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On Three Dimensional Real Hypersurfaces in Complex Space Forms

Dedicated to professor Hajime Urakawa on his sixtieth birthday

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Abstract. Three dimensional pseudo-symmetric Hopf hypersurfaces in complex projective plane and complex hyperbolic plane are classified.

1. Introduction

Locally symmetric Riemannian manifolds are characterized by the parallelity of the Riemannian curvature tensor R. As a generalization of locally symmetric Riemannian manifolds, the notion of semi-symmetric Riemannian manifold is introduced as follows.

A Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$, where $R \cdot R$ is the derivative of R by R. Local structures of semi-symmetric Riemannian manifolds are systematically investigated by Z. I. Szabo.

Study of semi-symmetric spaces was initiated by E. Cartan, A. Lichnerowicz, R. S. Couty and N. S. Sinjukov.

In 1968, K. Nomizu proposed a question [28]:

Are there complete, irreducible and simply connected semi-symmetric Riemannian manifolds which are not symmetric ?"

The first positive answer was given by H. Takagi [32]. Takagi and K. Sekigawa [30] constructed semi-symmetric hypersurfaces in Euclidean space. Szabó obtained a full intrinsic classification of semi-symmetric spaces. O. Kowalski obtained a full classification of 3-dimensional semi-symmetric spaces.

Classifications of semi-symmetric hypersurfaces in Euclidean space were obtained by Szabó [31].

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According to [13], a Riemannian manifold (M, g) is said to be *pseudo-symmetric* if there exists a function L such that $R(X, Y) \cdot R = L\{(X \wedge Y) \cdot R\}$ for all vector fields X and Y on M. Here $(X \wedge Y)$ is a tensor field of type $(1, 1)$ defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y .$$

In particular, a pseudo-symmetric space is called a *pseudo-symmetric space of constant type* if L is a constant. Clearly, semi-symmetric spaces are pseudo-symmetric spaces of constant type with $L = 0$. A pseudo-symmetric space is said to be *proper* if M is not semi-symmetric.

In dimension 3, pseudo-symmetry plays a special role. As is well-known, in 3-dimensional Riemannian geometry, constancy of the sectional curvature is equivalent to the Einstein condition, *i.e.*, $\rho_1 = \rho_2 = \rho_3$ for eigenvalues $\{\rho_j\}$ of the Ricci tensor. Moreover, the pseudo-symmetry is equivalent to the condition: the Ricci tensor has at most two eigenvalues, in dimension 3. Thus the pseudo-symmetry is a natural generalization of constant curvature property or Einstein condition.

The pseudo-symmetry condition naturally arises in the study of isometrically deformable hypersurfaces in 4-dimensional real space forms. V. Hajkova, O. Kowalski and M. Sekizawa [14] investigated such hypersurfaces in terms of pseudo-symmetry.

In the differential geometry of real hypersurfaces in complex space forms, it is well known that there are no locally symmetric real hypersurfaces in complex projective or hyperbolic spaces.

In [27], R. Nigerball and P. J. Ryan proved the non-existence of 3-dimensional semi-symmetric Hopf hypersurfaces and 3-dimensional Einstein real hypersurfaces in complex projective plane $P_2(\mathbf{C})$ and complex hyperbolic plane $H_2(\mathbf{C})$.

Thus both "Einstein" and "semi-symmetry" are too strong restriction for 3-dimensional real hypersurfaces.

In addition, one can see that geodesic spheres and horospheres in non-flat complex space forms are proper pseudo-symmetric spaces.

These observations show that pseudo-symmetry is more suitable than semi-symmetry or Einstein property for real hypersurfaces in complex space forms. Note that in our previous works [8]–[10], we investigated pseudo-symmetric almost contact metric 3-manifolds. In [15], pseudo-symmetric simply connected 3-dimensional Lie groups are classified.

The purpose of this paper is to investigate 3-dimensional pseudo-symmetric real hypersurfaces in non-flat complex space forms. The main result of the present paper is:

THEOREM 1. *The pseudo-symmetric Hopf hypersurfaces in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$ are locally holomorphically congruent to a horosphere in $H_2(\mathbf{C})$, a geodesic sphere in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$, a homogeneous tube over $H_1(\mathbf{C})$ in $H_2(\mathbf{C})$, a non-homogeneous real hypersurface which is realized as a tube over a certain holomorphic curve in $P_2(\mathbf{C})$ with radius $\pi/\sqrt{4c}$, where c is the holomorphic sectional curvature of the ambient space or a Hopf hypersurface in $H_2(\mathbf{C})$ with $A\xi = 0$.*

On the other hand, Y. Maeda [24] showed that, the shape operator A of a real hypersurface in a complex projective space $P_n(\mathbf{C})$ of constant holomorphic sectional curvature $c > 0$ ($n \geq 2$) satisfies $\|A\|^2 \geq c^2(n-1)/2$. This inequality also holds for real hypersurfaces in complex hyperbolic space (Chen-Ludden-Montiel [6]). Thus there are no real hypersurface in non-flat complex space forms with parallel A.

S. Maeda [23] generalized this non-existence theorem. More precisely, he showed that there are no real hypersurfaces in $P_n(\mathbf{C})$ ($n \geq 3$) with semi-parallel A, *i.e.*, $R\cdot A = 0$. However S. Maeda's proof can not hold for $P_2(\mathbf{C})$ and $H_n(\mathbf{C})$ with $n \geq 2$. R. Niebergall and P. J. Ryan [27] proved the non-existence of real hypersurfaces in $P_2(\mathbf{C})$ and $H_2(\mathbf{C})$ with $R \cdot A = 0$ by a method different from Maeda's one.

