QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXI, NUMBER 3
SEPTEMBER 2013, PAGES 549–572
S 0033-569X(2013)01324-5
Article electronically published on May 20, 2013

ON THREE METHODS FOR ANALYTIC LAPLACE INVERSION IN THE FRAMEWORK OF BROWNIAN MOTION AND THEIR EXCURSIONS

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Abstract. Working in a framework originating with Brownian motion and its excursions, this paper establishes a two-step Laplace inversion method for determining a function which is known through its transform after a convolution with another function with a known transform. The first step here has as its domain the class of parabolic cylinder functions, and it develops analytic Laplace inversion of their reciprocals. The second step pertains to convolutions on the positive reals with analytic factors where one of them is of exponential-order decay to zero at the origin; it develops two Laplace-inversion-based methods for handling these by asymptotic expansions. The results are shown to have applications to finance, yielding series representations and asymptotic expansions for the valuation and hedging of Parisian barrier options.

1. Introduction. This paper is about three mutually connected issues. First, from an analytic point of view, a two-step approach is established for isolating a function out of a convolution in the situation of a known Laplace transform of the convolution and of the complementary factor of the function therein, and an integral part of this approach includes two methods for handling convolutions by way of asymptotic expansions. The results have immediate applications to finance, and the second issue to be considered is the means for the valuation and hedging of a class of barrier options, the Parisian barrier options, they afford. Structurally, the first issue originates with Brownian motion, and the third contribution is to shed light on the explicit structure of minimal-length excursions of this stochastic process.

The functions to be studied, to be denoted by $H_{\alpha,\gamma}$, are defined on the positive reals. They depend on two complex parameters, α and γ, with α together with its square to be contained in the right-hand half-plane, and they are given only by their Laplace

Received October 4, 2011.
2010 *Mathematics Subject Classification.* Primary 44A10, 41Axx, 33C15; Secondary 60J65, 91G20.
Key words and phrases. Brownian motion subject to excursion conditions, constructive methods for handling convolutions, analytic Laplace inversion, Parisian barrier options.

transforms, namely:

$$\mathcal{L}(H_{\alpha,\gamma})(z) \stackrel{\text{def}}{=} \int_0^\infty \exp(-uz) H_{\alpha,\gamma}(u)\, du\,, \quad \mathrm{Re}(z) \gg 0\,.$$

These Laplace transforms originate in the explicitly given product:

$$\Psi(\sqrt{z}\,)\mathcal{L}(H_{\alpha,\gamma})(z) = \frac{\exp(-\alpha\sqrt{z})}{\sqrt{z}(\sqrt{z}+\gamma)}\,\Psi(-\sqrt{z}\,)\,, \quad \mathrm{Re}(z) \gg 0\,,$$

where Ψ is the parabolic cylinder function given by

$$\Psi(w) = \int_0^\infty x \exp\left(-\tfrac{1}{2}x^2 + wx\right) dx\,,$$

for any complex w. The defining equation of the Laplace transforms originates with a study of Azéma's martingales in [1], [2] by way of [10] and [11], where this theory is combined with that of the Brownian meander. It suffices to say that the distinguished role of the function Ψ here is that it encodes the motivating excursion-theoretic aspects. Its reciprocal is equal to the moment generating function of the random variable which characterizes occurence, or non-occurrence, of a minimal-length excursion of Brownian motion below the value 0, within a given time interval.

A first step in obtaining the functions $H_{\alpha,\gamma}$ explicitly consists in seeking the inversion of the transcendental function denominator, $\Psi(\sqrt{z}\,)$, of their Laplace transforms. This is addressed in Section 4, where we exploit the symmetry of the problem, as expressed by a functional equation for Ψ. We perform a Laplace inversion of this reciprocal using a geometric series. The result for $H_{\alpha,\gamma}$ is a finite-length expansion in terms of convolutions of the form

$$\varphi * \nu$$

with φ, morally, the Laplace inverse of the right-hand side of the Laplace equation defining $H_{\alpha,\gamma}$ and ν as a function real-analytic on $(0, \infty)$.

As a second step of our analytic inversion we develop expansions for these convolutions. The first method is to expand the factor $\mathcal{L}(\nu)$ of $\mathcal{L}(\varphi * \nu)$ into its Taylor series and establish when term-by-term Laplace inversion affords a convergent series representation of $\varphi * \nu$ (see Section 5 for details, in particular for how the latter series are in terms of the integral-order derivatives of φ). The second method to be established also focuses on the factor $\mathcal{L}(\nu)$ of $\mathcal{L}(\varphi * \nu)$ and demonstrates how its asymptotic expansions near ∞ induce, under Laplace inversion, asymptotic expansions near 0 of $\varphi * \nu$ which keep error terms effective (see Section 6 for details).

The results obtained yield series representations and asymptotic expansions for the valuation of Parisian barrier options (see [10] and [11]) as well as their hedging. We illustrate the situation of the so-called 'first standard case' in Section 3. This is done in order to connect with the second issue of the paper, in view of the recent increase in applications of the notion of Parisian-style delays, now also in insurance (see for example [9]).

2. A list of functions. The following three functions in the real variable u on $(0,\infty)$, the positive reals, are of constant use in this paper. For any complex α, γ with $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha^2)>0$,

$$g_{\alpha,\gamma}(u)=\exp(\alpha\gamma+\gamma^2u)\mathrm{Erfc}\left(\eta_{\alpha,\gamma}(u)\right)\quad\text{where}\quad \eta_{\alpha,\gamma}(u)=\frac{\alpha/2}{\sqrt{u}}+\beta\sqrt{u}\,,$$

and with $\mathrm{Erfc}\,(z)=(2/\sqrt{\pi}\,)\int_{[z,\infty)}\exp(-x^2)\,dx$ and

$$f_{\alpha,\gamma}(u)=\frac{1}{\sqrt{\pi u}}\exp\Big(-\frac{(\alpha/2)^2}{u}\Big)-\gamma\;g_{\alpha,\gamma}(u)\,,$$

and, for any integer $n\geq 0$, the functions ν_n given as n-fold convolutions by

$$\nu_n=\nu^{*(n)}\quad\text{where}\quad \nu(u)=(2/\sqrt{\pi}\,)\,\frac{\sqrt{u}}{2u+1}\,,$$

with the convention that ν_0 is the Dirac delta function at 0; see also Appendix A.

3. A motivation from finance. Paris barrier options furnish a particular class of barrier options distinguished by seeking to cushion the impact of the price S of their underlying security hitting their barrier $L\geq 0$. The new idea for this is as follows: these options require S to spend a minimum time D uninterruptedly above or below L before the option is knocked in or knocked out.

Drawing on [11], Sections 2, 4.2, and 5, the valuation of Paris barrier options reduces to that of the Paris down-and-in call. Referred to as the *Paris barrier option* in the sequel, this is the following European style contingent claim on S written at time t_0 and with maturity date T. Its payoff at T is that of a call option on S with exercise price K, namely $\phi(S_T)=\max\{0,S_T-K\}$ if the following holds: S is below L during a connected time subperiod of length at least $D>0$ of the monitoring period $[t_0,T]$ of the option.

For studying Paris barrier options we work in the Black-Scholes framework where the price S of a risky security is assumed to follow the process

$$S_u=\exp((r-\delta-\tfrac{1}{2}\sigma^2)u+\sigma B_u)\,,\quad u\in[0,\infty)\,.$$

Here B is a Brownian motion with respect to the risk neutral measure on the probability space underlying the model. To explain the constants, $r>0$ is the continuously compounding interest rate of a riskless security, a bond; δ depends on the risky security modelled and, for example, is equal to the dividend rate if S is the price of a stock; finally, $\sigma>0$ is the volatility of S. Combining the arbitrage pricing principle of risk neutral valuation with the basic valuation identity of [10], Section 4 (which technically speaking also entails changing by a Girsanov transformation from the given risk neutral measure to the measure Q for which the process $X=(1/\sigma)\log S$ is a Q-Brownian motion), the value $C_{\text{d-i}}$ of the Paris barrier option at an arbitrary time t of the monitoring period is then given by

$$C_{\text{d-i}}=e^{-(r+\varpi^2/2)\tau}\int_{\upsilon(S_t)}^{\infty}e^{\varpi x}(S_te^{\sigma x}-K)h_b^*(\tau,x)\,dx\,.$$

Here $\varpi=(1/\sigma)(r-\delta-\sigma^2/2)$ and $\tau=T-t$, we set $\upsilon(S_t)=(1/\sigma)\log(K/S_t)$, and h_b^*, with $b=(1/\sigma)\log(L/S_t)=a-X_t$, is a function on $[0,\infty)\times\mathbf{R}$, the normalized excursion law. This law is the object of study of [10], following [11], where it was determined at the level of Laplace transforms. In the present paper we concentrate on the situation: today's stock price S_t is equal to or above the options' barrier $L > 0$, whence $b \leq 0$, and the barrier is equal to or placed below the options' strike price $K > 0$, whence $v(S_t) \geq b$; we exclude the case where L and K are both equal to S_t. Then, [10], Proposition 2, p. 858 asserts

$$h_b(u, y) = (1/(2\sqrt{D}))\mathcal{L}^{-1}\Big(\frac{1}{\Psi(\sqrt{z})}\,\frac{\exp\big((b/\sqrt{D})\sqrt{z}\big)}{\sqrt{z}}\,I_-(y, z)\Big)\Big(\frac{u}{2D}\Big),$$

for any real number $u > 0$, where for $y \geq v(S_t)$,

$$I_-(y, z) := \int_0^\infty x \exp\Big(-\tfrac{1}{2}x^2 - \big|x + \frac{y-b}{\sqrt{D}}\,\big|\sqrt{z}\Big)\,dx = \exp\Big(-\frac{y-b}{\sqrt{D}}\sqrt{z}\Big)\Psi(-\sqrt{z}),$$

and the following reduction of the valuation problem occurs by direct computation.

Proposition 3.1. With the assumptions of this section we have the representation

$$\exp((r + \varpi^2/2)\tau)\,C_{\text{d-i}} = S_t H^*_{\varpi+\sigma}\big(\tau/(2D)\big) - K H^*_{\varpi}\big(\tau/(2D)\big),$$

where H^*_c, for any real c, is defined by $H_c(u) = \int_{[v(S_t),\infty]} \exp(cy) h^*_b(2Du, y)\,dy$ as a map on $[0, \infty)$, and, on setting $\alpha(y) = -(2b - y)/\sqrt{D}$, is given as a Laplace inverse by

$$H^*_c(u) = \tfrac{1}{2}\exp\big(cv(S_t)\big)\,\mathcal{L}^{-1}\Big(\frac{\exp\big(-\alpha\big(v(S_t)\big)\sqrt{z}\big)}{\sqrt{z}\,(\sqrt{z} - c\sqrt{D})}\,\frac{\Psi(-\sqrt{z})}{\Psi(\sqrt{z})}\Big)(u), \quad u \in (0, \infty).$$

The hedging of Paris barrier options is demonstrated by way of their Delta hedging as follows.

