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ON TIME-DEPENDENT QUADRATIC QUANTUM HAMILTONIANS*

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Abstract. We apply the techniques of canonical transforms to equations of the type

$$[A(t)\mathbb{P}^2+B(t)\{\mathbb{P}\mathbb{Q}+\mathbb{Q}\mathbb{P}\}+C(t)\mathbb{Q}^2+D(t)\mathbb{Q}+E(t)\mathbb{P}+F(t)\mathbb{1}]\psi(q,t)=-i\partial_t\psi(q,t),$$

where $\mathbb{Q}$ and $\mathbb{P}$ are the quantum position and momentum operators. The time-dependent parameters of the $W \wedge SL(2, R)$ evolution operator are found through linear differential equations. In terms of these we give explicitly the Green's function, all separating coordinates and similarity solutions of the equation. We analyze the behavior of Gaussian and coherent-state initial conditions in closed form and present a new interpretation of all the Lewis-Riesenfeld constants of motion.

1. Introduction. There has been sustained interest in the description of quantum systems with time-dependent Hamiltonians. These systems have been used to model, for example, the motion of charged particles in time-dependent electromagnetic fields and coherent states in lasers. (See the list of references given in [3] and [11].)

Günther [5], [6] and Leach [9]–[14] have used time-dependent canonical transformations to reduce some of the above problems to time-independent ones, mainly for classical mechanics. They have been able to extend their methods to quantum systems for the cases when the canonical transformation is linear and real. In quantum mechanics, one has to be aware [9], [10] that not all Hamiltonians can be mapped meaningfully into each other, not even all quadratic ones: there exist distinct orbits in the vector space of the latter under the action of real linear canonical transformations. These orbits are characterized by (among other things) the spectrum of the operators in each equivalence class. Here, we take up their suggestion that the techniques of canonical transforms which we developed in [19], [20], [22], [24], [25] can be used to extend and simplify the analysis of differential equations of the type

$$\mathbb{H}(t)\psi(q,t)=-i\partial_t\psi(q,t), \qquad q,t\in\mathcal{R}, \tag{1.1a}$$

$$\mathbb{H}(t):=A(t)\mathbb{P}^2+B(t)\{\mathbb{P},\mathbb{Q}\}_+ +C(t)\mathbb{Q}^2+D(t)\mathbb{Q}+E(t)\mathbb{P}+F(t)\mathbb{1}, \tag{1.1b}$$

$$(\mathbb{P}\varphi)(q):=-i\partial\varphi(q)/\partial q, \quad (\mathbb{Q}\varphi)(q):=q\varphi(q), \qquad \{\mathbb{P},\mathbb{Q}\}_+:=\mathbb{P}\mathbb{Q}+\mathbb{Q}\mathbb{P}. \tag{1.1c}$$

(By $x := X$ we indicate that the symbol x is defined as the expression X.) The solution to this problem consists in finding the evolution operator, and its Green's function integral representative, such that

$$\psi(q,t):=\exp[i\mathbb{K}(t)]\psi(q,0)=\int_{-\infty}^{\infty}dq'G(q,q';t)\psi(q,0) \tag{1.2}$$

be a solution to (1.1). In addition, we would like to have a clear understanding of (1.2) so as to be able to know the families of similarity solutions of the system as well as the behavior of, say, Gaussian and coherent-state initial conditions, without resorting to long integrations.

The six linearly independent operators in (1.1b) (viz., $\mathbb{P}^2$, $\{\mathbb{P},\mathbb{Q}\}_+$, $\mathbb{Q}^2$, $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{1}$) constitute a basis for a wsl $(2, R)$ algebra generating the WSL$(2, R)$ group, the semidirect product of the Heisenberg-Weyl group [21] and SL$(2, R)$. In the six-dimensional wsl $(2, R)$ vector space, $\mathbb{H}(t)$ defines a time-dependent magnitude and direction which in

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the WSL $(2, R)$ manifold translates into a line parameterized by time. The points on this line belong to a subgroup only when the direction of $\mathbb{H}(t)$ is fixed, i.e., only when $\mathbb{H}(t) = h(t)\mathbb{H}_0$. In this case the evolution generator $\mathbb{K}(t)$, for a unit subgroup parameter, lies in the same direction as $\mathbb{K}(t) = k(t)\mathbb{H}_0$, with $\dot{k}(t) = h(t)$ and $k(0) = 0$. (Time derivatives will be indicated by dots.) For the general time-dependent system, $\mathbb{H}(t)$ and $\mathbb{K}(t)$ are not parallel and do not commute. After a brief recapitulation of formulae in § 2, in § 3 we find the evolution operator $\exp[i\mathbb{K}(t)]$ satisfying

$$-i\partial_t \exp[i\mathbb{K}(t)] = \mathbb{H}(t) \exp[i\mathbb{K}(t)], \tag{1.3}$$

obtained by replacing (1.2) in (1.1a). Having found the time-dependent parameters of this operator through a sequential set of linear ordinary differential equations, we can obtain the Green's function immediately.

In § 4 we follow the program we have previously implemented for the time-independent case [22]: finding all (up to equivalence) separating variables and similarity solutions. The behavior of Gaussian and oscillator coherent states under time-dependent quadratic potentials is given in § 5, while in § 6 we examine the similarity group and constants of motion of the system. Here we shall see that the Lewis-Riesenfeld invariants [16] are classified completely through the orbit structure of our time-dependent Lie algebra. Finally, in § 7 we give a list of problems for which the present results can be applied with only minor substitutions.

2. WSL(2, *R*) canonical transforms. There exist local isomorphisms between three sets of objects: (i) elements of the WSL $(2, R)$ group, (ii) exponentials of up-to-second order differential operators and (iii) a class of integral transform kernels. The latter are the kernels for inhomogeneous linear real canonical transforms [25, Part IV].