Analogous to pseudo-symmetry, we shall study real hypersurfaces in $P_2(\mathbf{C})$ and $H_2(\mathbf{C})$ which satisfies the following *pseudo-parallel condition*;

$$(R(X, Y) \cdot A) = L\ (X \wedge Y) \cdot A$$

for all vector fields X and Y.

THEOREM 2. *Let M be a real hypersurface with pseudo-parallel shape operator A, i.e., $R \cdot A = L\ \mathcal{Q}(g, A)$ for some function L. Then M is locally holomorphically congruent to either*

- *a horosphere in $H_2(\mathbf{C})$,*
- *a geodesic hypersphere in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$, or*
- *a homogeneous tube over $H_1(\mathbf{C})$ in $H_2(\mathbf{C})$.*

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2. Preliminaries

2.1. Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. Denote by R the Riemannian curvature of M:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M. A tensor field F of type $(1, 3)$;

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is said to be *curvature-like* provided that F has the symmetric properties of R. For example,

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad X, Y \in \mathfrak{X}(M) \tag{1}$$

defines a curvature-like tensor field on M.

Every curvature-like tensor field F acts on the algebra $\mathcal{T}^1_s(M)$ of all tensor fields on M of type $(1, s)$ as a derivation:

$$(F\cdot P)(X_1,\dots,X_s;Y,X)= F(X,Y)\{P(X_1,\dots,X_s)\} - \sum_{j=1}^{s} P(X_1,\dots,F(X,Y)X_j,\dots,X_s)\,,$$
$$X_1,\dots,X_s\in\mathfrak{X}(M)\,,\quad P\in\mathcal{T}^1_s(M)\,.$$

The derivative $F\cdot P$ of P by F is a tensor field of type $(1, s+2)$.

For a tensor filed P of type $(1,s)$, we denote the by $\mathcal{Q}(g,P)$ the derivative of P with respect to the curvature-like tensor defined by (1).

A tensor field P is said to be *semi-parallel* if $R\cdot P=0$. More generally, P is said to be *pseudo-parallel* if there exists a function L such that $R\cdot P = L\,\mathcal{Q}(g,P)$. In particular, a pseudo-parallel tensor field P is said to be *proper* if $L\neq 0$.

2.2. A Riemannian manifold (M,g) is said to be *pseudo-symmetric* if R is pseudo-parallel, *i.e.*,

$$R\cdot R = L\,\mathcal{Q}(g,R)$$

for some function L. A pseudo-symmetric space is said to be *proper* if it is not semi-symmetric.

2.3. The *Ricci tensor* ρ of a Riemannian manifold (M,g) is defined by

$$\rho(X,Y)=\mathrm{trace}(Z\mapsto R(Z,X)Y)\,,\quad X,Y\in\mathfrak{X}(M)\,.$$

The tensor field S of type $(1,1)$;

$$\rho(X,Y)=g(SX,Y)\,,\qquad X,Y\in\mathfrak{X}(M)$$

metrically associated to ρ is called the *Ricci operator* of M. The trace s of S is called the *scalar curvature* of M.

A Riemannian manifold is said to be *Einstein* if $\rho = cg$ for some constant c. In this case, $c=\mathrm{s}/\dim M$.

On can see that every Einstein manifold has parallel Ricci tensor, *i.e.*, $\nabla\rho=0$ (equivalently $\nabla S=0$). More generally, Einstein manifolds have *semi-parallel* Ricci tensor ($R\cdot\rho=0$).

The Riemannian curvature R of a 3-dimensional Riemannian manifold (M,g) is expressed as

$$R(X,Y)Z=\rho(Y,Z)X-\rho(Z,X)Y+g(Y,Z)SX-g(Z,X)SY-\frac{\mathrm{s}}{2}(X\wedge Y)Z \tag{2}$$

for all $X,Y,Z\in\mathfrak{X}(M)$.

The formula (2) implies that a Riemannian 3-manifold is Einstein if and only if it is of constant curvature. Moreover we have

PROPOSITION 2.1. *On a Riemannian* 3*-manifold, the derivative* $R \cdot R$ *is given by*

$$\begin{aligned}(R(U,V)\cdot R)(X,Y)Z = {} & (R(U,V)\cdot \rho)(Y,Z)X - (R(U,V)\cdot \rho)(Z,X)Y \\ & + g(Y,Z)(R(U,V)\cdot S)X - g(Z,X)(R(U,V)\cdot S)Y\,.\end{aligned}$$

Let (M, g) be a Riemannian 3-manifold with pseudo-parallel Ricci operator such that $R \cdot S = LQ(g, S)$. Then by Proposition 2.1, we get $R \cdot R = LQ(g, R)$. Hence M is pseudo-symmetric.

COROLLARY 2.2. *A Riemannian* 3*-manifold* M *is pseudo-symmetric if and only if* M *has pseudo-parallel Ricci tensor. In particular* M *is semi-symmetric* ($R \cdot R = 0$) *if and only if* $R \cdot S = 0$.

