \quad \Upsilon_P(a, b) = \Psi_P(a + b, b). \tag{3}$$

Thus we may compute $\Psi_P(a, b)$ as follows. We associate a *weight* $z_1 \cdots z_n = w(x_1, \ldots, x_k)$ to every chain $x_1 < \cdots < x_k$ in $P \setminus \{\hat{0}, \hat{1}\}$ by the rule

$$z_i = \begin{cases} b & \text{if } i \in \{\rho(x_1), \rho(x_2), \ldots, \rho(x_k)\}, \\ a - b & \text{otherwise.} \end{cases}$$

We also allow $k = 0$, and associate $(a - b)^n$ to the empty chain. Now the ab-index $\Psi_P(a, b)$ is the sum of the weights of all strictly increasing chains in $P \setminus \{\hat{0}, \hat{1}\}$:

$$\Psi_P(a, b) = \sum_{k=0}^{n} \sum_{\hat{0} < x_1 < \cdots < x_k < \hat{1}} w(x_1, \ldots, x_k). \tag{4}$$

It was pointed out by Stanley [11], based on the results of Bayer, Billera [1], and Fine [2], that the ab-index $\Psi_P(a, b)$ of a Eulerian poset P may be written as a polynomial in $c \stackrel{\text{def}}{=} a + b$ and $d \stackrel{\text{def}}{=} ab + ba$. (We remind the reader that a ranked poset is Eulerian if, for every subinterval $[x, y] \subseteq P$, the value of the Möbius function μ on $[x, y]$ is given by $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$.) In this case we call $\Phi_P(c, d) \stackrel{\text{def}}{=} \psi_P(a, b)$ the *cd-index* of P.

2. Preliminaries on André and Simsun Permutations

It was first observed by Purtill in [10] that the cd-index of the Boolean algebra may be computed by summing the *cd-variation monomials* of *André permutations*. As mentioned by Stanley in [11], this result may be generalized to *André permutations of the second kind* and *simsun permutations* (Stanley calls them Sundaram permutations). In this section we summarize the definition and basic properties of these permutations, and we add a few elementary observations which will be necessary later.

First we recall the definition of André permutations given in [6] and in [10]. Let X be a linearly ordered set with $|X| = n$. A *permutation* of X is a word $\pi(1)\pi(2)\cdots\pi(n)$ such that every letter of X occurs exactly once. We denote the set of permutations of X by $\mathcal{S}_X$. When $X = \{1, 2, \ldots, n\}$ then we write $\mathcal{S}_n$ instead of $\mathcal{S}_{\{1,2,\ldots,n\}}$. We allow X to be the empty set, the only element of $\mathcal{S}_\emptyset$ is the empty word, which we call the *empty permutation*. Given a (possibly empty) subinterval $[i, j] \subseteq \{1, 2, \ldots, n\}$, we define the *restriction of* π *to* $[i, j]$ to be the permutation $\pi|_{[i,j]} \in \mathcal{S}_{\{\pi(i),\pi(i+1),\ldots,\pi(j)\}}$ given by the formula $\pi|_{[i,j]}(k) \stackrel{\text{def}}{=} \pi(i+k-1)$. We say that $i \in \{1, 2, \ldots, n-1\}$ is a *descent* of π if we have $\pi(i) > \pi(i+1)$ (otherwise i is an *ascent*), and $i \in \{2, \ldots, n\}$ is a *trough* if we have $\pi(i-1) > \pi(i) < \pi(i+1)$.

Definition 1. A permutation $\pi \in \mathcal{S}_X$ is an *André permutation* if it satisfies the following conditions:

(i) π has no double descents, i.e., there is no $i \in \{2, \ldots, n-1\}$ with $\pi(i-1) > \pi(i) > \pi(i+1)$.
(ii) Let $j, j' \in \{2, \ldots, n\}$ satisfy $j < j'$, $\pi(j-1) = \max\{\pi(j-1), \pi(j), \pi(j'-1), \pi(j')\}$, and $\pi(j') = \min\{\pi(j-1), \pi(j), \pi(j'-1), \pi(j')\}$. Then there is a j'' such that $j < j'' < j'$ and $\pi(j'') < \pi(j')$ holds.

In particular, we consider the empty permutation to be an André permutation. We call an André permutation *augmented* if $\pi(n) = \max X$. We denote the set of augmented André permutations of X (resp. $\{1, 2, \ldots, n\}$), by $\mathcal{A}_X$ (resp. $\mathcal{A}_n$).

Condition (ii) of Definition 1 does not seem to be very intuitive.[1] In our calculations we mostly use an equivalent definition of André permutations which relies on the following property (see Propriété 3.6 of [6] and Proposition 5.5 of [10]):

Proposition 1. *A permutation* $\pi \in \mathcal{S}_X$ *is an André permutation if and only if for* $m \stackrel{\text{def}}{=} \pi^{-1}(\min X)$ *the permutations* $\pi|_{[1,m]}$ *and* $\pi|_{[m+1,n]}$ *are André permutations.*

This proposition determines an André permutation in terms of two André permutations on a smaller set, except for the case when $m = \min X$ is the last letter. In that case we may use Propriété 3.3 of [6], which may be restated as follows:

[1] Foata and Schützenberger in [6] give the following alternative wording, which is only a little less complicated: "If $j' > j$ are two troughs such that $\pi(j) > \pi(j')$ and $\pi(j-1) > \pi(j'-1)$ then there is a trough j'' such that $j < j'' < j'$ and $\pi(j'') < \pi(j')$, and the same holds when $j' = n$ and $j'-1$ is a descent."

Proposition 2. *Let $\pi \in \mathcal{S}_X$ be a permutation of a linearly ordered n-element set X, with last letter* $\min X$. *Then π is an André permutation if and only if $\pi|_{[1,n-1]}$ is an augmented André permutation.*

Observe that Propositions 1 and 2 could serve as an equivalent definition of André permutations. Although this formulation is recursive, it seems to be more natural than the nonrecursive definitions.