(i) The elements of the WSL $(2, R)$ group can be represented as

$$g := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (\varepsilon, \eta, \theta) \right\} =: \{\mathbf{M}, \mathbf{v}, \theta\}, \tag{2.1a}$$

$$\mathbf{M} := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1, \quad \mathbf{v} := (\varepsilon, \eta), \tag{2.1b}$$

with the composition rule

$$\{\mathbf{M}_2, \mathbf{v}_2, \theta_2\}\{\mathbf{M}_1, \mathbf{v}_1, \theta_1\} = \{\mathbf{M}_2\mathbf{M}_1, \mathbf{v}_2\mathbf{M}_1 + \mathbf{v}_1, \theta_2 + \theta_1 + \tfrac{1}{2}\mathbf{v}_2\mathbf{M}_1\mathbf{\Omega}\mathbf{v}_1^T\}, \tag{2.2a}$$

$$\mathbf{\Omega} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{v}^T := \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}. \tag{2.2b}$$

Hence, the identity is $e = \{\mathbf{1}, \mathbf{0}, 0\}$ and $g^{-1} = \{\mathbf{M}^{-1}, -\mathbf{v}\mathbf{M}^{-1}, -\theta\}$.

(ii) The six up-to-second order differential operators span a wsl $(2, R)$ algebra. Their exponentiation can be related with (2.1) through

$$\exp i[a\mathbb{P}^2 + b\{\mathbb{P}, \mathbb{Q}\}_+ + c\mathbb{Q}^2 + d\mathbb{Q} + e\mathbb{P} + f\mathbb{1}] =: \mathbb{I}\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (\varepsilon, \eta, \theta) \right\}, \tag{2.3}$$

$$\alpha = \cos 2s - bs^{-1} \sin 2s, \qquad s = \pm(ac - b^2)^{1/2}, \tag{2.4a}$$

$$\beta = -as^{-1} \sin 2s, \tag{2.4b}$$

$$\gamma = cs^{-1} \sin 2s, \tag{2.4c}$$

$$\delta = \cos 2s + bs^{-1} \sin 2s, \tag{2.4d}$$

(2.4e) $$\varepsilon=\tfrac{1}{2}(ce-bd)s^{-2}(1-\cos 2s)+\tfrac{1}{2}ds^{-1}\sin 2s,$$

(2.4f) $$\eta=\tfrac{1}{2}(be-ad)s^{-2}(1-\cos 2s)+\tfrac{1}{2}es^{-1}\sin 2s,$$

(2.4g) $$\theta=f-\tfrac{1}{4}(ad^2+ce^2-2bde)s^{-2}(1-\tfrac{1}{2}s^{-1}\sin 2s).$$

This relation is valid for all values of $a, b, \cdots, f$ including the cases $ac-b^2\leqq 0$. When s^2 is negative, we can use $s'=\pm is=\pm(b^2-ac)^{1/2}$, replacing $\cos 2s$ and $s^{-1}\sin 2s$ by $\cosh 2s'$ and $s'^{-1}\sinh 2s'$. For $s=0$ we can approach this class from nonzero values, obtaining $\cos 2s\to 0$, $s^{-1}\sin 2s\to 2$, $s^{-2}(1-\cos 2s)\to 2$ and $s^{-2}(1-\frac{1}{2}s^{-1}\sin 2s)\to\frac{2}{3}$. Equations (2.3)–(2.4) together with the composition rule (2.2) yield all Baker-Campbell-Hausdorff relations for wsl $(2, R)$.

(iii) The hyperdifferential operators (2.3) can be extended to $\mathscr{L}^2(\mathscr{R})$ through integral transforms which are unitary in this space, as

(2.5a) $$[\mathbb{I}\{g\}\varphi](q):=\left[\mathbb{I}\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, (\varepsilon,\eta,\theta)\right\}\varphi\right](q)=\int_{-\infty}^{\infty}dq'C_g(q,q')\varphi(q'),$$

$$C_g(q,q'):=(2\pi\beta)^{-1/2}e^{-i\pi/4}$$

(2.5b) $$\cdot\exp i\left[\alpha\frac{(q'-\eta)^2}{2\beta}-\left(\frac{q}{\beta-\varepsilon}\right)(q'-\eta)+\frac{\delta q^2}{2\beta}+\frac{\varepsilon\eta}{2}+\theta\right],$$

$$\beta\neq 0,\quad \arg\beta\in[-\pi,0].$$

When $\beta=0$, corresponding through (2.4b) to the absence of a $\mathbb{P}^2$-term in the exponentiated operator, $C_g(q,q')$ can be found as a limit of a sequence of functions converging weakly to the Dirac δ. In that case, (2.5) collapses to the Lie action

(2.6) $$\begin{aligned}[\mathbb{I}\{g_e\}\varphi](q)&=\left[\mathbb{I}\left\{\begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix}, (\varepsilon,\eta,\theta)\right\}\varphi\right](q)\\ &=\alpha^{-1/2}\exp i\left[\frac{\gamma q^2}{2\alpha}+\frac{\varepsilon q}{\alpha}+\frac{\varepsilon\eta}{2}+\theta\right]\varphi\left(\frac{q}{\alpha}+\eta\right).\end{aligned}$$

This five-parameter subgroup of WSL $(2, R)$ will be called the group of geometric transformations.

Canonical transforms (2.5) follow the group composition rule (2.2) modulo a sign; i.e., they are a two-valued ray representation of WSL $(2, R)$ [25, § 9.1.4].