2.4. The pseudo-parallelity of tensor fields of type $(1, 1)$ is characterized as follows (*cf.* [12]).

LEMMA 2.3. *Let* (M, g) *be a Riemannian* 3*-manifold and* B *a tensor field on* M *of type* $(1, 1)$ *which is self-adjoint with respect to* g. *Take a local orthonormal frame field* $\{e_1, e_2, e_3\}$ *which diagonalizes* B *so that* $Be_j = b_j e_j$ $(j = 1, 2, 3)$. *Assume that* M *is not of constant curvature and* B *is not of the form* $B = \mu I$ *for some function* μ, *where* I *denotes the identity transformation. Then* B *is pseudo-parallel such that* $R \cdot B = LQ(g, B)$ *for some function* L *if and only if the eigenvalues of* B *and the sectional curvature function* K *locally satisfy the following relations* (*up to numeration*):

$$b_1 = b_2 \neq b_3\,, \quad K_{13} = K_{23} = L\,.$$

Here $K_{ij} = K(e_i \wedge e_j)$ *denotes the sectional curvature of the plane* $e_i \wedge e_j$ *spanned by* e_i *and* e_j.

PROOF. Assume that M satisfies $R \cdot B = L\, Q(g, B)$ for some function L. Then from the definition, it follows that

$$\begin{aligned}& R(X,Y)BZ - BR(X,Y)Z \\ & \quad = L\,\{g(Y,BZ)X - g(X,BZ)Y - g(Y,Z)BX + g(X,Z)BY\}\end{aligned} \tag{3}$$

for any vector fields X, Y, Z on M.

Take a local frame field $\{e_1, e_2, e_3\}$ defined on a neighborhood $\mathcal{U}$ of x for any point $x \in M$ such that $Be_i = b_i e_i$ $(i = 1, 2, 3)$.

Then from (3) we obtain

$$b_j R(e_i, e_j)e_j - BR(e_i, e_j)e_j = L\,(b_j - b_i)e_i\,.$$

From this, we further have

$$(b_j - b_i)(g(R(e_i, e_j)e_j, e_i) - L) = 0 \tag{4}$$

for $i = 1, 2, 3$.

Let $\mathcal{U}_1 = \{p \in \mathcal{U} | \ b_1(p) = b_2(p) = b_3(p)\}$, $\mathcal{U}_2 = \{p \in \mathcal{U} | \ b_1(p) \neq b_2(p) \neq b_3(p) \neq b_1(p)\}$, $\mathcal{U}_3 = \{p \in \mathcal{U} | \text{ two of } b_i\text{'s are same}\}$. Then we see that $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$ is dense in $\mathcal{U}$. Now, we proceed our arguments in $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ in order.

- In $\mathcal{U}_1$, it is easily seen that $B = bI$ holds. Here we put $b_1 = b_2 = b_3 = b$.
- In $\mathcal{U}_2$, from (4) we get

$$K_{ij} = g(R(e_i, e_j)e_j, e_i) = L$$

for any $i \neq j$. Taking account that dim $M = 3$ (by virtue of Schur's lemma), we can see that M is of constant curvature L on $\mathcal{U}_2$.

By the assumption, $\mathcal{U} = \mathcal{U}_3$. Thus, we may assume that $b_1 = b_2 \neq b_3$. Then from (4) we get $g(R(e_1, e_3)e_3, e_1) = g(R(e_2, e_3)e_3, e_2) = L$.

Conversely, if B satisfies $b_1 = b_2 \neq b_3$ and $K_{13} = K_{23} = L$. Then by using (3), we get $R \cdot B = L\mathcal{Q}(g, B)$. □

Corollary 2.2 together with Lemma 2.3 imply the following criterion for pseudo-symmetry.

PROPOSITION 2.4. *A Riemannian* 3*-manifold* (M, g) *of non-constant curvature is pseudo-symmetric if and only if it is quasi-Einstein. Namely there exists a one-form* ω *such that the Ricci tensor field* ρ *has the form:*

$$\rho = a\ g + b\ \omega \otimes \omega\,.$$

Here a and b are functions. In this case M *satisfies* $R \cdot R = L\ \mathcal{Q}(g, R)$ *with* $2L = a + b$.

The preceding proposition can be rephrased as follows:

PROPOSITION 2.5. *A Riemannian* 3*-manifold of non-constant curvature is a pseudo-symmetric space with* $R \cdot R = L\ \mathcal{Q}(g, R)$ *if and only if the eigenvalues of the Ricci tensor locally satisfy the following relations* (*up to numeration*):

$$\rho_1 = \rho_2\,, \quad \rho_3 = 2L\,.$$

3. Real hypersurfaces

3.1. A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $\widetilde{M}_n(c)$. A complete and simply connected complex space form is a *complex projective space* $P_n(\mathbf{C})$, a *complex Euclidean space* $\mathbf{C}^n$ or a *complex hyperbolic space* $H_n(\mathbf{C})$, according as $c > 0$, $c = 0$ or $c < 0$.

3.2. Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$. Take a local unit normal vector filed N of M in $\widetilde{M}_n(c)$. Then the Levi-Civita connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following *Gauss formula* and *Weingarten formula*:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX\,, \quad X \in \mathfrak{X}(M)\,.$$

Here g is the Riemannian metric of M induced by the Kähler metric $\tilde{g}$ of the ambient space $\widetilde{M}_n(c)$. The $(1, 1)$-tensor field A is called the *shape operator* of M derived from N.