Proposition 3.2. With the assumptions of this section we have for $\Delta_{\text{d-i}} = \partial_{S_t} C_{\text{d-i}}$, the Delta of the Paris barrier option, the following representation:

$$\begin{aligned}\exp((r + \varpi^2/2)\tau)\,\Delta_{\text{d-i}} = &\big((1 - c/\sigma) H^*_c - 1/(\sigma\sqrt{D})\,L^*_c\big)\big|_{c=\varpi+\sigma}\big(\tau/(2D)\big)\\ &+ (K/S_t)\big((c/\sigma)\,H^*_c + 1/(\sigma\sqrt{D})\,L^*_c\big)\big|_{c=\varpi}\big(\tau/(2D)\big)\end{aligned}$$

with the definition $L^*_c = \sigma\sqrt{D}\,S_t\,\partial_{S_t} H^*_c + c\sqrt{D}\,H^*_c(u)$, for any real $c > 0$.

Hence a reduction occurs of $C_{\text{d-i}}$ and its partial derivatives to the functions H^*_c and their partial derivatives, related to the functions of the introduction by way of

$$2\,H^*_c = \exp\big(cv(S_t)\big)\,H_{\alpha,\gamma} \quad \text{where } \alpha = -(2b - v(S_t))/\sqrt{D} \text{ and } \gamma = -c\sqrt{D}.$$

We develop a two-layered approach to this in Sections 4 to 6.

4. First layer results: Finite convolution expansions. The first layer of our two layer representation demonstrates how to take care of the higher transendental function denominator of the Laplace transforms defining $H_{\alpha,\gamma}$, for any complex α, γ with $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha^2) > 0$, and constructs the functions $H_{\alpha,\gamma}$ as linear combinations of finitely many convolutions. Our precise result is as follows.

THEOREM 4.1. With the assumptions of this section we have for the value of an arbitrary function $H_{\alpha,\gamma}$ at $\tau \geq 0$ the following representation in terms of finitely many convolutions:

$$H_{\alpha,\gamma}(\tau) = \sum\nolimits_{1\leq n<2\tau} \frac{(-1)^{n-1}}{(2\pi)^{n/2}} (g_{\alpha,\gamma} * \nu_n)(\tau - \frac{n}{2}),$$

where the sum is over the finitely many integers $n \geq 1$ satisfying $n < 2\tau$.

This result is established in Section 4.2 using analytic Laplace inversion. The difficulties here arise from irrationalities encoding the excursion theoretic aspects of the construction as given expression to by the reciprocal of the function Ψ in the Laplace transform of $H_{\alpha,\gamma}$. The idea of our representation is to resolve these irrationalities by way of summing convolutions. This is done by adding at any integer time $n+1$ a new layer as follows: a fixed function is convolved with the n-fold convolution ν_n of another fixed function.

Referring to Section 3, for Paris barrier options our representations thus, in particular, identify how the effect of the excursion source on the Paris options' value and the options' Delta is built up over time. The precise result, in financial terms sufficient for addressing the valuation and Delta hedging of the instruments in the situation of Section 3 by way of Propositions 3.1 and 3.2 resepctively, is as follows.

COROLLARY 4.2. With the assumptions of this section, the values of the functions H_c^* and L_c^* at arbitrary real $\tau \geq 0$ are given in terms of finitely many convolutions as follows:

$$H_c^*(\tau) = \tfrac{1}{2}\exp(c\upsilon(S_t)) \sum\nolimits_{1\leq n<\frac{\tau}{D}} \frac{(-1)^{n-1}}{(2\pi)^{n/2}} (g_{\alpha(\upsilon(S_t)),-c\sqrt{D}} * \nu_n)(\frac{\tau}{2D} - \frac{n}{2}),$$
$$L_c^*(\tau) = \tfrac{1}{2}\exp(c\upsilon(S_t)) \sum\nolimits_{1\leq n<\frac{\tau}{D}} \frac{(-1)^{n-1}}{(2\pi)^{n/2}} (f_{\alpha(\upsilon(S_t)),-c\sqrt{D}} * \nu_n)(\frac{\tau}{2D} - \frac{n}{2}),$$

where the sums are over the finitely many integers $n \geq 1$ satisfying $n < \tau/D$.

4.1. *The key result for Laplace inversion.* The key difficulty in our analytic inversion proofs of the first layer representations is as follows: finding ways to handle the reciprocal of the function Ψ at square root arguments $\sqrt{z}$. In this section we therefore develop, first, how this reciprocal becomes a Laplace transform if suitably weighted. Second, we establish how the Laplace inversion of these functions is then effected by a series in terms of the Section A.2 functions ν_n. Furnishing a first and decisive step for establishing our representations, our precise result is as follows.

THEOREM 4.3. Let R be any function defined on a half-plane sufficiently deep within the right-hand complex half-plane, and assume that it is a Laplace transform and satisfies $R(z) = O(|z|^{-a})$ as z tends to ∞ in the right-hand half-plane with a real $a > 1/2$. For any real $\alpha \geq 0$, we then have

$$\mathcal{L}^{-1}\left(\frac{e^{-\alpha z}\, R(z)}{\Psi(\sqrt{z}\,)}\right) = \Upsilon_R,$$

with the function Υ_R for any real $u > 0$ given by

$$\Upsilon_P(u) = \sum\nolimits_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2\pi)^{n/2}}\, \mathbf{1}_{\left(\alpha+\frac{n}{2},\infty\right)}(u) \left(\mathcal{L}^{-1}\Big(\frac{R(z)}{\sqrt{z}}\Big) * \nu_{n-1}\right)\left(u-\alpha-\frac{n}{2}\right),$$

where only the summands corresponding to $n < 2(u-\alpha)$ are different from 0.

At the base of this result is the fact that Ψ satisfies a functional equation, the key identity, from Appendix B recalled to assert for any complex w:

$$\Psi(w) = \Psi(-w) + \sqrt{\pi}\, w \exp(\tfrac{1}{2}w)\,.$$

For the proof of the theorem, first reduce to the case $\alpha = 0$ using the shifting theorem for the Laplace transform. As the key step for the analytic Laplace inversion, develop the reciprocal of $\Psi(\sqrt{z}\,)$ into a geometric series in terms of the powers of $\Psi(-\sqrt{z}\,)$. Start by writing the above key identity for Ψ in the following equivalent form:

$$\Psi(\sqrt{z}\,) = \big(1/f(z)\big)\big(1 - p(z)\big),$$

with the definitions

$$f(z) = \exp(-z/2)/\sqrt{2\pi z} \quad \text{and} \quad p(z) = -f(z)\,\Psi(-\sqrt{z}\,).$$

Recall from Appendix B the leading term expansion $\Psi(-\sqrt{z}\,) = O(|z|^{-1})$ as z tends to ∞ in the complex plane with $\mathbf{R}_{\leq 0}$ deleted. Hence choose a_0 so large that $|f(z)| < 1$ and $|\Psi(-\sqrt{z}\,)| < 1$ for all $|z| > a_0$. Since then $|p(z)| < 1$, too, develop the fraction factor of

$$\frac{1}{\Psi(\sqrt{z}\,)} = f(z)\,\frac{1}{1 - p(z)}$$

as a geometric series. Thus obtain

$$1/\Psi(\sqrt{z}\,) = f(z)\sum\nolimits_{n=0}^{\infty} p^n(z) = f(z)\sum\nolimits_{n=0}^{\infty} (-1)^n f^n(z)\Psi^n(-\sqrt{z}\,).$$

Rewriting this series in terms of the Section A.2 functions N_n, for $|\arg(z)| < \pi$ equal to $N_n(z) = \Psi^n(-z^{1/2})/z^{n/2}$, the function to be Laplace inverted is given by the series

$$\frac{R(z)}{\Psi(\sqrt{z}\,)} = \frac{R(z)}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^{n/2}} \exp\Big(-\,\frac{n+1}{2}\,z\Big) N_n(z)\,.$$

Granting for a moment that Laplace inversion of this series can be effected term by term, the Laplace inverse is thus obtained as a series as follows:

$$\mathcal{L}^{-1}\Big(\frac{R(z)}{\Psi(\sqrt{z}\,)}\Big)(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^{n/2}}\,\mathbf{1}_{\left(\frac{n+1}{2},\infty\right)}(u)\,\mathcal{L}^{-1}\Big(\frac{R(z)}{\sqrt{2\pi z}}\,N_n(z)\Big)\Big(u - \frac{n+1}{2}\Big),$$

for any real $u > 0$. Using the product theorem for the Laplace transform, the Laplace inversion is reduced to that of the functions N_n. Applying Proposition A.1 of Appendix A, these correpond to the functions ν_n, and the expression for Υ_R of the theorem follows. It therefore remains to justify taking the inverse Laplace operator through the series.

For this justification, appeal to the complex inversion formula and interpret $\mathcal{L}^{-1}$ as given by contour integration over $\{\mathrm{Re}(z) = c\}$ for any sufficiently big positive real c:

$$\mathcal{L}^{-1}\Big(\frac{R(z)}{\Psi(\sqrt{z}\,)}\Big)(u) = (1/(2\pi i))\int_{c-i\infty}^{c+i\infty} e^{zu}\,\frac{R(z)}{\Psi(\sqrt{z}\,)}\,dz\,.$$

The idea is to construct a majorizing function for the integrand by adapting the contour in three stages as follows.

Having assured the convergence of the series in a half-plane $\{\mathrm{Re}(z) > a_0\}$, in particular, first choose c larger than this a_0. The convergence of the series to be Laplace inverted thus assured, majorizing it on the inversion contour term by term reduces to majorizing the N_n factors there. To get a hold of the Ψ-numerators of N_n, appeal once more to the leading term expansion of Ψ recalled above and further enlarge c if necessary such that $\Psi(-\xi^{1/2})$ is now bounded in absolute value by three times the absolute value of the leading term for any $\mathrm{Re}(\xi) = c$. The following majorization thus results:

$$\Big|\frac{1}{\Psi(\sqrt{z}\,)}\Big| \le \frac{\exp(-c/2)}{\sqrt{2\pi|z|}} \sum\nolimits_{n=0}^{\infty} q^n, \quad \text{where} \quad q = q(z) = \frac{3\exp(-c/2)}{\sqrt{2\pi|z|^3}}\,.$$

To be able to sum the geometric series, as a last step, further enlarge c if necessary such that $q(z) < 1$ for any complex z with $|z| \ge c$. Hence arrive at the majorization

$$\Big|\frac{1}{\Psi(\sqrt{z}\,)}\Big| \le \frac{\exp(-c/2)}{\sqrt{2\pi|z|}}\,\frac{1}{1-q(z)}\,,$$

valid in particular for any z with $\mathrm{Re}(z) = c$.

With c fixed, the second factor on the right-hand side of this last majorization converges to 1 for z going to ∞ on the contour. Still, the factor $z^{-1/2}$ does not give integrability of this majorizing function on any inversion contour. Multiplication with a function R such that $R(z) = O(|z|^{-a})$ as z tends to ∞, however, does give this integrability if $a > 1/2$. Then, finally, Laplace inversion term by term is justified using the Lebesgue Dominated Convergence Theorem. The proof of Theorem 4.3 is complete.