As a consequence of Propositions 1 and 2 we obtain the following recursive description of augmented André permutations (see Corollary 5.6 of [10]):

Corollary 1. *A permutation $\pi \in \mathcal{S}_X$ is an augmented André permutation if and only if for $m \stackrel{\text{def}}{=} \pi^{-1}(\min X)$ the permutations $\pi|_{[1,m-1]}$ and $\pi|_{[m+1,n]}$ are augmented André permutations and the letter* $\max X$ *belongs to $\pi|_{[m+1,n]}$.*

Another way to describe André permutations is given by Foata and Strehl in [7]. We sketch their approach in order to arrive at the concept of André permutations of the second kind.

Definition 2. Given a permutation $\pi = \pi(1)\cdots\pi(n) \in \mathcal{S}_n$ and a letter x, we call the *x-factorization of π* the 5-tuple $(w_1, w_2, w_3, w_4, w_5)$ of (possibly empty) words, defined by the following properties:

(i) The word $\pi(1)\cdots\pi(n)$ is equal to the word $w_1w_2w_3w_4w_5$.
(ii) w_3 consists only of the letter x.
(iii) All letters of w_2 and w_4 are greater than x.
(iv) The last letter of w_1 and the first letter of w_5 is less than x.

Every permutation has a unique x-factorization for every $x \in \{1, 2, \ldots, n\}$ by Lemma 1 of [7]. As in [7], we denote by φ_x the involution which exchanges the permutation of x-factorization $(w_1, w_2, w_3, w_4, w_5)$ with the permutation of x-factorization $(w_1, w_4, w_3, w_2, w_5)$. The involutions $\{\varphi_x\colon 1 \le x \le n-1\}$ generate a 2^{n-1} element commutative group [7, Corollary 3], which we denote by G_n. (Note that φ_n is the identity.) For each $\pi \in \mathcal{S}_n$ we define a function $\mathrm{m}_\pi\colon \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ given by

$$\mathrm{m}_\pi(x, y) \stackrel{\text{def}}{=} \begin{cases} \min \pi|_{[\pi^{-1}(x),\pi^{-1}(y)]} & \text{when} \quad \pi^{-1}(x) < \pi^{-1}(y), \\ \min \pi|_{[\pi^{-1}(y),\pi^{-1}(x)]} & \text{when} \quad \pi^{-1}(x) > \pi^{-1}(y). \end{cases}$$

Then by Proposition 2 of [7] two permutations $\pi, \pi' \in \mathcal{S}_n$ are in the same orbit of G_n iff $\mathrm{m}_\pi = \mathrm{m}_{\pi'}$ holds. The action of G_n on the symmetric group allows us to define the augmented André permutations as a natural set of orbit representatives, as it is done in [8], where augmented André permutations are called *André permutations of the first kind*. In the language of this section, the definition of an André permutation of the first kind given in [8] may be rephrased as follows:

Definition 3. Let X be a linearly ordered alphabet of n letters. A permutation $\pi \in \mathcal{S}_X$ is an *André permutation of the first kind* if it is empty, or satisfies the following:

(i) π has no double descents.
(ii) $n-1$ is not a descent, i.e., $\pi(n-1) < \pi(n)$.
(iii) If $i \in \{2, \ldots, n\}$ is a trough of π, then for the $\pi(i)$-factorization $(w_1, w_2, w_3, w_4, w_5)$ of π the maximum letter of w_2 is less than the maximum letter of w_4.

Augmented André permutations are exactly the same as André permutations of the first kind, because they both have the same recursive description given in Corollary 1.

We may reformulate Foata and Strehl's definition of André permutations of the second kind given in [8] in the following way.

Definition 4. Let X be a linearly ordered alphabet of n letters. A permutation $\pi \in \mathcal{S}_X$ is an *André permutation of the second kind* if it is empty or satisfies the following:

(i) π has no double descents.
(ii) $n-1$ is not a descent, i.e., $\pi(n-1) < \pi(n)$.
(iii) If $i \in \{2, \ldots, n\}$ is a trough of π, then for the $\pi(i)$-factorization $(w_1, w_2, w_3, w_4, w_5)$ of π the minimum letter of w_2 is larger than the minimum letter of w_4.

In analogy to Corollary 1, it is straightforward to see that André permutations of the second kind have the following recursive description:

Proposition 3. *A permutation $\pi \in \mathcal{S}_X$ is an André permutation of the second kind if and only if for $m \stackrel{\text{def}}{=} \pi^{-1}(\min X)$ the permutations $\pi|_{[1,m-1]}$ and $\pi|_{[m+1,n]}$ are André permutations of the second kind, and the letter $\min(X\backslash\{m\})$ belongs to $\pi|_{[m+1,n]}$.*

Simsun permutations play a role in describing the action of the symmetric group on the maximal chains of the lattice of partitions Π_n, as observed by Sundaram and Simion in [12] on p. 267. The name simsun permutation occurs first on p. 6 of [13]. They are defined as follows:

Definition 5. Let X be an n-element linearly ordered set. A permutation $\pi \in \mathcal{S}_X$ is a *simsun permutation* if for all i, after removing the i largest letters from $\pi(1) \cdots \pi(n)$, the remaining word has no double descents. Here i may be any nonnegative integer up to n. (In particular, for $i = 0$ we obtain that π itself has no double descents.)

Following Stanley in [11], we call a simsun permutation π *augmented* if the last letter of π is the largest.

It is worth noting that every André permutation of the second kind is also a simsun permutation. In fact, part (ii) of Proposition 12 of [8] asserts that removing the largest letter from an André permutation of the second kind we obtain an André permutation of the second kind. (This is a direct consequence of Definition 4.) Hence, by repeated removal of the largest letter we never leave the class of permutations containing no double descents, and thus we must have started with a simsun permutation.

On the other hand, part (i) of Proposition 12 of [8] states the following.[2] Let π be an André permutation of the second kind. If we insert a largest letter into π after a letter x for which w_2 and w_4 in the x-factorization of π are empty, or before an x for which w_2 is empty and w_4 is nonempty, we obtain an André permutation of the second kind. (This statement, too, is a straightforward consequence of Definition 4.) Hence we have the following proposition:

Proposition 4. *The only place to insert the letter $(n+1)$ into an André permutation of the second kind of $\{1, \ldots, n\}$ such that we obtain a simsun permutation which is not an André permutation of the second kind, is right before the last letter.*

Proof. Let π be an André permutation of the second kind on $\{1, \ldots, n\}$. Assume we insert the letter $(n+1)$ into π, such that we obtain a simsun permutation π'. Clearly, π' is simsun if and only if no double descent is created.