The adjoint action of the group on the algebra

(2.7a) $$\mathbb{H}\xrightarrow{g}\mathbb{H}_g:=\mathbb{I}\{g\}\mathbb{H}\mathbb{I}\{g\}^{-1}=:\mathrm{Ad}_g\mathbb{H}$$

can be described through indicating the linear combination parameters of $\mathbb{H}$ by A, B, $\cdots$, F as in (1.1b), those of $\mathbb{H}_g$ by $A_g, B_g, \cdots, F_g$ and letting them transform as the elements of column vectors $\mathbf{H}$ and $\mathbf{H}_g$ under the 6×6 representation $\boldsymbol{\Gamma}(g)$ of WSL $(2, R)$/center WSL $(2, R)$:

(2.7b) $$\mathbf{H}_g=\boldsymbol{\Gamma}(g)\mathbf{H},\qquad g=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, (\varepsilon,\eta,\theta)\right\},$$

(2.7c) $$\boldsymbol{\Gamma}(g):=\begin{pmatrix}\alpha^2 & -2\alpha\beta & \beta^2 & 0 & 0 & 0\\ -\alpha\gamma & \alpha\delta+\beta\gamma & -\beta\delta & 0 & 0 & 0\\ \gamma^2 & -2\gamma\delta & \delta^2 & 0 & 0 & 0\\ 2\gamma\varepsilon & -2(\gamma\eta+\delta\varepsilon) & 2\delta\eta & \delta & -\gamma & 0\\ -2\alpha\varepsilon & 2(\alpha\eta+\beta\varepsilon) & -2\beta\eta & -\beta & \alpha & 0\\ \varepsilon^2 & -2\varepsilon\eta & \eta^2 & \eta & -\varepsilon & 1\end{pmatrix}.$$

Under (2.7), wsl $(2, R)/\mathbb{1}$ divides into five orbits whose representatives can be conveniently taken to be the Schrödinger Hamiltonians for the time-independent harmonic (h) and repulsive (r) oscillators, the linear potential (l), the free particle (f) and $\mathbb{P}$. The first four are

$$\mathbb{H}^h := \tfrac{1}{2}\mathbb{P}^2 + \tfrac{1}{2}\mathbb{Q}^2 \quad \text{for } AC - B^2 > 0, \tag{2.8a}$$

$$\mathbb{H}^r := \tfrac{1}{2}\mathbb{P}^2 - \tfrac{1}{2}\mathbb{Q}^2 \quad \text{for } AC - B^2 < 0, \tag{2.8b}$$

$$\mathbb{H}^l := \tfrac{1}{2}\mathbb{P}^2 + \mathbb{Q} \quad \text{for } AC - B^2 = 0, \quad BD - CE \neq 0, \tag{2.8c}$$

$$\mathbb{H}^f := \tfrac{1}{2}\mathbb{P}^2 \quad \text{for } AC - B^2 = 0, \quad BD - CE = 0, \tag{2.8d}$$

while the last one obtains for $A = B = C = 0$. This short list of results on canonical transforms contains some new expressions as (2.3)–(2.4) and adapts old ones as (2.7) to the wsl $(2, R)$ basis as displayed in (1.1b), which seems to be better suited for our purpose at hand than the spherical basis used elsewhere [22], [25 Part IV].

3. The evolution operator. We now turn to the task of finding the evolution operator $\exp[i\mathbb{K}(t)]$ for the solutions of the system (1.1)–(1.2) through the property (1.3). The problem and its solution through Baker-Campbell-Hausdorff relations, the Leibnitz rule, and the adjoint action of the group on the algebra are readily appreciated when we perform the task explicitly on the Heisenberg-Weyl part of WSL $(2, R)$:

$$\begin{aligned}
-i\partial_t \mathbb{W}\{\varepsilon, \eta, \theta\} &:= -i\partial_t \exp i[\varepsilon(t)\mathbb{Q} + \eta(t)\mathbb{P} + \theta(t)\mathbb{1}] \\
&= -i\partial_t \exp(i\varepsilon\mathbb{Q}) \exp(i\eta\mathbb{P}) \exp(i[\theta + \tfrac{1}{2}\varepsilon\eta]\mathbb{1}) \\
&= \dot{\varepsilon}\mathbb{Q}\mathbb{W}\{\varepsilon, \eta, \theta\} + \exp(i\varepsilon\mathbb{Q})\dot{\eta}\mathbb{P}\exp(i\eta\mathbb{P})\exp(i[\theta + \tfrac{1}{2}\varepsilon\eta]\mathbb{1}) \\
&\qquad\qquad + \mathbb{W}\{\varepsilon, \eta, \theta\}\partial_t(\theta + \tfrac{1}{2}\varepsilon\eta)\mathbb{1} \\
&= [\dot{\varepsilon}\mathbb{Q} + \dot{\eta}Ad_{(\varepsilon,0,0)}\mathbb{P} + \partial_t(\theta + \tfrac{1}{2}\varepsilon\eta)]\mathbb{W}\{\varepsilon, \eta, \theta\} \\
&= [\dot{\varepsilon}\mathbb{Q} + \dot{\eta}\mathbb{P} + (\dot{\theta} + \tfrac{1}{2}\dot{\varepsilon}\eta - \tfrac{1}{2}\varepsilon\dot{\eta})\mathbb{1}]\mathbb{W}\{\varepsilon, \eta, \theta\}.
\end{aligned} \tag{3.1}$$

Application of $-i\partial_t$ on the full WSL $(2, R)$ exponentiated operator (2.3) splits the semidirect product through the Leibnitz rule into an SL $(2, R)$ summand and the W summand above. The former is then brought to the product of three factors generated by $\mathbb{P}^2$, $\{\mathbb{P}, \mathbb{Q}\}_+$ and $\mathbb{Q}^2$. Through (2.7) we move all algebra elements to the left and sum them in a manner entirely analogous to (3.1), which is finally set equal to $\mathbb{H}(t)$ in (1.3); i.e.