4.2. *Proof of Theorem* 4.1. The proof of the convolution expansion for $H_{\alpha,\gamma}$ of Theorem 4.1 proceeds by the Laplace inversion of its defining Laplace transform. This Laplace transform is the product of the quotient of $\Psi(-z^{1/2})$ over $\Psi(z^{1/2})$ and $G = G_{\alpha,\gamma}$, where

$$G_{\alpha,\gamma}(z) = \frac{\exp(-\alpha\sqrt{z}\,)}{\sqrt{z}\,(\sqrt{z}+\gamma)} = \mathcal{L}(g_{\alpha,\gamma})(z)\,, \quad \mathrm{Re}(z) \gg 0\,,$$

from Section A.1. Applying Theorem 4.3 with $R(z) = G(z)\Psi(-\sqrt{z}\,)$ and with α there equal to zero, then gives, after an index shift by 1 of the summation,

$$H_{\alpha,\gamma}(x) = \sum\nolimits_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2\pi)^{n/2}}\,\mathbf{1}_{\left(\frac{n}{2},\infty\right)}(x)\,\mathcal{L}^{-1}\big(GN_n\big)\Big(x-\frac{n}{2}\Big),$$

for any real $x > 0$. Here recall N_n as the nth power of the function $N(z) = z^{-1/2}\Psi(-z^{1/2})$ on the right-hand complex half-plane. Using the product theorem for the Laplace transform, the Laplace inverse of any product GN_n is the convolution of the Laplace inverse $g_{\alpha,\,\gamma}$ of G with that of N_n. From Proposition A.1 the Laplace inverse of N_n is the function ν_n. The proof of Theorem 4.1 is complete.

5. Series representation for the convolutions off $[0,\frac{1}{2}]$**.** Continuing from Section 4.1, this section demonstrates a first principal approach to expansions of convolutions

$$\varphi * \nu_n, \quad \text{where} \quad \varphi \in \{f_{\alpha,\gamma},\ g_{\alpha,\gamma}\}\,,$$

with integral $n \ge 0$ and complex α, γ such that $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha^2) > 0$. It is series representations in terms of integral-order derivatives of the factor φ that are to be developed here as follows.

THEOREM 5.1. In the situation of this section, with α, γ complex such that $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha^2) > 0$ in particular, we have for any integer $n \geq 1$ the following representations of the convolutions $g_{\alpha,\gamma} * \nu_n$ and $f_{\alpha,\gamma} * \nu_n$ on $(\frac{1}{2}, \infty)$ by a pointwise absolutely convergent series in terms of the integral-order derivatives of $g_{\alpha,\gamma}$ and $f_{\alpha,\gamma}$ of Appendix C:

$$g_{\alpha,\gamma} * \nu_n = \sum\nolimits_{\ell=-\ell_0}^{\infty} a_{n,n+2\ell+1}\, f_{\alpha,\gamma}^{(\ell)} + \sum\nolimits_{\ell=-\ell_1}^{\infty} a_{n,n+2\ell}\, g_{\alpha,\gamma}^{(\ell)}\,,$$

where $\ell_0 = [(n+1)/2]$ and $\ell_1 = [n/2]$, and

$$f_{\alpha,\gamma} * \nu_n = \sum\nolimits_{k=-k_0}^{\infty} a_{n,n+2k}\, f_{\alpha,\gamma}^{(k)} + \sum\nolimits_{k=-k_1}^{\infty} a_{n,n+2k-1}\, g_{\alpha,\gamma}^{(k)}\,,$$

where $k_0 = [n/2]$ and $k_1 = [(n-1)/2]$. The coefficients $a_{n,k}$ here are in terms of gamma function values and, for any integer $n \geq 0$, are recursively defined by

$$a_{1,k} = (-1)^k (2^{k/2}/k!)\Gamma\big(k/2+1\big) \quad \text{and} \quad a_{n+1,k} = \sum\nolimits_{\ell=0}^{k} a_{1,\ell}\, a_{n,k-\ell}\,,$$

for any integer $k \geq 0$.

This result is to be established in Section 5.4. It furnishes a paradigm consequence of the principal approach to analytic Laplace inversion based on the deformation of contours to be developed first in Sections 5.1 to 5.3. This approach applies to products of Laplace transforms where one factor, $\mathcal{L}(\varphi)$, is decreasing to 0 at an exponential rate near ∞ and the remaining ones combine into a function analytic on a sliced complex plane off of a compact neighbourhood of 0. The idea is to expand this second factor into a Taylor series, and for the series for the product thus obtained establish the validity of Laplace inversion term-by-term in a way that preserves effectiveness of remainder terms.

For φ equal to $f_{\alpha,\gamma}$ or $g_{\alpha,\gamma}$, convergence of the resulting series can be established at positive real arguments off of the interval $[0, \frac{1}{2}]$, as reported in Theorem 5.1. The result is to be complemented by an expansion valid on that piece of the non-negative reals, as it is to be developed in Section 6.

5.1. *A contour integral formula for Laplace inversion.* This section develops the alternative perspective on Laplace inversion provided by deformations of inversion contours.

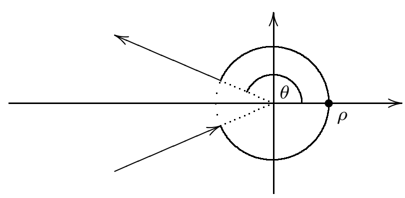

FIG. 5.1. The Hankel contour $C_{\theta,\rho}$.

We work with the Hankel contours $C_{\theta,\rho}$ with angles θ in $(\frac{\pi}{2}, \pi]$ and radii $\rho > 0$ as depicted in Figure 5.1, and the contours $C_\rho = C_{\pi,\rho}$ in particular. In this framework we have the following version of the complex inversion formula.

PROPOSITION 5.2. Let R be any meromorphic function on $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$, the complex plane with the non-positive real line deleted, for which there is a real $a > 1$ such that $R(z) =$

$O(|z|^{-a})$ as z tends to ∞ there. If R moreover represents a Laplace transform on the right-hand half-plane, its Laplace inverse at any real $u > 0$ is given by the contour integral

$$\mathcal{L}^{-1}(R)(t) = (1/(2\pi i)) \int_C e^{uz} R(z)\, dz\,,$$

where C is any Hankel contour $C_{\theta,\rho}$ with $\rho \geq \rho_0$, or their limits C_ρ.

To prove this result, choose ρ_0 as any sufficiently large positive real such that outside the open ball U in the origin with radius ρ_0, the function R is analytic and its absolute value is bounded satisfying $|R| \leq m_U$. Choose an arbitrary contour $C = C_{\theta,\rho}$ with $\rho \geq \rho_0$, and apply the complex inversion formula to R in a way that its contour of integration $\iota = \iota_{\xi_0}$ traces out the parallel to the imaginary axis through the real $\xi_0 > \rho_0$. Deform ι into C in the standard way by connecting ι and C_-, the contour C travelling in a reverse direction, with two subarcs of arbitrary radii $P > \xi_0$ and considering the closed contours γ_P thus obtained. By the global Cauchy theorem, the integral of $\exp^u R$ over any such γ_P is equal to 0. By the hypothesis on the growth of R (and distinguishing the situation in the left-hand and the right-hand half-plane), the integrals of $\exp^u R$ over the subarcs of γ_P go to 0 as P tends to ∞, and the proof of Proposition 5.2 is complete.

REMARK 5.3. Proposition 5.2 applies in particular to functions on $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$ of the form $R = MN_n$, where M is a meromorphic function bounded in absolute value off of some compact neighbourhood of 0 and $N_n(z) = (\Psi(-\sqrt{z}\,)/\sqrt{z}\,)^n$, for integral $n \geq 1$, as in Section A.2. This is a consequence of the leading term expansion for Ψ of Appendix B.

5.2. *A contour integral series expansion for Laplace inversion.* Following Section 5.1, this section studies when a series representation for the Proposition 5.2 Laplace inverses can be obtained by taking the integration through the Taylor series of one of its factors. Recalling the coefficients $a_{n,k}$ from Theorem 5.1, our result is as follows.

PROPOSITION 5.4. Let M be any meromorphic function on $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$, the complex plane with the non-positive real line deleted, whose absolute value is bounded outside some compact neighborhood of zero. Then any product MH_n is a Laplace transform. Its Laplace inverse at any real $u > \frac{1}{2}$ is given by the absolutely convergent series of contour integrals,

$$\mathcal{L}^{-1}(MN_n)(u) = \sum\nolimits_{k=0}^{\infty} a_{n,k}\, (1/(2\pi i)) \int_C e^{uz} M(z)\, z^{(k-n)/2}\, dz\,,$$

where C is any Hankel contour $C_{\theta,\rho}$ with $\rho \geq \rho_0$. If in addition M is obtained as a Laplace transform on the right-hand half-plane and is exponentially decreasing to 0 with the absolute value of its argument going to ∞, then we have for any real $u > \frac{1}{2}$ the absolutely convergent series representation

$$\mathcal{L}^{-1}(MH_n)(u) = \sum\nolimits_{k=0}^{\infty} a_{n,k}\, \mathcal{L}^{-1}\big(M(z)\, z^{(k-n)/2}\big)(u)\,.$$

At the base of the proposition's series is the fact that $a_{n,k}$ is the kth Taylor coefficient of the nth power of $\Psi(-w)$. The precise result, whose proof is omitted here, is as follows.

LEMMA 5.5. Any function N_n has the absolutely and compactly convergent series representation: $N_n(z) = \sum_{k=0}^{\infty} a_{n,k}\, z^{(k-n)/2}$ for any z in $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$.

Proof of Proposition 5.4. The first idea is to apply the Proposition 5.2 complex inversion formula and express the Laplace inverse of $R = MN_n$ as a contour integral over a Hankel contour C_ρ with any suitably big radius ρ:

$$\mathcal{L}^{-1}(MN_n)(u) = (1/(2\pi i)) \int_C e^{uz} M(z) N_n(z)\, dz\,.$$

This is justified since, as a consequence of the Appendix B leading term expansion for Ψ, we have $N_n(z) = O(|z|^{-3n/2})$ as z tends to ∞ in $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$.

The second idea is then to develop the factor N_n of the integrand into its Lemma 5.5 series and take the integral through the resulting series. For making this rigorous we proceed iteratively. As a first step, develop a single factor N_1 of N_n into its corresponding Lemma 5.5 series. On a formal term-by-term integration of the resulting series,

$$\mathcal{L}^{-1}(MH_n)(u) = \sum\nolimits_{k=0}^{\infty} \frac{a_{n,k}}{2\pi i} \int_{C_\rho} e^{uz} z^{(k-n)/2} M(z) N_{n-1}(z)\, dz\,,$$

and we now have to ask about two things. First, is taking this the integral through the summation justified and does it yield an (absolutely) convergent series? Second, can the construction be iterated in any summand, with each single factor H_1 remaining there?