If $(n+1)$ is inserted right before the last letter, then π' is a simsun permutation, but not André of the second kind, because condition (ii) of Definition 4 is violated. If $(n+1)$ is inserted as the last letter, then π' is an André permutation of the second kind. In all other cases we insert the letter $(n+1)$ before a letter $x = \pi(i)$ which is followed by at least one more letter $\pi(i+1)$. We must have $\pi(i) < \pi(i+1)$, otherwise we have created a double descent in π'. Thus the w_4 part in the x-factorization of π is nonempty. If the w_2 part is empty as well, then π' is an André permutation of the second kind by part (i) of Proposition 12 of [8]. Hence we are left with the case when the w_2 is nonempty either, i.e., x is a trough, preceded by a letter $y = \pi(i-1)$ satisfying $y > x$. There are no double descents in π, and so both the w_2 and w_4 parts of its y-factorization must be empty. Therefore π' is again an André permutation of the second kind by part (i) of Proposition 12 of [8]. □

Hence André permutations of the second kind may be characterized as exactly those simsun permutations which do not end with a descent, even after deleting the i largest letters.[3] From here it is easy to obtain the following analogue of Propositions 1 and 2.

Proposition 5. *A permutation $\pi \in \mathcal{S}_X$ is a simsun permutation if and only if for $m \stackrel{\text{def}}{=} \pi^{-1}(\min X)$ the permutation $\pi|_{[1,m-1]}$ is an André permutation of the second kind and $\pi|_{[m+1,n]}$ is a simsun permutation.*

In particular, augmented simsun permutations have the following recursive description:

Corollary 2. *A permutation $\pi \in \mathcal{S}_X$ is an augmented simsun permutation if and only if for $m \stackrel{\text{def}}{=} \pi^{-1}(\min X)$ the permutation $\pi|_{[1,m-1]}$ is an André permutation of the second*

[2] We warn the reader that there is typographical error in the statement. The article [8] uses the terms "peak," "trough," "rise," and "descent" in a nonstandard way. In order to avoid confusion, we do not cite the proposition literally.

[3] In the manuscript of Foata and Schützenberger [5, Définition 4.11] André permutations of the second type *are defined* by this property.

kind, $\pi|_{[m+1,n]}$ *is an augmented simsun permutation, and the letter* $\max X$ *belongs to* $\pi|_{[m+1,n]}$.

Remark. All permutations introduced in this section have a very similar recursive structure. This similarity may be made explicit by using the language of the theory of *André complexes* which is developed in the paper by Foata and Schützenberger [6].

3. The cd-Index of a Boolean Algebra

In this section we give an elementary proof of Purtill's result about the cd-index of a Boolean algebra. We then refine the argument to show Conjecture 3.1 of [11] for augmented André permutations. At the end of this section, we also indicate how to modify the presented proofs in order to get a proof of this conjecture for André permutations of the second kind and for augmented simsun permutations.

Let $\pi \in \mathcal{S}_n$. The *ab-variation monomial* $V_{ab}(\pi)$ of π is a word $v_1 \cdots v_{n-1}$ in noncommuting variables a and b such that for every $i \in \{1, 2, \ldots, n-1\}$ we have

$$v_i = \begin{cases} a & \text{if } i \text{ is an ascent,} \\ b & \text{if } i \text{ is a descent.} \end{cases}$$

The *cd-variation monomial* $V_{cd}(\pi)$ is the word in noncommuting variables c and d obtained from $V_{ab}(\pi)$ by first replacing every pair ba with d and then replacing the remaining letters with c. For the empty permutation, we define its ab-variation monomial, as well as its cd-variation monomial, to be 1. These variations were first defined by Foata and Schützenberger in [6] to encode the peak and trough statistics of permutations with no double descents.[4]

For a set $\mathcal{C}$ of permutations without double descents, we define its *cd-variation polynomial* $V_{cd}(\mathcal{C})$ to be the sum

$$V_{cd}(\mathcal{C}) \stackrel{\text{def}}{=} \sum_{\pi \in \mathcal{C}} V_{cd}(\pi).$$

Now we may state Purtill's result [10, Theorem 6.1] as follows:

Theorem 1. *The cd-index of the Boolean algebra B_n is the sum of the cd-variation monomials of the augmented André permutations of $\{1, 2, \ldots, n\}$. That is, we have*

$$\Phi_{B_n}(c, d) = V_{cd}(\mathcal{A}_n).$$

We present an elementary proof to this statement, which—unlike Purtill's original argument—does not involve constructing a bijection.

[4] In [6] the characters $+$, $-$, s, and t are used instead of a, b, c, and d, respectively. We modified the notation the same way as Purtill does in [10] and Ehrenborg and Readdy in [4], in order to bring it closer to the standard notation used in the study of cd-indices of Eulerian posets.

Proof. We denote $\Phi_{B_n}(c, d)$ by U_n and $V_{cd}(\mathcal{A}_n)$ by $\tilde{U}_n$. We only need to establish the following recursion formula for U_n:

$$U_{n+2} = \sum_{i=1}^{n} \binom{n}{i} \cdot U_i \cdot d \cdot U_{n+1-i} + c \cdot U_{n+1} \qquad \text{for} \quad n \geq 1. \tag{5}$$

In fact, it is an easy consequence of Corollary 1 that the polynomials $\tilde{U}_n$ satisfy

$$\tilde{U}_{n+2} = \sum_{i=1}^{n} \binom{n}{i} \cdot \tilde{U}_i \cdot d \cdot \tilde{U}_{n+1-i} + c \cdot \tilde{U}_{n+1} \qquad \text{for} \quad n \geq 1.$$

(This is Lemme 3.7 and Propriété 3.10 of [6] or Corollary 5.8 of [10].) Thus, considering also the trivial equalities $U_1 = \tilde{U}_1 = 1$ and $U_2 = \tilde{U}_2 = c$, we are done by induction.