An eigenvector X of the shape operator A is called a *principal curvature vector*. The corresponding eigenvalue λ of A is called a *principal curvature*. As is well known, the Kähler structure $(J, \tilde{g})$ of the ambient space induces an almost contact metric structure (ϕ, ξ, η, g) on M. In fact, the *structure vector field* ξ of M and its dual 1-form η are defined by

$$\eta(X) = g(\xi, X) = \tilde{g}(JX, N), \quad X \in \mathfrak{X}(M).$$

The $(1, 1)$-tensor field ϕ is defined by

$$g(\phi X, Y) = \tilde{g}(JX, Y), \quad X, Y \in \mathfrak{X}(M).$$

One can easily check that this structure (ϕ, ξ, η, g) is an almost contact structure on M, that is, it satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{5}$$

From these conditions, one can deduce that

$$\phi\xi = 0, \quad \eta \circ \phi = 0.$$

It follows that

$$\nabla_X \xi = \phi AX.$$

Let $\widetilde{R}$ and R be the Riemannian curvature tensors of $\widetilde{M}_n(c)$ and M, respectively. From the expression of the curvature tensor $\widetilde{R}$ of $\widetilde{M}_n(c)$, we have the following *equations of Gauss and Codazzi*:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

A real hypersurface M is said to be *η-umbilical* if there exist functions λ and μ such that $A = \lambda I + \mu\eta \otimes \xi$.

3.3. By the Gauss equation, the Ricci tensor ρ of a real hypersurface M is described as

$$\rho(X, Y) = \frac{c}{4}((2n + 1)g(X, Y) - 3\eta(X)\eta(Y)) + hg(AX, Y) - g(A^2X, Y), \tag{6}$$

where h denotes the trace of the shape operator A.

A real hypersurface M is said to be *pseudo-Einstein* if the Ricci operator S has the form $S = aI + b\eta \otimes \xi$ with real constants a and b.

It is well known that there are no Einstein real hypersurfaces in $\widetilde{M}_n(c)$ with $c \neq 0$ and $n \geq 2$.

Recently, pseudo-Einstein real hypersurfaces in $P_2(\mathbf{C})$ and $H_2(\mathbf{C})$ are classified by the first named author, T. Ivey, H. S. Kim and Ryan (This gives a complete answer to [26, Question 9.5] posed by Niebergall and Ryan). In particular it is shown that every pseudo-Einstein real hypersurface is a Hopf hypersurface. Note that a real hypersurface $M \subset \widetilde{M}_n(c)$ is said to be *Hopf* if ξ is a principal curvature vector field.

THEOREM 3.1. ([7], [16], [18]) *The pseudo-Einstein real hypersurfaces in* $P_2(\mathbf{C})$ *and* $H_2(\mathbf{C})$ *are locally holomorphically congruent to one of the following hypersurfaces*:

- *a geodesic hypersphere in* $P_2(\mathbf{C})$ *or* $H_2(\mathbf{C})$,
- *a horosphere in* $H_2(\mathbf{C})$,
- *a tube of totally geodesic* $H_1(\mathbf{C}) \subset H_2(\mathbf{C})$,
- *a non-homogeneous tube of a certain holomorphic curve in* $P_2(\mathbf{C})$ *of radius* $\pi/\sqrt{4c}$ *or*
- *a Hopf hypersurface in* $H_2(\mathbf{C})$ *with* $A\xi = 0$ *which are constructed by a pair of Legendre curves in the unit* 3*-sphere.*

Clearly every 3-dimensional pseudo-Einstein real hypersurface is pseudo-symmetric (see Proposition 2.4).

3.4. Here, we recall the following two fundamental results (See *eg.*, [26]).

LEMMA 3.2. *If* ξ *is a principal curvature vector, then the corresponding principal curvature* α *is locally constant.*

LEMMA 3.3. *Assume that* ξ *is a principal curvature vector and the corresponding principal curvature is* α. *If* $AX = \lambda X$ *for* $X \perp \xi$, *then we have* $(2\lambda - \alpha)A\phi X = (\alpha\lambda + \frac{c}{2})\phi X$.

3.5. R. Takagi [33], [34] classified the homogeneous real hypersurfaces of $P_n(\mathbf{C})$ into six types. T. E. Cecil and Ryan [5] extensively studied a Hopf hypersurface, which is realized as tubes over certain submanifolds in $P_n(\mathbf{C})$, by using its focal map φ_r. By making use of those results and the mentioned work of Takagi, M. Kimura [19] proved the local classification theorem for Hopf hypersurfaces of $P_n(\mathbf{C})$ all of whose principal curvatures are constant.

THEOREM 3.4. ([19]) *Let* M *be a Hopf hypersurface of* $P_n(\mathbf{C})$. *Then* M *has constant principal curvatures if and only if* M *is locally congruent to one of the following*:

(A_1) *a geodesic hypersphere of radius* r, *where* $0 < r < \frac{\pi}{2}$,

(A_2) *a tube of radius* r *over a totally geodesic* $P_\ell(\mathbf{C})(1 \leq \ell \leq n-2)$, *where* $0 < r < \frac{\pi}{2}$,

(B) *a tube of radius* r *over a complex quadric* Q^{n-1} *and totally geodesic and Lagrangian imbedded real projective space* $P_n(\mathbf{R})$, *where* $0 < r < \frac{\pi}{4}$,

(C) *a tube of radius* r *over* $P_1(\mathbf{C}) \times P_{(n-1)/2}(\mathbf{C})$, *where* $0 < r < \frac{\pi}{4}$ *and* $n (\geq 5)$ *is odd*,

(D) *a tube of radius* r *over a complex Grassmannian* $\mathrm{G}_{2,5}(\mathbf{C})$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 9$,

(E) *a tube of radius* r *over a Hermitian symmetric space* SO(10)/U(5), *where* $0 < r < \frac{\pi}{4}$ *and* $n = 15$.