Setting $R = MN_{n-1}$ where $N_0 = 1$, develop the remaining factor N_1 of MN_n into its above series to obtain

$$(MN_n)(z) = \frac{R(z)}{\sqrt{z}} \sum\nolimits_{k=0}^{\infty} a_k z^{k/2}\,,$$

in particular for $z = x\exp(i\pi)$. Summing the absolute values of the terms of the series for $\Psi(-w)$ gives $\Psi(|w|)$. Thus the absolute value of the series factor in the above identity can be majorized by $\Psi(x^{1/2})$ when $z = x\exp(i\pi)$. In Appendix B combining the key identity for Ψ with its leading-term expansion, $\Psi(x^{1/2})$ equals the sum of $2x^{-1}+R_2(ix^{1/2})$ and $(2\pi x)^{1/2}\exp(x/2)$. For $z = x\exp(i\pi) = -x$ we thus obtain

$$\begin{aligned}\Big| \exp(-ux)\frac{R(-x)}{i\sqrt{x}} \sum\nolimits_{k=0}^{\infty} a_k\,(-x)^{k/2} \Big| \\ \leq \big|R(-x)\big| \,\frac{\exp(-ux)}{|x|^{1/2}} \Big(\frac{2}{x} + \big|r_2(ix^{1/2})\big| + \sqrt{2\pi x}\,\exp(x/2)\Big).\end{aligned}$$

With R of at most polynomial growth as x tends to ∞, the asymptotic behaviour for large x on the right-hand side of this inequality is determined by that of the product of its two exponential function factors. The majorizing function thus decays to 0 as x tends to ∞ iff $u > \frac{1}{2}$, and in this case it is integrable with respect to x on any interval $[\rho, \infty)$. Using the Lebesgue Dominated Convergence Theorem, this implies that for $u > \frac{1}{2}$ the inversion integral over any Hankel contour C_R of the above series $R(z)\sum_k a_k z^{(k-1)/2}$ can be computed term by term. The result is an absolutely convergent series with kth terms given by

$$a_k = (1/(2\pi i)) \int_{C_R} e^{uz} \big((MN_{n-1})(z) z^{k-1/2}\big) dz\,.$$

If $n > 1$, iterate the above argument with the function R equal to $(MN_{n-2})(z)z^{k-1/2}$, for any z in $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$. Iterating n times, the first series of the proposition results in collecting terms according to powers of z.

The second series of the proposition follows if the contour integrals in the first series are deformations of complex inversion integrals. If M is exponentially decreasing to 0 with the absolute value of its argument growing to ∞, this is true. Indeed, just reverse the deformation argument of Proposition 5.2 mutatis mutandis. The proof of Proposition 5.4 is complete.

5.3. *Two intermediate contour integral series.* As a second step of the argument, this section specializes the series of Proposition 5.4 to the functions $F_{\alpha,\gamma}$ and $G_{\alpha,\gamma}$ of Section A.1; namely, $F_{\alpha,\gamma}(z) = \exp(-\alpha\sqrt{z})/(\sqrt{z}+\gamma)$ and $G_{\alpha,\gamma}(z) = F_{\alpha,\gamma}(z)/\sqrt{z}$, for real complex α, γ with $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha^2) > 0$. Our result is as follows.

Lemma 5.6. With the assumptions of this section we have for any integer $n \geq 1$ and any real $u > 1/2$ the following two absolutely convergent series representations:

$$f_{\alpha,\gamma} * \nu_n = \sum\nolimits_{k=-k_0}^{\infty} \frac{a_{n,n+2k}}{2\pi i} \int_C e^{uz} F_{\alpha,\gamma}(z) z^k \, dz + \sum\nolimits_{k=-k_1}^{\infty} \frac{a_{n,n+2k-1}}{2\pi i} \int_C e^{uz} G_{\alpha,\gamma}(z) z^k \, dz \,,$$

where $k_0 = [n/2]$ and $k_1 = [(n-1)/2]$, and

$$g_{\alpha,\gamma} * \nu_n = \sum\nolimits_{\ell=-\ell_0}^{\infty} \frac{a_{n,n+2\ell+1}}{2\pi i} \int_C e^{uz} F_{\alpha,\gamma}(z) z^\ell \, dz + \sum\nolimits_{\ell=-\ell_1}^{\infty} \frac{a_{n,n+2\ell}}{2\pi i} \int_C e^{uz} G_{\alpha,\gamma}(z) z^\ell \, dz \,,$$

where $\ell_0 = [(n+1)/2]$ and $\ell_1 = [n/2]$.

Simplifying notation to $F = F_{\alpha,\gamma}$ and $G = G_{\alpha,\gamma}$, our argument proceeds by reduction to the series of Proposition 5.4 with M there equal to F and G respectively. Indeed, F and G are meromorphic functions on $\mathbf{C} \setminus \mathbf{R}$ bounded outside an arbitray disc centered at 0 which contains their singularities. With the assumptions of Proposition 5.4 thus satisfied, we obtain the following two absolutely convergent series representations:

$$f_{\alpha,\gamma} * \nu_n(u) = \mathcal{L}^{-1}(FN_n)(u) = \sum\nolimits_{k=0}^{\infty} \frac{a_{n,k}}{2\pi i} \int_C e^{uz} F(z) z^{\frac{1}{2}(k-n)} \, dz \,,$$

$$g_{\alpha,\gamma} * \nu_n(u) = \mathcal{L}^{-1}(GN_n)(u) = \sum\nolimits_{k=0}^{\infty} \frac{a_{n,k}}{2\pi i} \int_C e^{uz} F(z) z^{\frac{1}{2}(k-(n+1))} \, dz \,,$$

where C is any Hankel contour as specified in Proposition 5.2.

For the bookkeeping of the half-integral exponents of these series, note the following. For any non-negative integer m, the difference $k-m$ is even and equal to 2ℓ for integral ℓ iff $k = m+2\ell$, and the minimal such ℓ is the negative of [m/2], the Gauß bracket of $m/2$. Analogously, $k-m$ is odd and equal to $2\ell-1$ for integral ℓ iff $k = m+2\ell-1$, and the minimal such ℓ is the negative of $[(m-1)/2]$.

Applying this with $m = n$ finishes the first series of the lemma; applying this with $m = n+1$, the second one follows, and the proof of Lemma 5.6 is complete.

5.4. *Proof of Theorem* 5.1. Proceeding by reduction to Lemma 5.6, there are two steps in establishing the series representations of Theorem 5.1. First, interpret the contour integrals in the Lemma 5.6 series as Laplace inverses. Second, interpret these Laplace inverses as derivatives. Indeed, since $\mathrm{Re}(\alpha) > 0$ the functions $F_{\alpha,\gamma}$ and $G_{\alpha,\gamma}$ there are exponentially decreasing to 0 near ∞, and an application of Proposition 5.2, as for proving the second series in Proposition 5.4, shows:

$$(1/(2\pi i)) \int_C e^{uz} F_{\alpha,\gamma}(z) z^k \, dz = \mathcal{L}^{-1}(F_{\alpha,\gamma}(z) z^k)(u)\,,$$
$$(1/(2\pi i)) \int_C e^{uz} G_{\alpha,\gamma}(z) z^k \, dz = \mathcal{L}^{-1}(G_{\alpha,\gamma}(z) z^k)(u)\,,$$

for any integer k. From Section A.1 recall $f_{\alpha,\gamma}$ as the Laplace inverse of $F_{\alpha,\gamma}$ and $g_{\alpha,\gamma}$ as that of $G_{\alpha,\gamma}$. Once more since $\mathrm{Re}(\alpha) > 0$, the derivatives to arbitrary non-negative order of these two Laplace inverses are checked to be 0 at the origin $u = 0$. For any integer $k \geq 0$, the Laplace inverse of z^k times any of the functions F or G is thus equal to the kth derivative of the respective functions $f_{\alpha,\gamma}$ and $g_{\alpha,\gamma}$. Referring to Appendix C, this holds true for integers $k < 0$, and the proof of Theorem 5.1 is complete.

6. Asymptotic expansions for the convolutions on $[0, \frac{1}{2}]$.

Continuing from Section 4.1 in a way complementary to Section 5, this section demonstrates a second principal approach to expansions of convolutions of the form

$$\varphi * \nu_n \quad \text{where} \quad \varphi \in \{f_{\alpha,\gamma},\ g_{\alpha,\gamma}\}\,,$$

with integral $n \geq 0$ and complex α, γ such that $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha^2) > 0$. To be developed are asymptotic expansions near 0 of a particular shape, namely in terms of negative-integral-order derivatives of the factor φ. To be established in Section 6.3, and with some notation to be explained after the statement, our precise result is as follows.

THEOREM 6.1. In the setting of this section, any convolution $g_{\alpha,\gamma} * \nu_n$ on $(0, \infty)$ with integral $n \geq 1$ has an asymptotic expansion near zero in terms of the negative-integral-order derivatives of $g_{\alpha,\gamma}$ and $f_{\alpha,\gamma}$ of Appendix C as follows. For n even, we have

$$g_{\alpha,\gamma} * \nu_n = \sum\nolimits_{k=n}^{N+n-1} c_{N,k}^{(n)}\, {g_{\alpha,\gamma}}^{(-(k+\frac{1}{2}n))} + R_{n,N}\,,$$

while for n odd we have

$$g_{\alpha,\gamma} * \nu_n = \sum\nolimits_{k=n}^{N+n-1} c_{N,k}^{(n)}\, {f_{\alpha,\beta}}^{(-(k+\frac{1}{2}(n+1)))} + R_{n,N}\,,$$

for any integer $N \geq 0$. For fixed real $\varepsilon > 0$ and if $N \geq 2$, the remainder terms satisfy for any real x in $(0, \varepsilon)$ if n is even:

$$\bigl|R_{n,N}(x)\bigr| \leq e\, C_{n,N}(\varepsilon)\, \frac{x^{N+3n/2}}{2N+3n}\, \exp\Bigl(-\frac{(\alpha/2)^2}{x}\Bigr),$$

and if n is odd:

$$\bigl|R_{n,N}(x)\bigr| \leq e\, C_{n,N}(\varepsilon)\, \Bigl(|\gamma| + \frac{1}{\sqrt{\pi x}}\Bigr)\, \frac{x^{N+(3n+1)/2}}{2N+3n+1}\, \exp\Bigl(-\frac{(\alpha/2)^2}{x}\Bigr).$$

To explain the notation, fix any integers n, $N \geq 0$. Using multi-index notation, the coefficients $c_{n,k}$, for any integer $k \geq 0$, are then defined as follows:

$$c_{N,k}^{(n)} = \sum \tbinom{n}{I} c^I,$$

where the sum is taken over all N-tuples $I = (i_1, \ldots, i_N)$ of integers $i_\ell \geq 0$ with $|I| = \sum_{\ell=1}^N i_\ell = n$ and $\sum_{\ell=1}^N i_\ell \ell = k$. For these I then

$$c^I = c_1^{i_1} \cdots c_N^{i_N} \qquad \text{where} \qquad c_m = (-1)^{m+1} \frac{2^m}{\sqrt{\pi}} \Gamma\Big(m + \frac{1}{2}\Big),$$

for any integer $m \geq 0$. In the estimate for the remaining terms, the $C_{n,N}(\varepsilon)$ is given by

$$C_{n,N}(\varepsilon) = \sum_{k=n+N}^{nN} |c|_{N,k}^{(n)} \varepsilon^{k-(n+N)} + \sum_{k=1}^{n} \binom{n}{k} \Big(\sum\nolimits_{\ell=1}^{N} \varepsilon^{\ell-1} |c_\ell|\Big)^{n-k} \Big(2\left((2N+1)!/N!\right)\Big)^k,$$

with the definition $|c|_{N,k}^{(n)} = \sum \binom{n}{I} |c|^I$, where the sum is taken over all N-tuples $I = (i_1, \ldots, , i_N)$ of integers $i_\ell \geq 0$ such that $|I| = n$ and $\sum_{1 \leq l \leq N} i_\ell \ell = k$, and where $|c|^I$ is multi-index notation for the product $|c_1|^{i_1} \cdots |c_N|^{i_N}$.