To show (5) we use (4). We represent B_{n+2} as the set of subsets of $\{1, 2, \ldots, n+2\}$, ordered by inclusion. We divide the chains $\kappa_1 \subset \kappa_2 \subset \cdots \subset \kappa_k$ of proper subsets of $\{1, 2, \ldots, n\}$ into two classes: the chains of the first kind are those which contain an element κ_l such that $n+2 \in \kappa_l$ but $n+1 \notin \kappa_l$, while the other chains (including the empty chain) are of the second kind.

Given a chain $\kappa_1 \subset \cdots \subset \kappa_k$ of the first kind, let λ denote the smallest element of the chain which contains $n+2$, and set $i \stackrel{\text{def}}{=} |\lambda \backslash \{n+2\}|$. The possible values of i are $0, 1, \ldots, n$. For a fixed i, there are $\binom{n}{i}$ ways to choose $\lambda \backslash \{n+2\}$. For a fixed λ, the elements of the chain below $\lambda \backslash \{n+2\}$ belong to the Boolean algebra on the set $\lambda \backslash \{n+2\}$ (or there is nothing below λ if $i = 0$ and $\lambda = \{n+2\}$). When i is positive, the element $\lambda \backslash \{n+2\}$ may or may not be included in the chain, independently of the other decisions. The elements of the chain above λ are identifiable with proper subsets of $\{1, 2, \ldots, n\} \backslash \lambda$. Thus the sum of weights of the chains of the first kind is

$$\sum_{i=1}^{n} \binom{n}{i} \cdot U_i \cdot a \cdot b \cdot U_{n+1-i} + b \cdot U_{n+1}. \tag{6}$$

Note that the factors b in formula (6) are contributed by the λ's, while the factors $a = (a-b) + b$ correspond to the $\lambda \backslash \{n+2\}$'s which may or may not be included in the chain, independently of all the other choices.

The calculation of the total weight of the chains of the second kind may be done in a very similar way. Here we denote by λ the last element of the chain not containing $n+1$, and $|\lambda|$ by i. (If there is no such λ, we set $\lambda \stackrel{\text{def}}{=} \emptyset$.) Now we have $\lambda \subseteq \{1, 2, \ldots, n\}$, and $0 \leq i \leq n$. The elements of the chain below λ belong to the Boolean algebra on the set of λ, those above $\lambda \cup \{n+1\}$ are identifiable with subsets of $\{1, 2, \ldots, n, n+2\} \backslash \lambda\}$, while the set $\lambda \cup \{n+1\}$ may or may not be included in the chain, independently of all other choices. Therefore the sum of weights of the chains of the second kind is

$$\sum_{j=1}^{n} \binom{n}{i} \cdot U_i \cdot b \cdot a \cdot U_{n+1-i} + a \cdot U_{n+1}. \tag{7}$$

Here the b's correspond to the λ's, and the a's are produced by the $\lambda \cup \{n+1\}$'s.

The polynomial U_{n+2} is the sum of (6) and (7), and so we obtain (5). $\square$

Remark. Ehrenborg and Readdy [4, Section 3, equation (1)] found a shorter proof of (5) using R-labeling.

In Conjecture 3.1 of [11] Stanley makes a conjecture which refines Theorem 1. He fixes any shelling $\sigma_0, \sigma_1, \ldots, \sigma_n$ of the boundary complex Λ^n of an n-dimensional simplex. Then he denotes the semisuspension of the face poset of $\overline{\sigma_0} \cup \overline{\sigma_1} \cup \cdots \cup \overline{\sigma_i}$ by Λ^i_n. He defines $\check{\Phi}^n_i$ to be the difference of the cd-index of Λ^n_i and the cd-index of Λ^n_{i-1}, i.e., he sets $\check{\Phi}^n_i \overset{\text{def}}{=} \Phi_{\Lambda^n_i} - \Phi_{\Lambda^n_{i-1}}$. His conjecture is that $\check{\Phi}^n_i$ is equal to the sum of variations of those augmented André permutations in $\mathcal{S}_{n+1}$ which satisfy $\pi(n) = n - i$.

We advise the reader to refer to [11] for the exact definition of the terms involved in stating Stanley's conjecture. Here we give a reformulation, which does not involve the notion of shelling or semisuspension, but is easily seen to be equivalent to his wording.

Definition 6. For $i = 0, 1, \ldots, n-1$ we denote by σ_i the coatom $\{1, 2, \ldots, n\} \backslash \{i\}$ in the Boolean algebra B_n of the subsets of $\{1, 2, \ldots, n\}$. For $k = 1, 2, \ldots, n$ we define $B_{n,k}$ to be the poset obtained from the subposet $[\hat{0}, \sigma_1] \cup [\hat{0}, \sigma_1] \cup \cdots \cup [\hat{0}, \sigma_k] \cup \{\hat{1}\}$ by adding an extra coatom σ'_k which covers exactly the sets of the form $\{1, 2, \ldots, n\} \backslash \{l, m\}$ with $l \leq k$ and $m > k$. We denote the cd-index of $B_{n,k}$ by $U_{n,k}$.

Note that in the terms of Stanley's notation $U_{n,k}$ is obviously equal to the cd-index of Λ^{n-1}_{k-1}, or to the sum $\check{\Phi}^{n-1}_0 + \check{\Phi}^{n-1}_1 + \cdots + \check{\Phi}^{n-1}_{k-1}$. Observe also that $B_{n,n-1}$ is isomorphic to B_n and so we have

$$U_{n,n-1} = U_n \qquad \text{for} \quad n \geq 2. \tag{8}$$

Definition 7. For $1 \leq i \leq n-1$ we define $\mathcal{A}^i_n$ to be the set of those augmented André permutations π of $\{1, 2, \ldots, n\}$ which satisfy $\pi(n-1) = n - i$.

Now we can reformulate Stanley's conjecture about the refinement of Theorem 1 as follows:

Theorem 2. *We have*

$$U_{n,k} = \sum_{i=1}^{k} V_{cd}(\mathcal{A}^i_n) \qquad \textit{for} \quad 1 \leq k < n. \tag{9}$$

In words, the cd-index of $B_{n,k}$ is equal to the sum of the cd-variation monomials of all augmented André permutations $\pi \in \mathcal{A}_n$ satisfying $\pi(n-1) \geq n-k$.