$$\begin{aligned}
-i\partial_t \mathbb{I}&\left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (\varepsilon, \eta, \theta)\right\} \\
&= [A\mathbb{P}^2 + B\{\mathbb{P}, \mathbb{Q}\}_+ + C\mathbb{Q}^2 + D\mathbb{Q} + E\mathbb{P} + F\mathbb{1}]\mathbb{I}\left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (\varepsilon, \eta, \theta)\right\},
\end{aligned} \tag{3.2}$$

and for the coefficients we obtain the set of ordinary differential equations

$$A = -\alpha^2\dot{\mu}, \qquad \mu := \beta/2\alpha, \tag{3.3a}$$

$$B = \alpha\gamma\dot{\mu} - \dot{\alpha}/2\alpha, \tag{3.3b}$$

$$C = -\gamma^2\dot{\mu} + 2\nu\dot{\alpha}/\alpha + \dot{\nu}, \qquad \nu := \gamma/2\alpha, \tag{3.3c}$$

$$D = \delta\dot{\varepsilon} - \gamma\dot{\eta}, \tag{3.3d}$$

$$E = \alpha\dot{\eta} - \beta\dot{\varepsilon}, \tag{3.3e}$$

$$F = \theta + \tfrac{1}{2}\dot{\varepsilon}\eta - \tfrac{1}{2}\varepsilon\dot{\eta}, \tag{3.3f}$$

which determine $\mathbb{I}\{g(t)\}$ subject to the boundary condition $g(0)=e$. Equations (3.3) can be inverted for the time-dependent parameters of g through a sequence of linear ordinary differential equations as

$$\ddot{\alpha}-\dot{\alpha}\dot{A}/A+\alpha[4(AC-B^2)+2\dot{B}-2B\dot{A}/A]=0, \qquad \alpha(0)=1, \quad \dot{\alpha}(0)=-2B(0), \tag{3.4a}$$

$$\dot{\beta}-\beta\dot{\alpha}/\alpha+2A/\alpha=0, \qquad \beta(0)=0, \tag{3.4b}$$

$$\gamma=-\alpha B/A-\dot{\alpha}/2A, \tag{3.4c}$$

$$\delta=(1+\beta\gamma)/\alpha, \tag{3.4d}$$

$$\dot{\varepsilon}=D\alpha+E\gamma, \qquad \varepsilon(0)=0, \tag{3.4e}$$

$$\dot{\eta}=D\beta+E\delta, \qquad \eta(0)=0, \tag{3.4f}$$

$$\dot{\theta}=F+\tfrac{1}{2}\varepsilon\dot{\eta}-\tfrac{1}{2}\dot{\varepsilon}\eta, \qquad \theta(0)=0. \tag{3.4g}$$

This is the key step in solving our original equation (1.1): given $A, B, \cdots, F$ in $\mathbb{H}(t)$, (3.4) provides us with the evolution operator in (1.2) which determines the subsequent time development of the initial conditions. As a check on the calculation, we can consider the case when the coefficients of $\mathbb{H}$ are constant: the solution of (3.4) then leads exactly to (2.4) with $a=At$, $b=Bt, \cdots, f=Ft$ and $s=(AD-B^2)^{1/2}t$, as a consequence of which the time dependence for $\alpha, \beta, \cdots, \eta$ appears only in the argument of the trigonometric functions, while θ exhibits also a linear dependence on t. The $AC-B^2=0$ orbits let terms linear in t appear for β and ε, quadratic ones for η and cubic ones for θ. The Green's function for the system (1.2) is now found simply by substituting the results of (3.4) in (2.5b). Although it is not necessary for the solution of (1.1), the explicit form for $\mathbb{K}(t)$ in (1.2) can be obtained from that of $\exp[i\mathbb{K}(t)]$ through inverting (2.4) for $a, b, \cdots, f$.

An important particular case pertains the Schrödinger equation with a time-dependent quadratic + linear potential term:

$$\mathbb{H}_1(t)=\tfrac{1}{2}\mathbb{P}^2+C(t)\mathbb{Q}^2+D(t)\mathbb{Q}+F(t)\mathbb{1}; \tag{3.5}$$

i.e., $A_1=\frac{1}{2}$, $B_1=0$, $E_1=0$. In that case (3.4) reduce to

$$\ddot{\alpha}+2C\alpha=0, \qquad \alpha(0)=1, \quad \dot{\alpha}(0)=0, \tag{3.6a}$$

$$\ddot{\beta}+2C\beta=0, \qquad \beta(0)=0, \quad \dot{\beta}(0)=-1, \tag{3.6b}$$

$$\gamma=-\dot{\alpha}, \qquad \delta=-\dot{\beta}, \qquad \dot{\alpha}\beta-\alpha\dot{\beta}=1, \tag{3.6c}$$

$$\dot{\varepsilon}=\alpha D, \quad \varepsilon(0)=0, \qquad \dot{\eta}=\beta D, \quad \eta(0)=0, \tag{3.6d}$$

$$\dot{\theta}=F+\tfrac{1}{2}(\alpha\eta+\beta\varepsilon)D, \qquad \theta(0)=0. \tag{3.6e}$$

One last consequence of our general construction is the relation

$$-i\partial_t\mathbb{I}\{g^{-1}\}=-\mathbb{I}\{g^{-1}\}\mathbb{H}(t), \tag{3.7}$$

where $\mathbb{I}\{g^{-1}\}=\mathbb{1}\{g\}^{-1}$ is the transformation inverse to (3.2).

4. Similarity solutions and separation of variables. The elementary separation of variables procedure for a manifestly separated equation $\mathbb{H}\psi(q,t)=-i\partial_t\psi(q,t)$ with $\mathbb{H}$ constant, searches for the (possibly generalized) eigenfunctions $\Psi_\lambda(q)$ of $\mathbb{H}$, whose time development consists in multiplication by a phase $\exp(i\lambda t)$. Similarity methods allow us to replace the eigenfunctions $\Psi_\lambda(g)$ of $\mathbb{H}$ by the eigenfunctions of a different operator

$\mathbb{H}'$, $\Psi'_\lambda(q)$, with the condition that $\mathbb{H}$ and $\mathbb{H}'$ lie in a finite-dimensional Lie algebra. In the case of the wsl $(2, R)$ algebra we can choose among any linear combinations of the six generators. In particular, it is useful to select the four orbit representative Schrödinger Hamiltonians $\mathbb{H}^\omega$, $\omega = h, r, l, f$ given in (2.8). Their (Dirac) normalized eigenfunctions and spectra are

(4.1a) $\Psi^h_\lambda(q) := (2^h n!\pi^{1/2})^{-1/2} \exp(-\tfrac{1}{2}q^2)H_n(q), \qquad \lambda = n+\tfrac{1}{2}, \quad n = 0, 1, 2, \cdots,$