Note the following small-N examples for the coefficients $c_{N,k}^{(n)}$:

$$c_{1,n}^{(n)} = c_1^n\,, \quad c_{2,n+k}^{(n)} = \tbinom{n}{k} c_1^{n-k} c_2^k\,,$$

for $0 \leq k \leq (2-1)n = n$,

$$c_{3,n+k}^{(n)} = \sum\nolimits_{\ell=\mathrm{ceil}(k/2)}^{n} \tbinom{n}{\ell} \tbinom{\ell}{k-\ell} c_1^{n-\ell} c_2^{2\ell-k} c_3^{k-\ell}\,,$$

for $0 \leq k \leq (3-1)n = 2n$, where $\mathrm{ceil}(a) = \{z \in \mathbf{Z} | a \leq z\}$, for arbitrary real a, is the smallest integer above a, and where $c_1 = 1$, $c_2 = -3$, and $c_3 = 15$.

To comment on our result, Theorem 6.1 gives expression to the perspective taken as follows: first work on the level of the explicitly given Laplace transforms where $\mathcal{L}(\nu_n)$ possesses an asymptotic expansion near ∞; see Section 6.2 for all this. The representing-near-∞ properties of this expansion are preserved on multiplication by factors $\mathcal{L}(\varphi)$, as these decay to 0 at an exponential rate near ∞. As a second step, then use Laplace inversion to thus induce an asymptotic expansion near 0 of $\varphi * \nu_n$ which preserves effectiveness of error estimates; details for this are to be worked out in Section 6.1. Refer to Remark 6.6 for the case $\varphi = f_{\alpha,\gamma}$.

6.1. *Preserving error terms on Laplace inversion.* The thrust of the approach is to derive the asymptotic behaviour for smaller arguments of a function from the asymptotic behaviour of its Laplace transform for larger values by keeping the error term effective on Laplace inversion. Our key result for this, to be established in this section, is as follows:

Proposition 6.2. Let f, h be functions of exponential type whose Laplace transforms are holomorphic on the right-hand half-plane and there satisfy

$$\mathcal{L}(f)(z) = \mathcal{L}(h)(z) + r(z), \quad \mathrm{Re}(z) > 0\,.$$

Suppose there is a real $\alpha > 1$ such that for any real $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that $|r(z)| \leq C(\varepsilon) \cdot |z|^{-\alpha}$ if $|z| \geq 1/\varepsilon$. Then f and h satisfy

$$f(t) = h(t) + R(t), \quad t \in (0, \infty)\,,$$

where the remainder term R satisfies

$$|R(t)| \leq C(\varepsilon) \max\Big\{\frac{e}{2}, \frac{e}{\pi}\,\frac{1}{\alpha-1}\Big\}\, t^{\alpha-1}, \quad t \in (0, \varepsilon)\,.$$

REMARK 6.3. If f, h are holomorphic only on a half-plane $\{z|\mathrm{Re}(z) > a_0\}$ with $a_0 > 0$, Proposition 6.2 remains valid restricting to ε with $0 < \varepsilon < 1/a_0$ and with the estimate

$$|R(t)| \le C(\varepsilon)\ \max\left\{\frac{e}{2}, \frac{e}{\pi}\,\frac{1}{(\alpha-1)}\left(1-(\varepsilon a_0)^2\right)^{\frac{1-\alpha}{2}}\right\} t^{\alpha-1}, \quad t \in (0,\varepsilon)\,.$$

Proofs. Subtracting h from f, a reduction occurs to the case with h and $\mathcal{L}(h)$ the respective functions 0. Put $F = \mathcal{L}(f)$, and for fixed real $\varepsilon > 0$ let $a_0 > 0$ be an arbitrary real with $a_0 < 1/\varepsilon$. In the complex inversion formula take the parallel $\{\mathrm{Re}(z) = a_0\}$ through a_0 to the imaginary axis as the path of integration. Consider the following deformation. For any $0 < t < \varepsilon$ let $S = S_t$ be the point on the line $\{\mathrm{Re}(z) = a_0\}$ with absolute value $\rho = 1/t$ which is in the upper half-plane. Replace that piece of $\{\mathrm{Re}(z) = a_0\}$ between S and its complex conjugate by the corresponding part γ_ρ of the circle with center 0 and radius ρ travelled in counterclockwise direction.

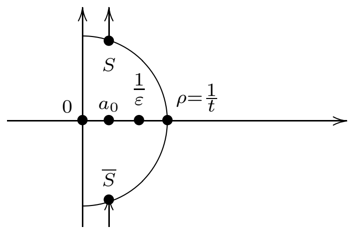

FIG. 6.1. Deformed inversion contour.

On γ_ρ the maximum of $|\exp(tz)|$ equals $\exp(\rho t) = e$. With $|z| = \rho > 1/\varepsilon$, majorize $|F(z)|$ by $C(\varepsilon)\,|z|^{-\alpha}$ times the length of γ_ρ. This length is at most one half of the circumference $2\pi\rho$ of the circle with radius ρ. Thus estimate the inversion integral over the contour γ_ρ by

$$\left|(1/(2\pi i))\int_{\gamma_\rho} e^{tz}\,F(z)\,dz\right| \le \frac{e}{2\pi}\,C(\varepsilon)|z|^{-\alpha}\,\frac{2\pi\rho}{2} = \frac{e}{2}\,C(\varepsilon)t^{\alpha-1},$$

recalling $|z| = \rho = 1/t$ for the last equality. To estimate the size of the inversion integral for F over the rest of the inversion contour, it is by symmetry sufficient to estimate the inversion integral over the subcontour from S to $a_0+i\infty$. For any z on this subcontour, $|z| \ge |S|$. Since $|S| = \rho \ge 1/\varepsilon$, the absolute value of $F(z)$ is majorized by $C(\varepsilon)|z|^{-\alpha}$. Moreover, $t < \varepsilon$ implies $\mathrm{Re}(tz) = ta_0 < t\varepsilon^{-1} < 1$, whence $|\exp(zt)| \le \exp(1) = e$. This combines to yield the estimate

$$\left|(1/(2\pi i))\int_S^{a_0+i\infty} e^{tz}\,F(z)\,dz\right| \le (e/(2\pi))\int_{\mathrm{Im}\,(S)}^{\infty} C(\varepsilon)|a_0+iy|^{-\alpha}\,dy\,.$$

Here consider the following consequences of $\alpha > 1$. The last integral can be majorized by suppressing a_0 in the denominator of its integrand, thus reducing to integrate $y^{-\alpha}$. A primitive of this function is $(1-\alpha)^{-1}y^{-\alpha+1}$, whence

$$\int_{\mathrm{Im}\,(S)}^{\infty} \frac{1}{|a_0+iy|^\alpha}\,dy \le \int_{\mathrm{Im}\,(S)}^{\infty} \frac{dy}{y^\alpha} = \left[\frac{y^{1-\alpha}}{1-\alpha}\right]_{y=\mathrm{Im}(S)}^{\infty}.$$

With its improper part killed, this last integral equals $\mathrm{Im}\,(S)^{1-\alpha}/(\alpha-1)$, whence

$$\Big|(1/(2\pi i))\int_S^{a_0+i\infty} e^{tz}\,F(z)\,dz\Big| \le \frac{e}{2\pi}\,\frac{C(\varepsilon)}{\alpha-1}\,\frac{1}{(\mathrm{Im}\,(S))^{\alpha-1}}\,.$$

To estimate the imaginary part of $S = S_t$ in terms of ε and a_0, let θ_t be the angle associated with S_t. Then $\sin(\theta_t)$ converges to $\sin(\theta_\varepsilon)$ from above as t converges to ε. By definition, $a_0 = |S_\varepsilon|\,\cos(\theta_\varepsilon) = \varepsilon^{-1}\,\cos(\theta_\varepsilon)$, whence $|\sin^2(\theta_\varepsilon)| = 1-\cos^2(\theta_\varepsilon) = 1-(a_0\varepsilon)^2$. This gives

$$\frac{1}{\big(\mathrm{Im}\,(S)\big)^{\alpha-1}} = \frac{1}{\big(|\sin\theta_t|\,\rho\big)^{\alpha-1}} \le \frac{t^{-(\alpha-1)}}{\big(1-(\varepsilon a_0)^2\big)^{(\alpha-1)/2}}\,,$$

recalling $\rho = 1/t$, and establishes Remark 6.3. To complete the proof of the proposition, recall that F is holomorphic in the entire right-hand half-plane, and thus any $a_0 > 0$ can be used in the contour for the complex inversion formula. Letting a_0 converge to 0, with ε fixed, kills the εa_0-term in the above estimate, and the proof of Proposition 6.2 is complete.

6.2. *Asymptotic expansions for N_n and its Laplace inverse.* As a first step in establishing Theorem 6.1, this section demonstrates the use of Proposition 6.2 by way of transposing an asymptotic expansion for the functions N_n at large complex arguments into an asymptotic expansion for ν_n at small positive real arguments by preserving effectiveness of remainder terms. Our precise result is as follows

Proposition 6.4. For the Laplace inverse $\mathcal{L}^{-1}(N_n) = \nu_n$ of N_n we have the following asymptotic expansion near zero:

$$\nu_n(t) = \mathcal{L}^{-1}(N_n)(t) = \sum\nolimits_{k=n}^{n+N-1} \frac{c^{(n)}_{N,k}}{\Gamma\big(k+n/2\big)}\,t^{k-1+n/2} + R_{n,N}(t)\,, \quad t\in(0,\infty)\,,$$

for any integer $N \ge 0$, and with coefficients $c_{n,k}$ as for Theorem 6.1. For any real $\varepsilon > 0$ and with $C_{n,N}(\varepsilon)$ as for Theorem 6.1, the remainder terms with $N \ge 2$ satisfy the estimate

$$\big|R_{n,N}(t)\big| \le \tfrac{1}{2}e\,C_{n,N}(\varepsilon)\,t^{N+3n/2-1}, \quad t\in(0,\varepsilon)\,.$$

The asymptotic expansion of the proposition is based on the following asymptotic expansion for N_n on the right-hand half-plane of independent interest:

Lemma 6.5. For any complex z with $\mathrm{Re}(z) > 0$, we have

$$N_n(z) = \frac{1}{z^{n/2}}\sum\nolimits_{k=n}^{n+N-1}\frac{c^{(n)}_{N,k}}{z^k} + \frac{1}{z^{n/2}}\,r_{n,N}(z)\,,$$

for any integer $N \ge 0$, where on fixing any real $a > 0$ and setting $\varepsilon_a = a^{-1}$ the remainder terms with $N \ge 2$ satisfy the estimate

$$\big|r_{n,N}(z)\big| \le \frac{C_{n,N}(\varepsilon_a)}{|z|^{n+N}}\,,$$

for any z with $\mathrm{Re}(z) > 0$ and $|z| \ge a$, and with $C_{n,N}(\varepsilon)$ as in Theorem 6.1.