For complex hyperbolic space $H_n(\mathbf{C})$, J. Berndt [2] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant.

THEOREM 3.5. ([2]) *Let* M *be a Hopf hypersurface of* $H_n(\mathbf{C})$. *Then* M *has constant principal curvatures if and only if* M *is locally congruent to one of the following*:

(A_0) *a horosphere*,

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* $H_{n-1}(\mathbf{C})$,

(A_2) *a tube over a totally geodesic* $H_\ell(\mathbf{C})$ $(1 \leq \ell \leq n-2)$,

(B) *a tube over a totally geodesic and Lagrangian imbedded real hyperbolic space* $H_n(\mathbf{R})$.

We call simply type (A) for real hypersurfaces of type (A_1), (A_2) in $P_n(\mathbf{C})$ and ones of type (A_0), (A_1) or (A_2) in $H_n(\mathbf{C})$.

3.6. Next, we recall a class of non-Hopf real hypersurfaces in $P_n(\mathbf{C})$ or $H_n(\mathbf{C})$ named as

(R): *a foliated real hypersurface whose leaves are complex hyperplanes* $P_{n-1}(\mathbf{C})$ *or* $H_{n-1}(\mathbf{C})$, *respectively in* $P_n(\mathbf{C})$ *or* $H_n(\mathbf{C})$.

These are realized as *ruled real hypersurfaces* in $P_n(\mathbf{C})$ or $H_n(\mathbf{C})$. Namely, let $\gamma : I \to \widetilde{M}_n(c)$ be a regular curve in a complex space form $\widetilde{M}_n(c)$. Then for each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurfaces which is orthogonal to holomorphic plane Span$\{\dot{\gamma}, J\dot{\gamma}\}$. We have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. These ruled real hypersurfaces are non-Hopf hypersurfaces in non-flat complex space form and particularly in $P_n(\mathbf{C})$ the ruled real hypersurfaces are non-complete (see [21] for the case $P_n(\mathbf{C})$ and [1] for $H_n(\mathbf{C})$, respectively).

Although all the homogeneous real hypersurfaces in $P_n(\mathbf{C})$ are Hopf, there exist homogeneous ruled hypersurfaces in $H_n(\mathbf{C})$ [3], [22].

3.7. To close this section we introduce the notion of pseudo-parallel real hypersurface.

DEFINITION 3.1. A real hypersurface M in $\widetilde{M}_n(c)$ is said to be

- *parallel* if A is parallel ($\nabla A = 0$);
- *semi-parallel* if A is semi-parallel ($R \cdot A = 0$);
- *pseudo-parallel* if A is pseudo-parallel ($R \cdot A = L\, \mathcal{Q}(g, A)$).

In particular M is said to be *proper pseudo-parallel* if M is pseudo-parallel and $R \cdot A \neq 0$.

We refer to the reader [26] about general theory of differential geometry of real hypersurfaces in complex space forms.

4. Three dimensional pseudo-parallel real hypersurfaces

4.1. In this section, we prove

THEOREM 4.1. *A real hypersurface M in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$ is pseudo-parallel, that is, M satisfies $R \cdot A = L\, \mathcal{Q}(g, A)$ for some function L if and only if M is η-umbilical.*

PROOF. Let M be a real hypersurface in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$. Suppose that M is pseudo-parallel, *i.e.*, M satisfies $R \cdot A = L\, \mathcal{Q}(g, A)$ for some function L. Take a local principal frame field $\{e_1, e_2, e_3\}$ defined on a neighborhood $\mathcal{U}$ of x for any point $x \in M$ such that $Ae_i = \lambda_i e_i$ $(i = 1, 2, 3)$.

In [27], Niegerball and Ryan proved there does not exist Einstein real hypersurface in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$. Furthermore, it is a well-known fact that $P_n(\mathbf{C})$ or $H_n(\mathbf{C})$ does not admit totally umbilical real hypersurfaces. Thus from Lemma 2.3, we may assume that $\lambda_1 = \lambda_2 (= \lambda) \neq \lambda_3$ and $K_{13} = K_{23} = L$. By using the equation of Gauss, one can show that $K_{13} = K_{23}$ is equivalent to

$$\frac{c}{4}(1 + 3\phi_{31}^2) + \lambda\lambda_3 = \frac{c}{4}(1 + 3\phi_{32}^2) + \lambda\lambda_3 \,.$$

Here we have put $\phi_{ij} = g(\phi e_i, e_j)$. From this, it follows that

$$g(\phi e_1, e_3)^2 = g(\phi e_2, e_3)^2 \,. \tag{7}$$

Using (7) with the formula $\phi^2 = -I + \eta \otimes \xi$ and the fact that $\phi e_1 \perp e_1$, we see that $\phi e_1 = e_2$ or e_3 up to sign. But, from (7) we find that $\phi e_1 = e_3$ is impossible. Hence, we have $\phi e_1 = e_2$ and $e_3 = \xi$ up to sign. This says that M is η-umbilical such that $A = \lambda I + \mu \eta \otimes \xi$, where $\mu = \lambda_3 - \lambda$. □

REMARK 4.1. Since $\phi\xi = 0$, in the proof above, we have $L = \frac{c}{4} + \lambda\lambda_3$. If in addition, M is Hopf, then M satisfies $\lambda\lambda_3 + \frac{c}{4} = \lambda^2 \neq 0$ (see [26, Corollary 2.3]). From there we get the non-existence of semi-parallel Hopf hypersurfaces in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$.