(4.1b) $\Psi^r_{\lambda,\sigma}(q) := \exp[i\pi(\tfrac{1}{2} - i\lambda)]2^{-3/4}\pi^{-1}\Gamma(\tfrac{1}{2} - i\lambda)D_{-1/2+i\lambda}(-\sigma 2^{1/2} e^{3i\pi/4} q),$

$\lambda \in R, \quad \sigma = \pm 1,$

(4.1c) $\Psi^l_\lambda(q) := 2^{1/3}\mathrm{Ai}(2^{1/3}[q-\lambda]), \qquad \lambda \in \mathcal{R},$

(4.1d) $\Psi^f_{\lambda,\sigma}(q) := (2\pi)^{-1/2} \exp(i\sigma[2\lambda]^{1/2} q), \qquad \lambda \in \mathcal{R}^+, \quad \sigma = \pm 1,$

and satisfy $\mathbb{H}^\omega \Psi^\omega_\lambda(q) = \lambda \Psi^\omega_\lambda(q)$.

Now, the evolution operator for the time-dependent system (with a nonzero $\mathbb{P}^2$-coefficient) can be written, for values of the parameters in a neighborhood of the identity, as

(4.2) $$\mathbb{I}\left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (\varepsilon, \eta, \theta)\right\} = \mathbb{I}\{g^\omega_e(t)\} \exp[i\tau^\omega(t)\mathbb{H}^\omega], \qquad \omega = h, r, l, f,$$

where $\mathbb{I}\{g^\omega_e\}$ is a time-dependent geometric transformation (2.6) and $\tau^\omega(t)$ will have the role of a transformed time variable. In applying the evolution operator (4.2) on the corresponding initial conditions (4.1), the rightmost factor will multiply the latter by a phase, while the action of the geometric transformation will change the argument of $\Psi^\omega_\lambda(q)$ to $q/\alpha^\omega_e(t) + \eta^\omega_e(t)$, and this will be the only λ-dependent function to contain this new variable. The other factor, stemming from the multiplier or modulation function in (2.6), will be common to the whole eigenfunction set. In this way, the solution R-separates (or separates with a modulation factor: see [17]) as

(4.3a) $$\Psi^\omega_\lambda(q, t) = R^\omega(v^\omega, t)T^\omega_\lambda(t)\Psi^\omega_\lambda(v^\omega),$$

(4.3b) $$v^\omega(q, t) := q/\alpha^\omega_e(t) + \eta^\omega_e(t),$$

(4.3c) $$R^\omega(v, t) := (\alpha^\omega_e)^{-1/2} \exp i[\tfrac{1}{2}\gamma^\omega_e\alpha^\omega_e v^2 + (\varepsilon^\omega_e - \alpha^\omega_e\gamma^\omega_e\eta^\omega_e)(v - \eta^\omega_e/2) + \theta^\omega_e],$$

(4.3d) $$T^\omega_\lambda(t) := \exp[i\lambda\tau^\omega(t)].$$

For $\omega = h, r, l, f$, these are four inequivalent similarity solutions, and $(v^\omega(q, t), t)$ are the corresponding separating coordinates for the time-dependent system (1.1). Performing the necessary substitutions [(2.8) in (2.3)–(2.4) for the rightmost factor in (4.2), the product rule (2.2) for the right-hand term and some algebra to solve for the parameters of $g^\omega_e(t)$ and $\tau^\omega(t)$] we find the constituent expressions of (4.3).

The separating variables are (v^ω, t), where

(4.4a) $$v^f(q, t) = (q - \beta\varepsilon)/\alpha + \eta,$$

(4.4b) $$v^l(q, t) = v^f(q, t) - \tfrac{1}{2}\beta^2/\alpha^2,$$

(4.4c) $$v^{r,h}(q, t) = (q + \alpha\eta - \beta\varepsilon)(\alpha^2 \mp \beta^2)^{-1/2}.$$

In all cases, $v^\omega(q, 0) = q$, as it should. The λ-and t-dependent phases (4.3d) are

(4.5a) $$T^f_\lambda(t) = T^l_\lambda(t) = \exp(-i\lambda\beta/\alpha),$$

(4.5b) $$T^r_\lambda(t) = [(\alpha - \beta)/(\alpha + \beta)]^{i\lambda/2},$$

(4.5c) $$T^h_\lambda(t) = [(\alpha - i\beta)/(\alpha + i\beta)]^{\lambda/2},$$

and the modulation factors (4.3c) are

$$\text{(4.6a)}\quad R^f(v,t)=\alpha^{-1/2}\exp i[\tfrac{1}{2}\alpha\gamma v^2+(\varepsilon+\gamma\{\beta\varepsilon-\alpha\eta\})(v-\tfrac{1}{2}\{\eta-\beta\varepsilon/\alpha\})+\theta],$$

$$\text{(4.6b)}\quad \begin{aligned}R^l(v,t)=R^f(v,t)\exp i[(\beta/\alpha)(1+\tfrac{1}{2}\beta\gamma)v+\\ +(\beta^2/2\alpha^2)(\varepsilon+\gamma\{\beta\varepsilon-\alpha\eta\}+\beta\{4+3\beta\gamma\}/12\alpha)],\end{aligned}$$

$$\text{(4.6c)}\quad \begin{aligned}R^{r,h}(v,t)=(\alpha^2\mp\beta^2)^{-1/4}\exp i[\tfrac{1}{2}\alpha^{-1}(\gamma\{\alpha^2\mp\beta^2\}\mp\beta)v^2+\alpha^{-1}(\varepsilon-\gamma\{\alpha\eta-\beta\varepsilon\})\\ \cdot(\{\alpha^2\mp\beta^2\}^{1/2}v-\tfrac{1}{2}\{\alpha\eta-\beta\varepsilon\})+\theta].\end{aligned}$$

When $\mathbb{H}(t)=\mathbb{H}^\upsilon$, $\upsilon=f, l, r, h$ the results obtained from (4.4)–(4.6) reduce to the separating variables and multipliers of the representatives of the four inequivalent orbits for the time-independent Schrödinger equations given in [25, Table 10.3]. [In this table, the entries are as referred to equation (10.53), where the exponent $S(v,t)$ includes all and only v-dependent, λ-independent terms.]