Proof of Lemma 6.5. Recalling from Section A.2 the definition $N_n(z) = N_1(z)^n$ where $N_1(z) = z^{-1/2}\Psi(-z^{1/2})$, for any z in $\mathbf{C} \setminus \mathbf{R}_{\leq 0}$, the argument is based on the following asymptotic expansion of $\Psi(-z^{1/2})$ on $\{\mathrm{Re}(z) > 0\}$:

$$\Psi(-\sqrt{z}\,) = \textstyle\sum_{k=1}^{N} c_k z^{-k} + \rho_{N+1}(\sqrt{z}\,)\,, \quad \mathrm{Re}(z) > 0\,,$$

for any integer $N \geq 0$, setting $c_k = (-1/\sqrt{\pi}\,)(-2)^k\Gamma(k+1/2)$, and where $|\rho_{N+1}(z^{1/2})| \leq c_{N+1}/|z|^{N+1}$ with the temporary definition $c_{N+1} = 2(2N+1)!/N!$. Setting $a_k = c_k z^{-k}$ for integral $k \leq N$ and $a_{N+1} = \rho_{N+1}(z^{1/2})$, an application of the multinomial theorem gives

$$\Psi^n(-\sqrt{z}\,) = \textstyle\sum_{|J|=n} \binom{n}{J} a^J,$$

with J ranging over all the $(N+1)$-tuples $(j_1, \ldots, j_{N+1})$ of integers $j_\ell \geq 0$ summing up to n. Decomposing such J in the form $J = (I, j_{N+1})$ obtain the identity

$$\binom{n}{J} = \frac{n!}{j_1! \cdots j_{N+1}!} = \binom{n}{j_{N+1}}\binom{n-j_{N+1}}{I}.$$

Breaking up the summation for the nth power of $\Psi(-z^{1/2})$ accordingly,

$$\Psi^n(-\sqrt{z}\,) = \textstyle\sum_{k=0}^{n} \binom{n}{k}\big(\sum_{\ell=1}^{N} a_\ell\big)^{n-k} a_{N+1}^k,$$

also using the multinomial theorem to reverse summation over the index sets I. The following terms of this sum are now collected into the error term $r_{n,N+1}(z)$. First, all summands of the last sum with index $k \geq 1$; they all have degrees in z^{-1} of at least $n-k+k(N+1) = n+kN$, i.e., at least $n+N$. Then also add all other terms with $k = 0$ having degree of at least $n+N$ in z^{-1}. The highest power order of z^{-1} common to all of these summands of the error term is $z^{-(n+N)}$.

For the estimate of the error term choose $|z| \geq a$. Apply the triangle inequality to the absolute value of $r_{n,N+1}(z)$ as often as possible. Using the estimate for $|a_{N+1}| = |\rho_{N+1}(z^{1/2})|$ bring the highest power $|z|^{-(n+N)}$ of $|z|^{-1}$ in front of the resulting expression. Thus obtain the majorization $|r_{n,N+1}(z)| \leq \Sigma_1(z)+\Sigma_2(z)$, where

$$|z|^{n/2}\Sigma_1(z) = |z|^{-(n+N)}\textstyle\sum_{k=1}^{n}\displaystyle\binom{n}{k}\Big(\textstyle\sum_{\ell=1}^{N}\displaystyle\frac{|c_\ell|}{|z|^{\ell-1}}\Big)^{n-k}\frac{c_{N+1}^k}{|z|^{(k-1)N}}\,,$$

$$|z|^{n/2}\Sigma_2(z) = |z|^{-(n+N)}\ \textstyle\sum_{k=n+N}^{n(N+1)}\sum_{|I_N|=n,\ \sum_{\ell=1}^{N}\ell j_\ell = k}\displaystyle\binom{n}{I_N}\frac{|c|^{I_N}}{|z|^{k-(n+N)}}\,,$$

using the multi-index notation for Theorem 6.1. All powers of $|z|^{-1}$ remaining in the pertinent sums majorize by $a^{-1} = \varepsilon_a$ to obtain the constant $C_{n,N}(\varepsilon_a)$. This completes the proof of Lemma 6.5.

Proof of Proposition 6.4. With $\nu_n = \mathcal{L}^{-1}(N_n)$ established in Proposition A.1, apply $\mathcal{L}^{-1}$ to the preceding Lemma 6.5 to obtain as a first step the expansion

$$\mathcal{L}^{-1}(N_n)(t) = \textstyle\sum_{k=0}^{n+N-1} c_{N,k}^{(n)}\mathcal{L}^{-1}(z^{-(k+n/2)})(t) + R_{n,N}(t)\,,$$

where

$$R_{n,N}(t) = \mathcal{L}^{-1}(r_{n,N}(z)z^{-n/2})(t)\,.$$

To proceed, note the reformulation $\Gamma(a)\mathcal{L}^{-1}\big(z^{-a}\big)(t) = t^{a-1}$, for any real $t > 0$ and complex a with $\mathrm{Re}(a) > 0$, of the identity $\mathcal{L}(t^a)(z) = \Gamma(a+1)z^{-(a+1)}$. From this formula it is sufficient to establish the majorization of error terms $|R_{n,N}(t)|$ of the proposition. For this, transfer the estimate of Lemma 6.4 by way of Proposition 6.2, applied with $C(\varepsilon) = C_{n,N}(\varepsilon)$ and $\alpha = n/2+(n+N)$, to obtain

$$|R_{n,N}(t)| \leq C_{n,N}(\varepsilon) \max\{e/2, e/((3n/2+N-1)\pi)\}\, t^{3n/2+N-1} .$$

The proof of Proposition 6.4 is complete.

6.3. *Proof of Theorem* 6.1. Our argument is patterned after that for Proposition 6.4, and demonstrates Theorem 6.1 as a consequence of Laplace inversion of the uniform asymptotic expansion of N_n of Lemma 6.5. Indeed, multiply this expansion by $G = G_{\alpha,\gamma}$. Distinguishing the case where n is even or odd, do or do not add the square root in the denominator of G to the half-integer powers of the reciprocal of z in this expansion to make them integral. For n even, not changing anything, this gives

$$\big(GN_n\big)(z) = G(z)\Big(\sum\nolimits_{k=n}^{n+N-1}\ c_{N,k}^{(n)}/z^{k+n/2} + r_{n,N}^{\mathrm{ev}}(z)\Big),$$

while for n odd, recalling $F = F_{\alpha,\gamma} = z^{1/2}G$ from Section A.1, we obtain

$$\big(GN_n\big)(z) = F(z)\Big(\sum\nolimits_{k=n}^{n+N-1}\ c_{N,k}^{(n)}/z^{k+(n+1)/2} + r_{n,N}^{\mathrm{odd}}(z)\Big),$$

for any complex number z with sufficiently big real part and any integer $N \geq 0$. On fixing any real $\varepsilon > 0$ and setting $a = \varepsilon^{-1}$, the remainder terms for $N \geq 2$ satisfy

$$\big|r_{n,N}^{\mathrm{ev}}(z)\big| \leq \frac{e}{2}\,\frac{C_{n,N}(\varepsilon)}{|z|^{n+N+n/2}} \qquad \text{and} \qquad \big|r_{n,N}^{\mathrm{odd}}(z)\big| \leq \frac{e}{2}\,\frac{C_{n,N}(\varepsilon)}{|z|^{n+N+(n+1)/2}}\,,$$

for any z in the right-hand half-plane such that $|z| \geq a$. For n even the expression in brackets is the asymptotic expansions of N_n, while for n odd it is that of $z^{-1/2}N_n$. Both of these functions are Laplace transforms. Using the product theorem the Laplace inverse of any GN_n is given as a convolution of the Laplace inverse of G, respectively F, with the respective Laplace inverse of the expression in brackets. The latter has been established in Proposition 6.4 with an explicit transfer of the error terms as the crucial point. With the single summands being Laplace transforms themselves, a Laplace inversion of the expression in brackets is effected term by term. Convolve with the Laplace inverse of G and F respectively. Then reverse the convolution using the product theorem to obtain for n even:

$$\mathcal{L}^{-1}\big(GN_n\big) = \sum\nolimits_{k=n}^{n+N-1}\ c_{N,k}^{(n)}\mathcal{L}^{-1}\left(\frac{G(z)}{z^{k+n/2}}\right) + g * R_{n,N}^{\mathrm{ev}}\,,$$

and for n odd:

$$\mathcal{L}^{-1}\big(GN_n\big) = \sum\nolimits_{k=n}^{n+N-1}\ c_{N,k}^{(n)}\mathcal{L}^{-1}\left(\frac{F(z)}{z^{k+(n+1)/2}}\right) + f * R_{n,N}^{\mathrm{odd}}\,,$$

in the obvious notation (with f denoting the Laplace inverse of F and g that of G). Expressing the remaining Laplace inverses as the respective negative-order derivatives of Appendix C yields the desired asymptotic expansion, and a reduction occurs to establishing the estimates for the remaining terms. For these majorize the value at any real $t > 0$ of the respective convolutions by the product of the following two magnitudes. First, the respective maximum $M(f,t)$ and $M(g,t)$ of the absolute value of the respective Laplace inverses f and g on $(0,t)$. Second, the integral on $(0,t)$ of the absolute value of the respective remainder term factors. The integrals majorize by majorizing integrands as for Proposition 6.4, by $(2^{-1}e)C_{n,N}(\varepsilon)$ times the respective power of the integration variable. The result is majorizing integrals which equal, for n even,

$$\frac{e}{2}\,C_{n,N}(\varepsilon)\frac{t^{N+3n/2}}{N+3n/2} \qquad \text{and} \qquad \frac{e}{2}\,C_{n,N}(\varepsilon)\frac{t^{N+(3n+1)/2}}{N+(3n+1)/2}$$

for n odd and positive. As a majorizing constant $M(g,t)$ take $\exp(-\alpha^2/(4t))$; for $M(f,t)$ take $(|\beta|+(\pi t)^{-1/2})\,M(g,t)$. The proof of Theorem 6.1 is complete. □

REMARK 6.6. An analogous asymptotic expansion for $f_{\alpha,\gamma} * \nu_n$ is obtained by interchanging the roles of even and odd as well as that of G and F in the above argument, with a consequent index shift by -1 in the resulting expansion and estimates.