4.2. Due to the results in [33], geodesic spheres in $P_n(\mathbf{C})$ are the only η-umbilical real hypersurfaces in $P_n(\mathbf{C})$ $(n \geq 2)$. Analogously, one can check that real hypersurfaces of type (A_0), (A_1) in $H_n(\mathbf{C})$ $(n \geq 2)$ are determined by η-umbilicity (see [25], [26]). Thus, we have

THEOREM 4.2. *Let M be a real hypersurface in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$. Suppose that M satisfies $R \cdot A = L\, \mathcal{Q}(g, A)$ for some function L. Then M is locally holomorphically congruent to one of the following*: (A_0) *a horosphere in $H_2(\mathbf{C})$*; (A_1) *a geodesic hypersphere in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$, a homogeneous tube over $H_1(\mathbf{C})$ in $H_2(\mathbf{C})$. In all the cases, the function L is a non-zero constant and hence all the examples above are non semi-parallel.*

REMARK 4.2. The tube of totally geodesic and Lagrangian imbedded $H_2(\mathbf{R}) \subset H_2(\mathbf{C})$ of radius $r = \ln(2 + \sqrt{3})$ is the only real hypersurface in non-flat complex space form $\widetilde{M}_2(c)$ with *two* distinct constant principal curvatures, but which is not pseudo-parallel (*cf.* [4, Proposition 3.2]).

COROLLARY 4.3. *There are no semi-parallel real hypersurfaces in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$.*

This result was proved in [27] in a different way.

In higher dimension ($n \geq 3$), the following non-existence theorem of semi-parallel real hypersurfaces are obtained by S. Maeda and M. Ortega.

THEOREM 4.4. ([23], [29]) *There are no semi-parallel real hypersurfaces in non-flat complex space form $\widetilde{M}_n(c)$.*

In view of this non-existence theorem together with our Theorem, the following problem naturally arises.

PROBLEM 4.1. *Classify pseudo-parallel real hypersurfaces in non-flat complex space forms $\widetilde{M}_n(c)$ with $n \geq 3$.*

5. Three dimensional pseudo-symmetric real hypersurfaces

5.1. In this section, we classify pseudo-symmetric Hopf hypersurfaces in $\widetilde{M}_2(c)$ with $c \neq 0$.

Let M be a Hopf hypersurface in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$. Then we may put $A\xi = \alpha\xi$ and

$$AU = \beta U, \quad A\phi U = \gamma\phi U$$

for a unit vector U orthogonal to ξ. Here, we remark that α is constant (see Lemma 3.2). By Lemma 3.3 we also have

$$(2\beta - \alpha)A\phi U = (\alpha\beta + c/2)\phi U\,. \tag{8}$$

From (6) it follows that

$$S\xi = p\xi,\ SU = qU,\ S\phi U = d\phi U\,, \tag{9}$$

where we have put $p = c/2 + h\alpha - \alpha^2,\ q = 5c/4 + h\beta - \beta^2,\ d = 5c/4 + h\gamma - \gamma^2$.

THEOREM 5.1. *A Hopf hypersurface M in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$ is pseudo-symmetric if only if $\alpha = 0$ or M is η-umbilical.*

PROOF. Suppose that M is pseudo-symmetric. Then from the relations (9) we may consider following three cases:

- $p = q$ if and only if

$$(\alpha - \beta)\gamma = 3c/4\,, \tag{10}$$

where we have used $h = \alpha + \beta + \gamma$. First, we look at the case $2\beta = \alpha$. Then together with (8) we can see that it occurs only in a horosphere in $H_2(\mathbf{C})$. Actually, $A = I + \eta \otimes \xi$ ($\alpha = 2\beta$ and $\beta = \gamma = \sqrt{-c}/2$) (*cf.* [2]). But, this does not satisfy (10). Thus, we assume that $2\beta \neq \alpha$. Then, together with (8) the equation (10) yields

$$\alpha\beta^2 + (2c - \alpha^2)\beta - 5/4\alpha c = 0\,.$$

Here, we can find at once that $\alpha \neq 0$. In fact, $\alpha = 0$ implies $c = 0$. We also see that it has at most three constant principal curvatures α, β_1, β_2. Thus, it suffices to consider a real hypersurface of type (A) or (B) in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$. But, we already know that for those cases $\beta_1\beta_2 = -c/4$ (*cf.* [2], [34]). After all, we conclude that this can not occur.

- $p = d$ if and only if $(\alpha - \gamma)\beta = 3c/4$. By similar arguments to the former case, we see that this case is also impossible.
- $q = d$ if and only if

$$\alpha(\beta - \gamma) = 0\,,$$

from which we get $\alpha = 0$ or $\beta = \gamma$. The latter case gives that M is η-umbilical, that is $A = \beta I + (\alpha - \beta)\eta \otimes \xi$. □

In our context, we give a simple proof of the following obtained in [27].