As stated above, the separating coordinates and similarity solutions displayed in (4.3)–(4.6) are inequivalent under the similarity group and are representatives of their orbit. The most general separating coordinates and solutions may be found through applying a geometric similarity transformation (2.7) to the separating operators $\mathbb{H}^\omega$, $\omega=f, l, r, h$. This amounts, for the coordinates (4.4), to the application of (2.6). As the added generality will involve four new parameters for all our expressions, we shall forego their display.

We should remark here that our construction is formally valid not only for Schrödinger equations (1.1) but, allowing for complex coefficients $A(t), B(t), \cdots, F(t)$, for any contracting parabolic equation as, for example, the diffusion and Fokker-Planck equations.

When faced with a problem with constant boundary conditions on moving boundaries, we can apply the "method of images" if we can find a set of separating coordinates $v^\omega(q,t)=c_1$ and $v^\omega(q,t)=c_2$ belonging to the same family which match these boundaries. The solution of the Sturm-Liouville problem for the operator $\mathbb{H}^\omega$ which produces this separation, on (c_1, c_2), yields the corresponding "best set" of similarity solutions in which to expand or approximate the initial conditions so that the boundary conditions are automatically satisfied.

5. Gaussians and coherent states. In addition to the similarity solutions of the last section, there are certain functions whose time evolution under time-dependent quadratic Hamiltonians can be reduced to a group-theoretic problem involving only matrix algebra. Among these we have Gaussians, oscillator coherent states and plane waves. The former can be written

$$\text{(5.1a)}\quad G_w(q-k):=(2\pi w)^{-1/2}\exp[-(q-k)^2/2w]=(\mathbb{I}\{\mathbf{G}_w,\mathbf{0},0\}\boldsymbol{\delta}_k)(q),$$

$$\text{(5.1b)}\quad \mathbf{G}_w=\begin{pmatrix}1 & \exp(-i\pi/2)w\\ 0 & 1\end{pmatrix},\qquad w>0,\qquad (\boldsymbol{\delta}_k)(q)=\delta(q-k).$$

They are Gaussians of width w, centered at k, of unit L^1-norm and L^2-norm $(4\pi w)^{-1/2}$. The oscillator coherent states are given in terms of Bargmann's transform [1] as

$$\text{(5.2a)}\quad \Upsilon_k(q):=\pi^{-1/4}e^{k^2/2}\exp[-(q-2^{1/2}k)^2/2]=(2\pi)^{1/4}(\mathbb{I}\{\mathbf{B},\mathbf{0},0\}\boldsymbol{\delta}_k)(q),$$

$$\text{(5.2b)}\quad \mathbf{B}=2^{-1/2}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}.$$

Lastly, plane waves can be defined in terms of the Fourier transform $\mathbb{F} = \exp(-i\pi/4)\mathbb{I}\{-\mathbf{\Omega}, \mathbf{0}, 0\}$, but we shall not discuss them here.

The time evolution of Gaussian initial conditions under a time-dependent equation (1.1) can be easily calculated through the WSL (2, R) composition rule for the matrix-vector representations of the evolution operator times $\mathbb{I}\{\mathbf{G}_w, \mathbf{0}, 0\}$, decomposed as a product of some $\mathbb{I}\{\mathbf{G}_{w'}, \mathbf{0}, 0\}$ times a geometric transform $\mathbb{I}\{g_e\}$, as given by (2.6). If the parameters of the latter are denoted by $\alpha_e, \gamma_e, \varepsilon_e, \eta_e, \theta_e$, we have

$$\begin{aligned} G_w(q-k, t) &= [\exp(i\mathbb{K}(t))\mathbb{I}\{\mathbf{G}_w, \mathbf{0}, 0\}\boldsymbol{\delta}_k](q) \\ &= [\mathbb{I}\{\mathbf{G}_{w'}, \mathbf{0}, 0\}\mathbb{I}\{g_e\}\boldsymbol{\delta}_k](q) \\ &= \alpha_e^{1/2} \exp i[\tfrac{1}{2}\alpha_e\gamma_e k^2 + (\varepsilon_e - \alpha_e\gamma_e\eta_e)(k - \tfrac{1}{2}\eta_e) + \theta_e]G_{w'}(q - \alpha_e[k - \eta_e]) \\ &= (\delta - i\gamma w)^{-1/2} \exp i\left[\frac{1}{2}\frac{\gamma k^2}{\delta - i\gamma w} + \left(\varepsilon - \frac{\gamma\{\varepsilon w + i\eta\}}{\gamma w + i\delta}\right)(k - \tfrac{1}{2}\{\eta - i\varepsilon w\}) + \theta\right] \\ &\qquad \cdot G_{w'}\left(q - \frac{\varepsilon w - i[k-\eta]}{\gamma w + i\delta}\right), \qquad w' = \frac{\alpha w + i\beta}{\delta - i\gamma w}, \end{aligned}$$

where only elementary algebraic manipulations have been used.

The oscillator coherent-states $\Upsilon_k(q)$ are coherent under the usual time-independent harmonic oscillator due to the property

(5.4a) $$\exp(it\mathbb{H}^h)\mathbb{I}\{\mathbf{B}, \mathbf{0}, 0\} = \mathbb{I}\{\mathbf{B}, \mathbf{0}, 0\}\mathbb{I}\{\operatorname{diag}(e^{it}, e^{-it}), \mathbf{0}, 0\},$$

where, from (2.3)–(2.4) and (2.8a) [or, in fact, from (3.5)–(3.6)],

(5.4b) $$\exp(it\mathbb{H}^h) = \mathbb{I}\left\{\begin{pmatrix}\cos t & -\sin t \\ \sin t & \cos t\end{pmatrix}, \mathbf{0}, 0\right\}.$$