7. Epilogue. In this paper, a two-step Laplace inversion approach has been developed which is aimed at transforms with transcedental function denominators. The idea of the approach is to effect a reduction to inverting Laplace transforms of convolutions as a first step, and to provide asymptotic expansion methods for thus handling convolutions as a second step. This approach has been demonstrated for a concrete class of Laplace transforms in terms of parabolic cylinder functions as they originate with Brownian motion and their excursions. With this structure given expression to by a functional equation satisfied by the denominators, the approach was then found to take the following form.

First, a finite-length convolution representation for the Laplace inverses was established in Section 4 as a consequence of the functional equation, effecting a reduction of the inversion to the study of finite-length convolutions with known Laplace transforms. The idea then was to proceed by asymptotically expanding part of the transforms and to effect an inversion term by term that preserves the effectiveness of the error terms of the expansions that were started with. Two realizations for this two-step approach were developed. In Section 5 representations of the convolutions by convergent series were thus provided, and in Section 6 representations by asymptotic expansions were established for them. The two representations are complementary by construction: the series converge off of a compact neighbourhood of 0, while the asymptotic expansions have their representation properties exactly on neighbourhoods of this type.

In this way, the results of the paper enable a basic implementation of the convolutions of step one, special-function-type, by way of combining a series and an asymptotic expansion. We thus conclude the paper by looking to what extent this enables one to shed light on the motivating example from finance in Section 3.

We more precisely consider one of the typical situations problematic for barrier options, namely when the current value of the underlying security, S_t, is approaching the barrier, L, such that large Deltas are building up. For the down-and-in barrier options with strikes $K \geq L$ considered in the present paper, the latter condition gives expression to the perceived likeliness of two events. First, to the preceived likeliness of the option being triggered into existence by the value of the underlying falling below the level L before maturity, at time T, of the barrier option. Second, the large Deltas are to be seen as giving expression as well to the perceived likeliness of the price of the underlying

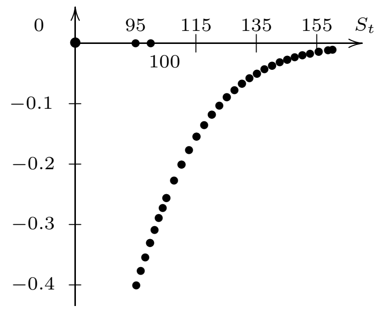

FIG. 7.1. Delta of Down-and-In call with L=95, K=100, σ=40% and τ=$\frac{1}{4}$ as a function of S_t.

to subsequently recover from levels below L, to positions above the strike K of the option triggered into existence, before maturity of the latter. Referring to 'practitioners' sources for formulas and further details (here quoting, hors collection, books such as J. Boissonade 'Les options exotiques' (Paris 2000)), this situation is depicted in Figure 7.1 as resulting from the choice of parameters $\tau = T - t = 1/12$ year and $\sigma = 9/10$ % p.a. in a situation with $L = 95$ and $K = 100$.

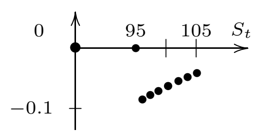

FIG. 7.2. Delta of Down-and-In Parisian call option with D=$\frac{1}{60}$ in the situation of Figure 7.1 as a function of S_t.

The built-in ability to avoid this building up of large Deltas near the barrier has been put forward as an argument in favour of the use of the Parisian barrier options. With a stylized choice of D=1/60 in addition, this is supported here in that we find the corresponding Deltas down by almost a factor 5 as depicted in Figure 7.2, which uses the same scale for the Deltas as in Figure 7.1.

Technically speaking, the curve of Figure 7.2 results from interpolation at the Deltas computed at $S_t = 96$, 100, 105 using the formula of Proposition 3.2, and hence by reduction to the finite-length convolution representations of Corollary 4.2. Since $\tau/D = 5$, the latter have 4 summands starting with length-2 convolutions at $u_1 = 2$, corresponding to $n = 1$, and terminating with length-5 convolutions at $u_4 = 1/2$, corresponding to $n = 4$. The contributions of the summands decrease to 0 rapidly, and we used the asymptotic expansions of Theorem 6.1 with $N = 3$ for the computation of the respective second half of them. We found the brunt of the contributions to originate wih the respective first two summands, if not just the ones corresponding to $n = 1$. Here, the series of Theorem 5.1 provided rapidly convergent means for computation. The computations were readily replicated by numerical integration for $n = 1$. The requirements of CPU-time for this type of independent verification, however, rapidly grew out of the potential of our vintage 2011 machine, even when using a most rapid computing system such as gp-PARI; with computation times in the CPU-days, independent verification was thus possible just for some of the $n = 2$ results.

These last findings of non-possibilities, on the other hand, are also indicative of a direction for future work, namely, to provide alternative ways for handling convolutions computationally and to provide methods for this based on alternative analytic Laplace inversion techniques, in particular. We hope to be able to address this in future work.

Appendix A. Laplace transform pairs. This appendix collects pertinent Laplace transform pairs. Here the Laplace transform is the linear operator $\mathcal{L}$ on the continuous functions of exponential type on $(0, \infty)$, the positive reals, defined as follows: it associates with any such function f the function $\mathcal{L}(f)$ given by

$$\mathcal{L}(f)(z) = \int_0^\infty \exp(-zu) f(u)\, du\,,$$

for any complex z in a half-plane contained sufficiently deep within the right-hand complex half-plane $\{z|\mathrm{Re}(z) > 0\}$. The maps $\mathcal{L}(f)$ then are analytic on such half-planes, and the operator $\mathcal{L}$ is an injection with inverse $\mathcal{L}^{-1}$, the inverse Laplace transform; see [3] or [4] for more details. We moreover work with the principal branch of the complex logarithm on $\mathbf{C} \setminus (-\infty, 0]$, the complex plane $\mathbf{C}$ cut along the non-positive reals $(-\infty, 0]$.

A.1. For any complex α with $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha^2) \geq 0$, we have on the complex half-plane $\{\mathrm{Re}(z{+}\beta) > 0\}$ the following two Laplace transforms:

$$\mathcal{L}(g_{\alpha,\beta})(z) = \frac{\exp(-\alpha\sqrt{z}\,)}{\sqrt{z}\,(\sqrt{z}+\beta)} \stackrel{\mathrm{def}}{=} G_{\alpha,\beta}(z), \text{ where } g_{\alpha,\beta}(u) = \exp(\alpha\beta+\beta^2 u)\,\mathrm{Erfc}\left(\eta_{\alpha,\beta}(u)\right),$$

$$\mathcal{L}(f_{\alpha,\beta})(z) = \frac{\exp(-\alpha\sqrt{z}\,)}{\sqrt{z}+\beta} \stackrel{\mathrm{def}}{=} F_{\alpha,\beta}(z), \text{ where } f_{\alpha,\beta}(u) = \frac{\exp(-(\alpha/2)^2/u)}{\sqrt{\pi u}} - \beta\, g_{\alpha,\beta}(u)\,,$$

for any real $u > 0$, setting $\eta_{\alpha,\beta}(u) = (\alpha/2)/\sqrt{u}+\beta\sqrt{u}$; see [5], Section 5.6, eqs. (12) on p. 246 and (16) on p. 247. These transform pairs are obtained using the two Laplace transforms on $\{\mathrm{Re}(z) > 0\}$: $\mathcal{L}(\psi_\alpha)(z) = \exp(-\alpha\sqrt{z}\,)$ where $2\sqrt{\pi u^3}\,\psi_\alpha(u) = \alpha\exp(-(\alpha/2)^2/u)$, and $\mathcal{L}(\chi_\alpha)(z) = \exp(-\alpha\sqrt{z}\,)/\sqrt{z}$ where $\sqrt{\pi u}\,\chi_\alpha(u) = \exp(-(\alpha/2)^2/u)$; see [4], Beispiel 8, p. 50ff; here α is any complex with $\mathrm{Re}(\alpha^2)$, $\mathrm{Re}(\alpha) \geq 0$ and with $\mathrm{Re}(\alpha) > 0$ for ψ_α.

A.2. As the first of two sets of functions to be considered define the functions N_n on $\mathbf{C} \setminus (-\infty, 0]$ for any integer $n \geq 0$ by

$$N_n = N_1^n, \quad \text{where} \quad N_1(z) = \Psi(-\sqrt{z}\,)/\sqrt{z}\,,$$

for any z in $\mathbf{C} \setminus (-\infty, 0]$. Following [11], the function Ψ here is the generalization of the normal distribution given by the integral

$$\Psi(w) = \int_0^\infty x \exp\left(-\tfrac{1}{2}x^2+wx\right) dx\,, \quad w \in \mathbf{C}\,;$$

it is further studied in Appendix B.

The second set is furnished by the functions $\nu_n = \nu^{*(n)}$ on $[0, \infty)$; here $\nu_0 = \nu^{*(0)}$ is the Dirac delta function at 0 and ν_n, for any integer $n \geq 1$, is the n-fold convolution on $[0,\infty)$ given by

$$\nu_n = \nu^{*(n)}, \quad \text{where} \quad \nu(u) = (2/\sqrt{\pi}\,)\sqrt{u}/(2u+1)\,, \quad u \in [0,\infty).$$

PROPOSITION A.1. We have $\mathcal{L}(\nu_n) = N_n$ on $\{\mathrm{Re}(z) > 0\}$, for any integer $n \geq 0$.

With the case $n = 0$ valid by convention, establishing this result reduces to the case $n = 1$ using the product theorem of the Laplace transform. For this, represent the numerator of $N_1(z)$ as the Laplace transform $\mathcal{L}(f)$ at $w = \sqrt{z}$ of the function $f(t) = t\exp(-t^2/2)$ on $(0,\infty)$. Then, $\mathcal{L}^{-1}(N_1)(t) = (1/\sqrt{\pi t})\int_0^\infty f(x)\exp(-x^2/(4t))\,dx$ by general principles, and the right-hand side can be checked to be equal to $\nu(t)$, as desired.

Appendix B. Further properties of the function Ψ. This appendix develops relevant properties of the function Ψ from Section A.2, given by the integral $\Psi(w) = \int_0^\infty x\exp\big(-\frac{1}{2}x^2+wx\big)\,dx$, for any complex w.