COROLLARY 5.2. *There does not exist a semi-symmetric* ($R \cdot R = 0$) *Hopf hypersurface in* $P_2(\mathbf{C})$ *or* $H_2(\mathbf{C})$.

PROOF. Suppose that M is semi-symmetric and Hopf. Then since M is 3-dimensional we can make use of the criterion (with $L = 0$), stated as Proposition 2.5. Thus it must satisfy $p = 0$ and $q = d$. First, we easily see that $\alpha = 0$ implies $c = 0$. So, we consider only the case $\beta = \gamma$. Then from the condition $c/2 + h\alpha - \alpha^2 = 0$ ($p = 0$) we get

$$\alpha\beta = -c/4\,, \tag{11}$$

where we have used $h = \alpha + \beta + \gamma$. And the equation (8) becomes

$$(2\beta - \alpha)A\phi U = c/4\phi U\,.$$

Multiplying α to both sides and using (11), then we get

$$(-c/2 - \alpha^2)A\phi U = \alpha c/4\,\phi U\,.$$

Since $\beta = \gamma$, taking the ϕU-component of this equation, then we get

$$(-c/2 - \alpha^2)\beta = c\alpha/4\,.$$

Multiplying α again and using (11), then we get $c = 0$, a contradiction. Thus, we have proved the assertion. □

Now we arraive at the main result of this paper.

THEOREM 5.3. *Let M be a Hopf hypersurface in* $P_2(\mathbf{C})$ *or* $H_2(\mathbf{C})$. *Suppose that M is pseudo-symmetric. Then M is locally isometric to one of the following*: (A_0) *a horosphere in* $H_2(\mathbf{C})$; (A_1) *a geodesic hypersphere in* $P_2(\mathbf{C})$ *or* $H_2(\mathbf{C})$, *a homogeneous tube over* $H_1(\mathbf{C})$ *in* $H_2(\mathbf{C})$; *or a non-homogeneous real hypersurface which is realized as a tube of radius* $\pi/\sqrt{4c}$ *over a certain holomorphic curve in* $P_2(\mathbf{C})$, *where the focal map* φ_r *has constant rank on M or a Hopf hypersurface in* $H_2(\mathbf{C})$ *with* $A\xi = 0$.

PROOF. Let M be a pseudo-symmetric Hopf hypersurface in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$. Then by Theorem 5.1, M is η-umbilical or $A\xi = 0$. We remark that there is a non-homogeneous Hopf hypersurface with $A\xi = 0$ which is a tube of radius $\pi/\sqrt{4c}$ over a certain Käher submanifold in $P_n(\mathbf{C})$, when its focal map has constant rank on M (*cf.* [5, Theorem 1], [21, Theorem]).

On the other hand, in $H_2(\mathbf{C})$, there exist Hopf hypersurfaces with $A\xi = 0$. Such hypersurfaces are constructed by pairs of Legendre curves in the unit 3-sphere and pseudo-Einstein [16].

Thus M is locally holomorphically congruent to a type (A_0) hypersurface, type (A_1) hypersurface, a non-homogeneous tube of radius $\pi/4$ over a certain holomorphic curve in $P_2(\mathbf{C})$ or a Hopf hypersurface with $A\xi = 0$ in $H_2(\mathbf{C})$.

Conversely these real hypersurfaces are pseudo-symmetric (*cf.* Theorem 3.1). □

REMARK 5.1. Hopf hypersurfaces in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$ with $A\xi = 0$ have non-constant principal curvatures. Moreover these hypersurfaces are non-homogeneous.

COROLLARY 5.4. *A Hopf hypersurface in* $P_2(\mathbf{C})$ *or* $H_2(\mathbf{C})$ *is pseudo-symmetric if and only if it is pseudo-Einstein.*

5.2. Let M be a ruled real hypersurface in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$. Since ξ is not a principal curvature vector field, we can define a (local) unit vector field V by

$$V = \frac{1}{|A\xi - \alpha\xi|}(A\xi - \alpha\xi), \quad \alpha = g(A\xi, \xi)\,.$$

Now we put $\nu = |A\xi - \alpha\xi| > 0$. Then one can see that $(\xi, V, \phi V)$ is a (local) orthonormal frame field of M. Then the shape operator A is expressed as

$$A\xi = \alpha\xi + \nu V \ (\nu \neq 0)\,, \tag{12}$$

$$AV = \nu\xi\,, \tag{13}$$

$$A\phi V = 0\,. \tag{14}$$

The principal curvatures $\{\lambda_1, \lambda_2, \lambda_3\}$ and their corresponding principal curvature vector fields $\{X_1, X_2, X_3\}$ of M are given by

- $\lambda_1 = (\alpha + \sqrt{\alpha^2 + 4\nu^2})/2$ with $X_1 = \nu V + \lambda_1\xi$,
- $\lambda_2 = (\alpha - \sqrt{\alpha^2 + 4\nu^2})/2$ with $X_2 = \nu V + \lambda_2\xi$,
- $\lambda_3 = 0$ with $X_3 = \phi V$.

Note that M has three distinct principal curvatures, because $\nu \neq 0$.