Indeed, the Bargmann matrix $\mathbf{B}$ diagonalizes the harmonic oscillator evolution operator matrix representative, turning $\exp(it\mathbb{H}^h)$ into a change-of-scale operator with $\alpha = e^{it}$; hence, acting with (5.4) on $\delta(q-k)$, we have $\Upsilon_k(q, t) = e^{-it/2}\Upsilon_{k\exp(it)}(q)$. The repetition of (5.3) with $\mathbf{B}$ in place of $\mathbf{G}_w$, however, is impossible unless the matrix representatives of $\exp(i\mathbb{K}(t))$ lie on a subgroup generated by $\mathbb{H}^h$. Oscillator coherent states will thus remain coherent only if the Hamiltonian of the system is of the type (3.5) with a fixed strength $C(t) = \frac{1}{2}$, i.e. when only the oscillator attracting center is allowed to move as $\kappa(t) = -D(t)$, so $F(t) = \frac{1}{2}\kappa(t)^2 + \varphi(t)$. In this case, (5.4), (2.6) and (3.6) quickly lead to

(5.5a) $$\Upsilon_k(q, t) = \exp i[\tfrac{1}{2}t + \bar{\varepsilon}(k + \tfrac{1}{2}\bar{\eta}) + \bar{\theta}]\Upsilon_{k\exp(it)}(q),$$

(5.5b) $$\bar{\varepsilon}(t) = -2^{-1/2}\int_0^t dt'\kappa(t')\,e^{it'}, \qquad \bar{\eta}(t) = 2^{-1/2}i\int_0^t dt'\kappa(t')\,e^{-it'},$$

(5.5c) $$\bar{\theta}(t) = \int_0^t dt'[\tfrac{1}{2}\kappa(t')^2 + \varphi(t') + \int_0^{t'} dt''\kappa(t'')\sin(t' + t'')].$$

One can define coherent states for Hamiltonians (1.1b) other than the harmonic oscillator. If the three leading coefficients are constant, the evolution operator will lie on an SL (2, R) subgroup and a diagonalizing matrix $\mathbf{B}'$ may exist which effects the analogue of (5.4a). For the $\mathbb{H}^h$ orbit this yields Gaussian functions of real width, while for the $\mathbb{H}^r$ orbit the width is imaginary and the L^2-norm infinite. The $\mathbb{H}^l$ and $\mathbb{H}^f$ orbits do not allow for diagonalization.

We may suggest two generalizations for the coherence property (5.4a). One of them is to allow in the rightmost factor for a geometric transform, with the result that the new coherent state may change its scale and be multiplied by an imaginary Gaussian phase factor. The second is to investigate the coherence or reformation of certain initial conditions at a given later time or times, so that the analogue of (5.4a) is required to hold only for the corresponding matrix or set of matrices.

6. The similarity algebra and constants of motion. The search for constants of motion $\mathbb{X}(t) \in \mathrm{wsl}\,(2, R)$ can be formulated through demanding that

$$[\mathbb{H}(t)+i\partial_t, \mathbb{X}(t)]=0. \tag{6.1}$$

The discovery of these is intimately related to finding the similarity group and separating coordinates for a class of time-independent partial differential equations [17, and references therein]. Indeed, if $\mathbb{X}(t)$ contains a $\mathbb{P}^2$-term with coefficient $A^x(t)$, we may turn the second derivative in q into a first derivative in t on the solution space of (1.1) through subtracting and adding $\chi(t)\mathbb{H}$, with $\chi(t)=A^x(t)/A(t)$, and replacing $\mathbb{H}$ by $-i\partial_t$ in the added term. Thus

$$\mathbb{X}^s(t) := \mathbb{X}(t)-\chi(t)[\mathbb{H}(t)+i\partial_t] \tag{6.2}$$

is a first order differential operator in q and t, satisfying

$$[\mathbb{H}(t)+i\partial_t, \mathbb{X}^s(t)]=-i\dot{\chi}(t)[\mathbb{H}(t)+i\partial_t] \tag{6.3}$$

and mapping solutions into solutions. The set of all operators with this property constitutes the similarity algebra of the differential equation.

Both (6.1) and (6.3) can be solved for the coefficients of $\mathbb{X}(t)$ which will satisfy certain coupled nonlinear differential relations. Finding a minimal set of parameters is the final step in identifying the similarity algebra, which should be $\mathrm{wsl}\,(2, R)$. As we have started here from this algebra and presume to know the evolution operator $\exp[i\mathbb{K}(t)]$ associated with $\mathbb{H}(t)$, we can find six linearly independent operators $\mathbb{X}(t)$ as

$$\mathbb{X}_n(t) := \mathbb{I}\{g\}\mathbb{X}_n^0\mathbb{I}\{g\}^{-1}, \qquad g(t)=\left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (\varepsilon, \eta, \theta)\right\}, \tag{6.4a}$$

$$\mathbb{X}_1^0=\mathbb{P}^2, \quad \mathbb{X}_2^0=\{\mathbb{P}, \mathbb{Q}\}_+, \quad \mathbb{X}_3^0=\mathbb{Q}^2, \quad \mathbb{X}_4^0=\mathbb{Q}, \quad \mathbb{X}_5^0=\mathbb{P}, \quad \mathbb{X}_6^0=\mathbb{1}, \tag{6.4b}$$

which satisfy (6.1) because of (3.2) and (3.7). In choosing the numbering (6.4b) for the $\mathbb{X}_n^0=\mathbb{X}_n(0)$ we can make use of the adjoint representation of the group (2.7) in order to write them more explicitly as

$$\mathbb{X}_n(t)=\sum_{m=1}^{6}\Gamma_{mn}(g)\mathbb{X}_m^0; \tag{6.5}$$

i.e., the time-dependent coefficients of $\mathbb{X}_n(t)$ in the $\mathbb{X}^0$-basis are given by the elements of the nth column of the matrix (2.7c), the latter satisfying (3.4). Among (6.5), $\mathbb{X}_6(t)=\mathbb{1}$ is trivial and only reflects the fact that the differential equation (1.1) is linear.