Developing the linear exponential factor of the integrand of Ψ in its series and integrating the resulting series term by term yields the following *series expansion*:

$$\Psi(w) = \sum\nolimits_{n=0}^\infty a_n\, w^n, \quad \text{where} \quad a_n = \frac{2^{n/2}}{n!}\,\Gamma\Big(\frac{n+2}{2}\Big).$$

This series is absolutely convergent for any complex number w, and its convergence is uniform on compact sets. As a first appplication, rearrange it in its even and odd order terms. Replacing w by its negative leaves unchanged the even part and produces a minus sign in the odd part. Using the duplication identity for the gamma function and redeveloping the square order exponential series that results as a factor then yields the *key identity*:

$$\Psi(w) = \Psi(-w) + \sqrt{\pi}\; w\;\exp\big(\tfrac{1}{2}w^2\big),$$

which connects the values of Ψ on the right-hand half-plane with those on the left-hand half-plane. Alternatively, the identity can be obtained by partial integration of the defining integrals for $\Psi(w)$ and $\Psi(-w)$ respectively by way of the partial integration identity

$$\Psi(-\sqrt{2}w) = 1 - \sqrt{2\pi}\,w\exp(w^2)\mathrm{Erfc}\,(w).$$

This identity is the basis of the *leading term expansion* for Ψ on the left-hand half-plane:

$$\Psi(-w) = \frac{1}{w^2} + R_2(w), \quad \text{where} \quad |R_2(w)| \leq \frac{6}{|w|^4}\,,$$

for any complex w with $\mathrm{Re}(w) > 0$. The point of the result, which is a special case of a general uniform asymptotic expansion of Ψ on the left-hand half-plane, is the validity of the remainder term estimate throughout the left-hand half-plane. This is established in two steps, for $|\arg w| \leq \pi/4$ and $\pi/4 < |\arg w| < 3\pi/4$, respectively. The argument proceeds along the lines of [8], Section 2.2, replacing the original contour of integration for Ψ by a ray emanating from the origin with a suitable angle $|\theta| < \pi/4$ in the second step.

Appendix C. Formulas for integral-order derivatives. This appendix provides explicit expressions for the integral-order derivatives of the functions $f_{\alpha,\beta}$ and $g_{\alpha,\beta}$, a notion for any sufficiently regular map φ on $\mathbf{R}_{\geq 0}$ to be denoted by $\varphi^{(m)}$, for any integer m, and defined as follows: if $m \geq 0$ it denotes the mth derivative of φ and if $m \leq 0$ set $\varphi^{(0)} = \varphi$ and recursively put $\varphi^{(-|m|-1)}(x) = \int_{[0,x]} \varphi^{(-|m|)}(u)\,du$, for any real x. We require these derivatives for real parameters $\alpha > 0$ and β, and a further reduction occurs by way of the representation

$$f_{\alpha,\beta} = (1/\sqrt{\pi})\,\phi_{\frac{1}{2},(\frac{\alpha}{2})^2} - \beta\, g_{\alpha\beta}$$

in terms of the functions on $(0,\infty)$ defined by

$$\phi_{a,c}(x) = x^{-a}\exp(-c/x), \quad x > 0,$$

and recalled to be defined by

$$g_{\alpha,\beta}(x) = \exp(\alpha\beta + \beta^2 x)\mathrm{Erfc}\,(\eta_{\alpha,\beta}(x)), \quad x > 0\,,$$

where $\mathrm{Erfc}\,(\xi) = (2/\sqrt{\pi})\int_{[\xi,\infty)}\exp(-x^2)\,dx$ and $\eta_{\alpha,\beta}(x) = (\alpha/2)/\sqrt{x} + \beta\sqrt{x}$. The integral order derivatives of the functions $f_{\alpha,\beta}$ thus become expressible in terms of those of the functions $\phi_{a,c}$ and $g_{\alpha,\beta}$, and it is hence sufficient to give expressions for the latter.

As summarized by the next four results, we give such expressions in terms of the incomplete gamma function. From [6], Section 9.2, pp. 134–143, for example, this function is given by $\Gamma(s,x) = \int_{[x,\infty)} w^{s-1}\exp(-w)\,dw$, for any real $x > 0$ and s. We also use the Pochhammer symbol $(\lambda)_k$ for any complex λ recalled to be recursively defined by $(\lambda)_0 = 1$ and $(\lambda)_{k+1} = (\lambda + k)(\lambda)_k$.

As a first step, the integral-order derivatives of the functions $\phi_{a,c}$ with $c > 0$ are computed in the following two results.

Proposition C.1. With the assumptions of this appendix we have for any integer $n \geq 0$ the representation

$$\phi_{a,c}^{(n)} = \sum\nolimits_{\ell=0}^{n} A_{n,\ell}\, c^{\ell}\, \phi_{a+n+\ell,c}\,,$$

where $A_{n,0} = (-1)^n (a)_n$, and for $\ell \geq 1$,

$$A_{n,\ell} = ((-1)^{\ell+n}/\ell!)\sum\nolimits_{k=\ell}^{n}(a)_{n-k}\binom{n}{k}\sum\nolimits_{m=0}^{\ell-1}(-1)^m\binom{\ell}{m}(\ell-m)_k\,.$$

Proposition C.2. With the assumptions of this appendix we have for any integer $n \geq 0$ and any real $u > 0$ the representation

$$\phi_{a,c}^{(-(n+1))}(u) = \sum\nolimits_{\ell=0}^{n}((-1)^{\ell}/\ell!)\,\frac{\Gamma(a-(\ell+1),c/u)}{c^{a-(\ell+1)}}\,\frac{u^{n-\ell}}{(n-\ell)!}\,.$$

Recalling the representation $\phi_{a,c}^{(-(n+1))}(u) = (1/\Gamma(n+1))\int_0^u (u-x)^n \phi_{a,c}(x)\,dx$ as a fractional integral (see for example [12], Chapter XII, §8, Eq. (8·1), p. 133), both results are established by direct computation.

As a second step, the integral order derivatives of the functions $g_{\alpha,\beta}$ are computed in the following two results. For the higher order derivatives we have the following result.

PROPOSITION C.3. With the assumptions of this appendix we have for any integer $n \geq 0$ the representation

$$g_{\alpha,\beta}^{(n)} = \beta^{2n} g_{\alpha,\beta} + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \tbinom{n}{k+1} \sum_{\ell=0}^{k} (-1)^{k-\ell} \tbinom{k}{l} \frac{\beta^{2n}}{\beta^{2(\ell+1)}} \big\{ (\alpha/2)\, \phi^{(\ell)}_{\frac{3}{2},(\frac{\alpha}{2})^2} - \beta\, \phi^{(\ell)}_{\frac{1}{2},(\frac{\alpha}{2})^2} \big\} \,.$$

For the higher negative order derivatives it needs to distinguish the case of parameters β being zero or not. This is encoded by the functions $I_\ell(g_{\alpha,\beta})$, for any integer $\ell \geq 0$, which we define on $(0,\infty)$ as follows. If $\beta \neq 0$ we put

$$\begin{aligned} &I_\ell(g_{\alpha,\beta})(u) \\ &\quad = \sum\nolimits_{k=0}^{\ell} \frac{(-\ell)_k}{\beta^{2(k+1)}} \{ u^{\ell-k} g_{\alpha,\beta}(u) - \frac{1}{\sqrt{\pi}} \frac{\Gamma(k-\ell+\frac{1}{2}, c_\alpha/u)}{c_\alpha^{k-\ell}} + \frac{\beta}{\sqrt{\pi}} \frac{\Gamma(k-\ell-\frac{1}{2}, c_\alpha/u)}{c_\alpha^{k-\ell-1/2}} \} \end{aligned}$$

and if $\beta = 0$ we define

$$I_\ell(g_{\alpha,0})(u) = \frac{u^{\ell+1}}{\ell+1} g_{\alpha,0}(u) - \frac{1/\sqrt{\pi}}{\ell+1}\, \Gamma\Big(-(\tfrac{1}{2}+\ell), \frac{c_\alpha}{u} \Big) c_\alpha^{1+\ell} \,,$$

where in both cases we set $c_\alpha = (\alpha/2)^2$. Proceeding along the lines of the argument for Proposition C.2, we then have in terms of these functions the following result.

PROPOSITION C.4. With the assumptions of this appendix we have for any integer $n \geq 0$ and any real $u > 0$ the representation

$$g_{\alpha,\beta}^{(-(n+1))}(u) = \sum\nolimits_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!} \frac{u^{n-\ell}}{(n-\ell)!} I_\ell(g_{\alpha,\beta})(u) \,.$$

REMARK C.5. The negative order derivatives of Proposition C.2, for the parameter a there equal to $1/2$ and $3/2$, as well as those of Proposition C.4 are in terms of incomplete gamma functions $\Gamma(s,x)$ at half integral arguments s. The incomplete gamma functions occurring there are hence expressible as finite linear combinations of weighted exponential functions plus an error integral value. More precisely, for integers $N \geq 0$ we have the representation

$$\Gamma(1/2+N, x) = \sum\nolimits_{k=0}^{N-1} \gamma_k\, x^{N-1/2-k} e^{-x} + \gamma_N \sqrt{\pi}\, \mathrm{Erfc}\,(\sqrt{x}\,) \,,$$

where $\gamma_k = (1/2)_N/(1/2)_{N-k}$. For integers $N \geq 1$ we have the representation

$$\Gamma(1/2-N, x) = \sum\nolimits_{k=0}^{N-1} \delta_k\, x^{-(N-k+1/2)} e^{-x} + \delta_N \sqrt{\pi}\, \mathrm{Erfc}\,(\sqrt{x}\,) \,,$$

where $\delta_k = (-1)^k (1/2)_{N-k}/(1/2)_N = (-1)^k/\gamma_k$. These formulas can be checked to reduce to standard integration formulas; for the latter see for example [7] Eq. 3b) and 5b), p. 109.

REFERENCES

[1] J. Azéma, M. Yor, *Étude d'une martingale remarquable*, Sém. Proba. **XXIII** (=LNMS 1372) (1989), 88–130. MR1022900 (91e:60139)

[2] J. Azéma, M. Yor, *Sur les zéros des martingales continues*, Sém. Proba. **XXVI** (=LNMS 1526) (1992), 248–306. MR1231999 (95m:60073)

[3] R. Beals, *Advanced mathematical analysis*, Springer, New York, 1973. MR0530403 (58:26624)

[4] G. Doetsch, *Handbuch der Laplace Transformation I*, Birkhäuser, Basel, 1971. MR0344808 (49:9547)

[5] A. Erdélyi et.al., *Tables of integral transforms I*, McGraw Hill, New York, 1954. MR0061695 (15:868A)

[6] A. Erdélyi et.al., *Higher transcendental functions II*, reprint, Krieger, Malabar, 1981. MR0698780 (84h:33001b)

[7] W. Gröbner und N. Hofreiter, *Integraltafel. Teil I: Unbestimmte Integrale*, 3. verb. Aufl., Springer, Wien, 1961.

[8] N. N. Lebedev, *Special functions and their applications*, Dover, New York, 1972. MR0350075 (50:2568)

[9] Z. Palmowski, I. Czarna, R. Loeffen, *Parisian ruin probabilities for spectrally negative Lévy processes,*, arXiv:1102.4055v1 (2011).

[10] M. Schröder, *Brownian excursions and Parisian barrier options: A note*, J. Appl. Prob. **40** (2003), 855–864. MR2012672 (2004h:60024)

[11] M. Yor, M. Jeanblanc–Picqué, M. Chesney, *Brownian excursions and Parisian barrier options*, Adv. Appl. Prob. **29** (1997), 165–184. MR1432935 (98b:90013)

[12] A. Zygmund, *Trigonometric series, II*, 2nd ed., Cambridge UP, Cambridge, 1977. MR0617944 (58:29731)