Now we study pseudo-symmetry of ruled real hypersurfaces. Since $h = \alpha$, from (6) and

(12), we have

$$S\xi = \rho_1 \xi\ , \quad \rho_1 = \frac{1}{2}c - \nu^2\ ,$$

$$SV = \rho_2 V\ , \quad \rho_2 = \frac{5}{4}c - \nu^2\ ,$$

$$S\phi V = \rho_3 \phi V\ , \quad \rho_3 = \frac{5}{4}c\ .$$

From these we see that

- $\rho_1 = \rho_2 \Longleftrightarrow c = 0$.
- $\rho_1 = \rho_2 \Longleftrightarrow \nu^2 = -3c/4$.
- $\rho_2 = \rho_3 \Longleftrightarrow \nu = 0$.

Since we assume that $c \neq 0$ and $\nu \neq 0$, we get the following result.

PROPOSITION 5.5. (i) *A ruled real hypersurface M in $P_2(\mathbf{C})$ does not admit pseudo-symmetric structure.*

(ii) *A ruled real hypersurface M in $H_2(\mathbf{C})$ is pseudo-symmetric if and only if* $0 < \nu^2 = -3c/4$. *In this case M is has constant Ricci eigenvalues* $(\rho_1, \rho_2, \rho_3) = (5c/4, 2c, 5c/4)$. *The principal curvatures $\{\lambda_1, \lambda_2, \lambda_3\}$ and their corresponding principal curvature vector fields $\{X_1, X_2, X_3\}$ of M are given by*

- $\lambda_1 = (\alpha + \sqrt{\alpha^2 - 3c})/2$ *with* $X_1 = \nu V + \lambda_1 \xi$,
- $\lambda_2 = (\alpha - \sqrt{\alpha^2 - 3c})/2$ *with* $X_2 = \nu V + \lambda_2 \xi$,
- $\lambda_3 = 0$ *with* $X_3 = \phi V$.

PROOF. The only possibility for a ruled real hypersurface M in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$ to be pseudo-symmetric is $0 < \nu^2 = -3c/4$. This implies that $c < 0$. □

PROBLEM 5.1. *Classify (or characterize) the base curve γ of a ruled real hypersurface M in $H_2(\mathbf{C})$ with $\nu^2 = -3c/4$.*

COROLLARY 5.6. *There are no pseudo-symmetric ruled real hypersurfaces with constant principal curvatures in $H_2(\mathbf{C})$.*

PROOF. The only ruled real hypersurfaces with constant primcipal curavtures in $H_2(\mathbf{C})$ are minimal ruled hypersurfaces induced by totally real horocycles [4]. Note that these hypersurfaces are the only homogeneous ruled real hypersurfaces in $H_2(\mathbf{C})$ (see [3], [22]). From [22, Theorem 6] (see also [4]), one can see that such real hypersurfaces do not satisfy the condition $\nu^2 = -3c/4$. □

5.3. In complex space forms, the following non-existence results due to S. Maeda, U-H. Ki, H. Nakagawa, Y. J. Suh are obtained.

THEOREM 5.7. ([20], [17]) *There are no Ricci semi-parallel real hypersurfaces in $\widetilde{M}_n(c)$ with $c \neq 0$ and $n \geq 3$.*

In dimension 3, since Ricci semi-parallelity is equivalent to semi-symmetry, there are no Ricci semi-parallel hypersurfaces in $\widetilde{M}_2(c)$ with $c \neq 0$.

Comparing these observations with our classification of pseudo-symmetric real hypersurfaces, the following problem would be of some interest and importance.

PROBLEM 5.2. *Classify real hypersurfaces with pseudo-parallel Ricci opeartor in* $\widetilde{M}_n(c)$ *with* $c \neq 0, n \geq 3$.

6. Concluding remarks

6.1. There are several generalizations of local symmetry other than semi-symmetry and pseudo-symmetry.

(N) Naturally reductive homogeneous spaces;

($\mathfrak{C}$) $\mathfrak{C}$-spaces, *i.e.*, Riemannian manifolds such that for any geodesic its corresponding Jacobi operator has constant eigenvalues along that geodesic;

(GO) Riemannian g.o spaces, *i.e.*, Riemannian manifolds all of whose geodesics are orbits of one-parameter subgroups of isometries;

(W) Weakly symmetric spaces, *i.e.*, Riemannian manifolds such that for any pair of points there exists an isometry interchanging these points;

(C) Commutative spaces , *i.e.*, Riemannian manifolds such that the algebra of all isometry-invariant differential operators is commutative;

(D) D'Atri spaces, *i.e.*, Riemannian manifolds whose geodesic symmetries are volume preserving up to sign.

The following inclusion relations are known;

$$\mathrm{N} \subset \mathrm{GO}, \quad \mathrm{W} \subset \mathrm{GO}, \quad \mathrm{W} \subset \mathrm{C}\,.$$

$$\mathrm{N}, \mathrm{GO}, \mathrm{W}, \mathrm{C} \subset \mathrm{D}, \quad \mathrm{N}, \mathrm{GO}, \mathrm{W}, \mathrm{C} \subset \mathfrak{C}\,.$$

Note that in dimension 3, N = GO. In the case of real hypersurfaces in $\widetilde{M}^n(c)$ $(c \neq 0)$, all of these classes are the same. Moreover the only real hypersurfaces of dimension ≥ 3, in the each class are type A hypersurfaces [11]. Our main theorem implies that the class of pseudo-symmetric real hypersurfaces in $\widetilde{M}_2(c)$ $(c \neq 0)$ and that of naturally reductive hypersurfaces has no inclusion relation.

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