Proof. In analogy to our proof of Theorem 1, we show the theorem by induction. We denote the right-hand side $\sum_{i=1}^{k} V_{cd}(\mathcal{A}^i_n)$ by $\tilde{U}_{n,k}$ for $1 \leq k < n$. The last letter of an augmented André permutation of $\{1, 2, \ldots, n\}$ is always n, and so we have $\mathcal{A}_n = \bigcup_{i=1}^{n-1} \mathcal{A}^i_n$, implying

$$\tilde{U}_{n,n-1} = \tilde{U}_n = U_n. \tag{10}$$

(The last equality follows from Theorem 1.)

We show that the polynomials $U_{n,k}$ and $\tilde{U}_{n,k}$ satisfy the same recursion formula. This is done in Propositions 6 and 7. Our induction basis is (8) and (10). □

Proposition 6. *The polynomials $U_{n,k}$ satisfy*

$$U_{n+2,k} = \sum_{i=0}^{k-1} \sum_{\substack{j=0 \\ j\neq -i}}^{n-k} \binom{k}{i} \cdot \binom{n-k}{j} \cdot U_{i+j} \cdot d \cdot U_{n+1-i-j,k-i} + c \cdot U_{n+1,k} \qquad \textit{for} \quad 1 \leq k \leq n. \tag{11}$$

(Observe that the condition in the second sum excludes only $i = j = 0$.)

Proof. As in the proof of Theorem 1, we use (4). Assume we are given $B_{n+2,k}$, where $1 \leq k \leq n$. We divide again the chains $\kappa_1 < \kappa_2 < \cdots < \kappa_k$ into the same two classes: the chains of the first kind are those which contain an element $\kappa_l \neq \sigma_k'$ such that $n+2 \in \kappa_i$ but $n+1 \notin \kappa_l$, and the other chains (including the empty chain) are of the second kind.

We compute the total weight of the chains of the first kind. Let λ denote the smallest element containing $n+2$ in a chain of the first kind. By the definition of the chains of the first kind, we must have $\lambda \neq \sigma_k'$ and $n+1 \notin \lambda$, implying $\lambda < \sigma_k'$. Let $i \stackrel{\text{def}}{=} |\lambda \cap \{1, 2, \ldots, k\}|$ and $j \stackrel{\text{def}}{=} |\lambda \cap \{k+1, \ldots, n\}|$. Then $\lambda \in B_{n+2,k}$ is equivalent to the conditions $0 \leq i \leq k-1$ and $0 \leq j \leq n-k$. For a fixed pair (i, j), there are $\binom{k}{i} \cdot \binom{cn-k}{j}$ ways to choose $\lambda \cap \{1, 2, \ldots, n\}$. (The set λ is then uniquely determined, because $n+1 \notin \lambda$ and $n+2 \in \lambda$.) When $i+j$ is positive then $\lambda \backslash \{n+2\}$ may be included in the chain or not, independently of the other decisions, and the elements of the chain below $\lambda \backslash \{n+2\}$ belong to a Boolean algebra of rank $i+j$. The interval $[\lambda, \hat{1}] \subset B_{n+2,k}$ is isomorphic to $B_{n+1-i-j,k-i}$. Hence the sum of weights of the chains of the first kind is

$$\sum_{i=0}^{k-1} \sum_{\substack{j=0 \\ j\neq -i}}^{n-k} \binom{k}{i} \cdot \binom{n-k}{j} \cdot U_{i+j} \cdot a \cdot b \cdot U_{n+1-i-j,k-i} + b \cdot U_{n+1,k}. \tag{12}$$

Again, the b's in the above formula correspond to the λ's and the a's to the $\lambda \backslash \{n+2\}$'s.

Consider now the chains of the second kind. Let λ be the largest element satisfying $\lambda \neq \sigma_k'$ and $n+1 \notin \lambda$ in such a chain. (We set $\lambda \stackrel{\text{def}}{=} \emptyset$ if there is no such element in the chain.) Then we must have $\lambda \subseteq \{1, 2, \ldots, n\}$, where equality is only possible if $k = n$. We set $i \stackrel{\text{def}}{=} |\lambda \cap \{1, 2, \ldots, k\}|$ and $j \stackrel{\text{def}}{=} |\lambda \cap \{k+1, k+2, \ldots, n\}|$. Again we must have $0 \leq i \leq k-1$ and $0 \leq j \leq n-k$. If $i+j \neq 0$ then the interval $[\hat{0}, \lambda] \subset B_{n+2,k}$ is isomorphic to B_{i+j}, contributing a factor U_{i+j}. Every element of the chain above λ is either σ_k' or a set containing $n+1$. Because $n+2 \notin \lambda$, the set $\lambda \cup \{n+1\}$ belongs to $B_{n+2,k}$, and the interval $[\lambda \cup \{n+1\}] \subset B_{n+2,k}$ is isomorphic to $B_{n+1-i-j,k-i}$, contributing a factor $U_{n+1-i-j,k-i}$. Thus the sum of weights of the chains of the second kind is

$$\sum_{i=0}^{k-1} \sum_{\substack{j=0 \\ j\neq -i}}^{n-k} \binom{k}{i} \cdot \binom{n-k}{k} \cdot U_{i+j} \cdot b \cdot a \cdot U_{n+1-i-j,k-i} + a \cdot U_{n+1,k}. \tag{13}$$

By (4), $U_{n+2,k}$ is the sum of (12) and (13) yielding (11). □

The last ingredient of the proof of Theorem 2 is the following statement:

Proposition 7. *The polynomials $\tilde{U}_{n,k}$ satisfy the recursion*

$$\tilde{U}_{n+2,k} = \sum_{i=0}^{k-1} \sum_{\substack{j=0 \\ j \neq -i}}^{n-k} \binom{k}{i} \cdot \binom{n-k}{k} \cdot \tilde{U}_{i+j} \cdot d \cdot \tilde{U}_{n+1-i-j,k-i} + c \cdot \tilde{U}_{n+1,k} \qquad \text{for} \quad 1 \le k \le n. \tag{14}$$

Proof. Consider a permutation $\pi \in \bigcup_{l=m}^{k} \mathcal{A}_{n+2}^{m}$. Let i denote the number of letters from the k-element set $\{n+2-k, n+3-k, \dots, n+1\}$ which occur before the letter 1 in the word $\pi(1)\cdots\pi(n+2)$. Let j be the number of the letters from the $(n-k)$-element set $\{2, 3, \dots, n+1-k\}$ appearing before the letter 1 in $\pi(1)\cdots\pi(n+2)$. Clearly we have $0 \le j \le n-k$ and since $\pi(n-1) \in \{n+2-k, n+3-k, \dots, n+1\}$ we must also have $0 \le i \le k-1$. By Corollary 1, the permutation π is an augmented André permutation if and only if the subword $\pi|_{[1,i+j]}$ of letters occurring before the letter 1 is an André permutation, and subword $\pi|_{[i+j+1,n]}$ of letters occurring after the 1 is an augmented André permutation. With the current assumptions the only condition that has to be added is that the penultimate letter of $\pi|_{[i+j+1,n]}$ must belong to the $k+1-i$ largest elements of $\{\pi(i+j+1), \pi(i+j+2), \dots, \pi(n)\}$. The contribution of the letter 1 to $V_{cd}(\pi)$ is c when 1 is the first letter, and d in all other cases. Thus if we sum the cd-variation of all π's belonging to a fixed pair (i, j), we obtain a contribution of $\binom{k}{i} \cdot \binom{n-k}{j} \cdot \tilde{U}_{i+j} \cdot d \cdot \tilde{U}_{n+1-i-j,k-i}$ for $i+j > 0$, and a contribution of $c \cdot \tilde{U}_{n+1,k}$ for $i = j = 0$. Summing over all possible values of i and j we obtain the desired formula. □

Remark. Conjecture 3.1 of [11] also contains two other similar statements, where the role of André permutations is taken over by *André permutations of the second kind*, and *augmented simsun permutations*, respectively. The proof of Theorem 2 can be easily modified such that it becomes the proof of these other two conjectures. In fact, we only need to prove the respective analogues of Proposition 7. This may be done because the role of Corollary 1 can be taken over by Proposition 3 for André permutations of the second kind and by Corollary 2 for augmented simsun permutations. We leave verification of the details to the reader.

Remark. Theorem 2 is important because of Theorem 3.1 of [11] which allows us to express the cd-index of an arbitrary simplicial Eulerian poset P of rank n in terms of its h-vector $(h_0, \dots, h_{n-1})$ and the polynomials $\{U_{n,k}\colon 1 \le k \le n\}$ as follows:

$$\Phi_P(c, d) = h_0 \cdot U_{n,1} + \sum_{i=1}^{n-2} h_i \cdot (U_{n,i+1} - U_{n,i}).$$

4. A Cubical Analogue of André Permutations

In this section we present a cubical analogue of augmented André permutations, together with a new notion of ab- and cd-variation for signed permutations. The cd-variation of our signed André permutation is equal to the cd-index of the face poset of the cube. This cubical analogue is different from the one defined by Purtill in [10] and allows us also to extend the commutative group-action defined by Foata and Strehl in [8] from permutations to signed permutations.

We remind the reader that the elements of the group of symmetries $\mathcal{B}_n$ of the standard n-cube may be represented as *signed permutations* $(\varepsilon, \pi) = (\varepsilon(1), \pi(1))(\varepsilon(2), \pi(2)) \cdots (\varepsilon(n), \pi(n))$ where $\pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$ is a permutation and each $\varepsilon(i)$ is from the set $\{-1, 1\}$. The product of (ε, π) and (ε', π') is $(\varepsilon'', \pi \cdot \pi')$ where we have $\varepsilon''(i) = \varepsilon'(i) \cdot \varepsilon(\pi''(i))$. Thus we introduce, for a nonempty n-set X, the set $\mathcal{B}_X$ of signed permutations of X to be the set of pairs (ε, π), where $\pi \in \mathcal{S}_X$ is a permutation of X and $\varepsilon\colon \{1, 2, \ldots, n\} \longrightarrow \{-1, 1\}$ is a sign function. For $X = \emptyset$ we still consider the only element of $\mathcal{B}_\emptyset$ to be the empty word. Given a signed permutation $(\varepsilon, \pi) \in \mathcal{B}_X$ we define the *restriction* $(\varepsilon, \pi)|_{[i,j]}$ of (ε, π) to a (possibly empty) interval $[i, j] \subseteq \{1, 2, \ldots, n\}$ by $(\varepsilon, \pi)|_{[i,j]}(k) \stackrel{\text{def}}{=} (\varepsilon(k+i-1), \pi(k+i-1))$.

Definition 8. We say that $(\varepsilon, \pi) \in \mathcal{B}_X$ is a *signed André permutation* if π is an André permutation and we have $\varepsilon(i) = 1$ for all i satisfying $\pi(i) = \min\{\pi(i), \pi(i+1), \ldots, \pi(n)\}$. We call a signed André permutation (π, ε) *augmented* if it satisfies $\pi(n) = n$. We denote the set of augmented signed André permutations of X (resp. $\{1, 2, \ldots, n\}$) by $\mathcal{A}_X^{\pm}$ (resp. $\mathcal{A}_n^{\pm}$).

In words, $(\varepsilon, \pi) \in \mathcal{B}_X$ is a signed André permutation if $\pi \in \mathcal{S}_X$ is an André permutation and the sign of every letter which is a minimum from the right is positive.

Definition 9. Let X be an n-element set. We say that $i \in \{1, 2, \ldots, n-1\}$ is an *ascent, descent,* or *trough* of $(\varepsilon, \pi) \in \mathcal{B}_X$, respectively, if and only if i is an ascent, descent, or trough of the underlying permutation π. We define the ab-variation (resp. cd-variation) monomials of a signed permutation (resp. signed André permutation) (ε, π) to be equal to the ab-variation (resp. cd-variation) of their underlying permutation π. That is, we set

$$V_{ab}((\varepsilon, \pi)) \stackrel{\text{def}}{=} V_{ab}(\pi) \qquad \text{for all} \quad (\pi, \varepsilon) \in \mathcal{S}_X,$$

and

$$V_{cd}((\varepsilon, \pi)) \stackrel{\text{def}}{=} V_{cd}(\pi) \qquad \text{for all} \quad (\varepsilon, \pi) \in \mathcal{A}_X^{\pm}.$$

Observe that $\mathcal{S}_X$ may be considered as the subset

$$\{(\varepsilon, \pi) \in \mathcal{B}_X\colon \varepsilon(i) = 1 \text{ for } i = 1, 2, \ldots, n\} \text{ of } \mathcal{B}_X.$$

Under this identification the definitions of signed André permutation, ab-variation, and cd-variation become the extensions of their signless counterparts. Proposition 1 has the following extension to signed André permutations:

Proposition 8. *For any $(\varepsilon, \pi) \in \mathcal{B}_X$ where $\pi \in \mathcal{S}_X$ has no double descents, (ε, π) is a signed André permutation if and only if for $m \stackrel{\text{def}}{=} \pi^{-1}(\min X)$ the permutation $\pi|_{[1,m]} \in \mathcal{S}_X$ is an André permutation and $(\varepsilon, \pi)|_{[m,n]}$ is a signed André permutation.*

Proof. By Proposition 1, the underlying permutation π is André if and only if $\pi|_{[1,m]}$ and $\pi|_{[m+1,n]}$ are André permutations, so we only need to check when the condition about the signs is satisfied. All minimums from the right belong to the interval $[m, n]$. Hence given an André permutation π, the pair (ε, π) is a signed André permutation if and only if $(\varepsilon, \pi)|_{[m,n]}$ is a signed André permutation. □

Using again Propriété 3.3 of [6] we obtain the following signed generalization of Corollary 1.

Corollary 3. *For any $(\varepsilon, \pi) \in \mathcal{B}_X$ where $\pi \in \mathcal{S}_X$ has no double descents, (ε, π) is an augmented signed André permutation if and only if for $m \stackrel{\text{def}}{=} \pi^{-1}(\min X)$ the permutation $\pi|_{[1,m-1]} \in \mathcal{S}_X$ is an augmented André permutation and $(\varepsilon, \pi)|_{[m+1,n]}$ is an augmented signed André permutation, and the letter $\max X$ belongs to $\pi|_{[m+1,n]}$.*

Now we show a signed extension of Theorem 1. Given a cd-word $w = w_1 w_2 \cdots w_n$, we denote the reversed word $w_n w_{n-1} \cdots w_1$ by w^{rev}. We extend the operation $w \mapsto w^{\text{rev}}$ to cd-polynomials linearly.

Theorem 3. *The cd-index of the face lattice C_n of the $(n-1)$-cube is the sum of the reversed cd-variation monomials of the augmented André permutations of $\{1, 2, \ldots, n\}$. That is, we have*

$$\Phi_{C_n}(c, d) = \sum_{(\varepsilon,\pi)\in\mathcal{A}_n^{\pm}} V_{cd}^{\text{rev}}((\varepsilon, \pi)).$$

Proof. Let V_n denote the cd-index of the face lattice of the $(n-1)$-cube and let $\tilde{V}_n$ denote the sum of the cd-variations of the signed augmented André permutations of $\{1, 2, \ldots, n\}$.

We proceed by induction on n. Observe first that in analogy with Lemme 3.7 and Propriété 3.10 of [6] and Corollary 5.8 of [10], Corollary 3 implies the following recursion formula for the polynomials $\tilde{V}_n$:

$$\tilde{V}_{n+2} = \sum_{i=0}^{n-1} \binom{n}{i} \cdot 2^{n-i} \cdot \tilde{U}_{n-i} \cdot d \cdot \tilde{V}_{i+1} + c \cdot V_{n+1} \qquad \text{for} \quad n \geq 1. \tag{15}$$

Hence the theorem follows from the fact that the polynomials V_n also satisfy

$$V_{n+2} = \sum_{i=0}^{n-1} \binom{n}{i} \cdot 2^{n-i} \cdot V_{i+1} \cdot d \cdot U_{n-i} + V_{n+1} \cdot c \qquad \text{for} \quad 1 \leq k \leq n, \tag{16}$$

as was shown by Ehrenborg and Readdy Section 5, equation (3) of [4], using R-labeling.

In fact, our statement then follows by induction from $V_1 = \tilde{V}_1 = 1$, $V_2 = \tilde{V}_2 = c$, and from the fact that, for every $n \geq 1$, the polynomial U_n is not only equal to $\tilde{U}_n$ but also to U_n^{rev}. The equality $U_n = U_n^{\text{rev}}$ holds because the reverse of the cd-index of a poset is the cd-index of the dual poset, and the Boolean algebra B_n is self-dual. □

Remark. Independently of Ehrenborg and Readdy, we have found another proof of (16). This proof is not included, because it is longer than the R-labeling argument, and it uses (4) in a completely analogous way to the proofs of Theorem 1 and Proposition 6. In the omitted proof, the chains are split into those containing a face of the form $\kappa = (u_1, \dots, u_n, 0)$, and those which do not. (Here we encode the the nonempty faces in the same way as Rota and Metropolis in [9]: we identify each nonempty face κ with a vector $(u_1, u_2, \dots, u_{n+1})$, where all u_i belong to the set $\{0, 1, *\}$. We set $u_i = 0$ or $u_i = 1$, respectively, if the ith coordinate of all vertices of κ is 0 or 1, respectively, otherwise we set $u_i = *$.)

Now we describe how to extend the Foata–Strehl group action defined in Definition 2 to signed permutations, such that our signed André permutations become a natural set of orbit representatives.

Given a signed permutation $(\varepsilon, \pi) \in \mathcal{B}_n$ and $x \in \{1, 2, \dots, n\}$, we define the *x-factorization of* (ε, π) to be the 5-tuple of signed permutations $((w_1, s_1), (w_2, s_2), (w_3, s_3), (w_4, s_4), (w_5, s_5))$ such that

(i) $(w_1, w_2, w_3, w_4, w_5)$ is the x-factorization of π, and
(ii) for each $i \in \{1, 2, 3, 4, 5\}$ if $w_i = \pi|_{[k_i, l_i]}$, then $(w_i, s_i) = (\varepsilon, \pi)|_{[k_i, l_i]}$.

In words, we take the x-factorization of the underlying permutation π keeping the signs unchanged. For $x \in \{1, 2, \dots, n\}$ we extend the reflection φ_x to signed permutations by requiring that it exchanges the signed permutation of x-factorization $((w_1, s_1), (w_2, s_2), (w_3, s_3), (w_4, s_4), (w_5, s_5))$ with the signed permutation of x-factorization $((w_1, s_1), (w_4, s_4), (w_3, s_3), (w_2, s_2), (w_5, s_5))$. By abuse of notation, we denote the extended reflection by the same symbol φ_x. The proof of the fact that the reflections φ_x generate a commutative group carries over, *mutatis mutandis*, to this case, and so we have the same 2^{n-1} element commutative group G_n acting on the signed permutations. It is also clear that for every orbit there is exactly one $(\varepsilon, \pi) \in \mathcal{B}_n$ in the orbit such that π is an (unsigned) augmented André permutation, i.e., $\pi \in \mathcal{A}_n$. It is not guaranteed, however, that the ε would satisfy the sign condition of Definition 8.

We introduce new reflections φ'_y for $y = 1, 2, \dots, n$ such that the generated group remains commutative, and the signed augmented André permutations became a system of orbit representatives. The operation φ'_y will leave the underlying permutation π of $(\varepsilon, \pi) \in \mathcal{B}_n$ unchanged, and reverse only the sign of the letter y if $\mathrm{m}_\pi(y, n) = k$. In other words, we set

$$\varphi'_y(\varepsilon, \pi) \stackrel{\text{def}}{=} (\varepsilon', \pi),$$

where

$$\varepsilon'(i) = \begin{cases} -\varepsilon(i) & \text{if } \pi(i) = y \text{ and } \mathrm{m}_\pi(y, n) = y, \\ \varepsilon(i) & \text{otherwise.} \end{cases}$$

Given the fact that the reflections φ_x leave the functions m_π unchanged and the reflections φ'_y change only the signs of pairwise different letters in a way depending only on m_π, it is easy to see that the set $\{\varphi_x\colon 1 \le i \le n-1\} \cup \{\varphi'_y\colon 1 \le j \le n\}$ generates a commutative group $G_n^{\pm}$.

Proposition 9. *The elements of $\mathcal{A}_n^{\pm}$ form a system of orbit representatives for $G_n^{\pm}$.*

Proof. As we noted before, to every orbit of $G_n = \langle \varphi_x\colon 1 \le i \le n-1 \rangle$ there belongs exactly one (ε, π) satisfying $\pi \in A_n$. The φ'_y's do not change the underlying permutation, so this property will hold even for the larger orbits of $G_n^{\pm} = \langle \{\varphi_x\colon 1 \le i \le n-1\} \cup \{\varphi'_y\colon 1 \le j \le n\} \rangle$.

Assume first that we are given an element $(\varepsilon, \pi) \in \mathcal{O}$ such that $\pi \in \mathcal{A}_n$. If for an $i \in \{1, 2, \ldots, n\}$ the letter $\pi(i)$ is a minimum from the right in the word $\pi(1) \cdots \pi(n)$, then we have $\mathrm{m}_\pi(\pi(i), n) = \pi(i)$. Applying $\varphi'_{\pi(i)}$ if necessary, we arrive at a signed permutation (ε', π) with $\varepsilon'(i) = 1$ in the same orbit. By repeating the same procedure for all $i \in \{1, 2, \ldots, n\}$, we obtain a signed augmented André permutation belonging to $\mathcal{O}$.

Assume now that there are even two signed augmented André permutations (ε, π), $(\varepsilon', \pi') \in \mathcal{A}_n^{\pm}$ belonging to the same orbit $\mathcal{O}$. Their underlying permutation must be the same, so we must have $\pi = \pi'$. The reflections φ_x and φ'_y commute, and $\mathcal{A}_n$ is a system of orbit representatives for the signless action of the φ_x's, hence the φ_x's must cancel in the product of φ_x's and φ'_y's which takes (ε, π) into (ε', π). For all $i \in \{1, 2, \ldots, n\}$ for which $\pi(i)$ is not a minimum from the right in π, we have $\mathrm{m}_\pi(\pi(i), n) \ne \pi(i)$ and so any φ_y leaves the sign $\varepsilon(i)$ unchanged. Thus we must have $\varepsilon(i) = \varepsilon'(i)$ for these i's. However, for the i's for which $\pi(i)$ is a minimum from the right in π we must have $\varepsilon(i) = \varepsilon'(i) = 1$ by the definition of augmented signed André permutations. Therefore we must have $\varepsilon = \varepsilon'$, showing that there is exactly one signed augmented André permutation in each orbit. □

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References

1. M. M. Bayer and L. J. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.
2. M. M. Bayer and A. Klapper, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991), 33–47.
3. A. Björner, Shellable and Cohen–Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* **260** (1980), 159–183.
4. R. Ehrenborg and M. Readdy, The **r**-cubical lattice and a generalization of the **cd**-index, *European J. Combin.*, to appear.
5. D. Foata and M. P. Schützenberger, Nombres d'Euler et permutations alternantes, Manuscript, University of Florida, Gainesville, FL, 1971.
6. D. Foata and M. P. Schützenberger, Nombres d'Euler et permutations alternantes. In: J. N. Srivastava *et al.*, *A Survey of Combinatorial Theory*, Amsterdam, North-Holland, 1973, pp. 173–187.

7. D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, *Math. Z.* **137** (1974), 257–264.
8. D. Foata and V. Strehl, Euler numbers and variations of permutations. In: *Colloquio Internazionale sulle Teorie Combinatorie*, 1973, Tome I (*Atti Dei Convegni Lincei* **17**, 119–131), Accademia Nazionale dei Lincei, Rome, 1976.
9. N. Metropolis and G.-C. Rota, Combinatorial structure of the faces of the n-cube, *SIAM J. Appl. Math.* **35**(4) (1978), 689–694.
10. M. Purtill, André permutations, lexicographic shellability, and the cd–index of a convex polytope, *Trans. Amer. Math. Soc.* **338**(1) (1993), 77–104.
11. R. P. Stanley, Flag f-vectors and the cd-index, *Math. Z.* **216** (1994), 483–499.
12. S. Sundaram, The homology representations of the symmetric group on Cohen–Macaulay subposets of the partition lattice, *Adv. in Math.* **104** (1994), 225–296.
13. S. Sundaram, The homology of partitions with an even number of blocks, *J. Algebraic Combin.* **4**(1) (1995), 69–92.

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