For the special case of Schrödinger equations with a time-dependent potential (3.5), the elements of $\Gamma_{mn}(g)$ are subject to restrictions (3.6c) and yield the constants of motion of the problem as

$$\mathbb{X}_4(t)=-\dot{\beta}\mathbb{Q}-\beta\mathbb{P}+\eta\mathbb{1}, \qquad \mathbb{X}_4(0)=\mathbb{Q}, \tag{6.6a}$$

$$\mathbb{X}_5(t)=\dot{\alpha}\mathbb{Q}+\alpha\mathbb{P}-\varepsilon\mathbb{1}, \qquad \mathbb{X}_5(0)=\mathbb{P}, \tag{6.6b}$$

$$\mathbb{X}_1(t)=\mathbb{X}_5(t)^2, \quad \mathbb{X}_2(t)=\{\mathbb{X}_5(t), \mathbb{X}_4(t)\}_+, \quad \mathbb{X}_3(t)=\mathbb{X}_4(t)^2, \tag{6.6c}$$

leaving α, β, ε and η to be determined out of linear differential equations (3.6a), (3.6b) and (3.6d).

In making these statements we are aware of the work by Lewis [15] and Riesenfeld [16] on the exact solutions of the quantum time-dependent harmonic oscillator, (3.5) with $D = F = 0$, where the search is directed for quadratic invariants of the form

$$\mathbb{X}(t) = \tfrac{1}{2}[x_1(t)\mathbb{P}^2 + x_2(t)\{\mathbb{P}, \mathbb{Q}\} + x_3(t)\mathbb{Q}^2], \tag{6.7}$$

through the replacement in (6.1) and equating the function coefficients of the operator terms. In solving for the $x_k(t)$, $k = 1, 2, 3$ one finds that these are given in terms of a single function $\rho(t)$ determined by a nonlinear differential equation:

$$x_1(t) = \rho^2, \quad x_2(t) = -2\rho\dot{\rho}, \quad x_3(t) = \rho^{-2} + \dot{\rho}^2, \tag{6.8a}$$

$$\ddot{\rho}(t) + 2C(t)\rho(t) = \rho(t)^{-3}. \tag{6.8b}$$

The expression (6.7) was also shown [16] to factorize into a "raising and lowering" operator pair. Since they explicitly asked ∂_t not to appear in (6.7), they worked with (6.1), and not with (6.3), as one would normally do in applying similarity methods to the problem. The equivalence is nevertheless guaranteed by (6.2).

It is easy to verify that if we take an arbitrary linear combination of (6.6c), $c_1\mathbb{X}_1(t) + c_2\mathbb{X}_2(t) + c_3\mathbb{X}_3(t)$ (with c_k independent of time), then the coefficients of the $\mathbb{P}^2$-term, for example, will be $c_1\alpha^2 - 2c_2\alpha\beta + c_3\beta^2$. Comparison with (6.7)–(6.8a) leads us to put this equal to some $\sigma(t)^2$, i.e.,

$$\sigma(t) := [c_1\alpha^2 - 2c_2\alpha\beta + c_3\beta^2]^{1/2}. \tag{6.9}$$

Equations (3.6a)–(3.6b) together with (6.9) imply that the Lewis-Riesenfeld equation (6.8b) is satisfied for $\sigma(t)$ with a factor of $c_1c_3 - c_2^2$ on the right-hand side. The two other coefficients in (6.8a) follow suit. Indeed, this feature of the Lewis-Riesenfeld invariant in classical mechanics has been noticed before by Eliezer and Gray [4], who asked α and β only to be the Cartesian coordinates of a two-dimensional oscillator with a radially symmetric time-dependent oscillator of strength $2C(t)$. In this context, the angular momentum is shown to be related to the constant value of (6.7).

The new algebraic interpretation we obtain for the quantities α and β is that of the linear combination coefficients for a time-dependent linear canonical transformation in quantum-mechanical phase space. They can thus be seen as drawing out in this space the motion of the Eliezer-Gray associated oscillator, the angular momentum constancy $\dot{\alpha}\beta - \alpha\dot{\beta} = 1$ being due to the unimodularity condition in (2.1b) which in turn is a consequence of asking $\mathbb{X}_4(t)$ and $\mathbb{X}_5(t)$ to remain as canonically conjugate operators under the time-dependent canonical transformation. The two inhomogeneous parameters ε and η are due to the movement of the original quantum oscillator center and do not appear in the Eliezer-Gray associated oscillator. Their presence in (6.6) underlines the fact that the quantum-mechanical phase space (the W group manifold [21]) is really three-dimensional, the third dimension being generated by $\mathbb{1}$, which can be equivalently regarded as the phase fibers on a base Euclidean phase space.

Returning finally to the set of invariants (6.5) for the more general problem posed in (1.1) and choices of linear (not functional) combinations among them, convenience may dictate that we build the invariants out of the time evolution (6.4a) of the four Schrödinger Hamiltonians (2.8). Call them $\mathbb{X}_\omega(t)$, $\omega = h, r, l, f$. The similarity solutions $\Psi^\omega(q, t)$ seen in § 4 are the eigenfunctions of the latter $\mathbb{X}_\omega(t)$ with time-independent eigenvalues λ [16]. Our advantage in using Lie algebraic methods is the knowledge that, corresponding to the five orbits in wsl $(2, R)/\mathbb{1}$ we have the same number of algebraically inequivalent constants of motion which include first-order terms in addition to (6.7). These, moreover, in the form (6.2) provide us with a large class of realizations of the wsl $(2, R)$ algebra as first order two-variable operators (in q and t) which can be readily exponentiated to a multiplier realization of the group on space-time. The technique is straightforward:

$$\begin{aligned}\varphi(q, t) &\overset{g}{\mapsto} \varphi_g(q, t) := [\exp(i\mathbb{K}(t))\mathbb{I}\{g\}\varphi](q) \\ &= [\mathbb{I}\{g_e\} \exp(i\mathbb{K}(t_g(t)))\varphi](q) = \mu(g, t; g_e)\varphi(q'(q, t), t_g(t)).\end{aligned} \tag{6.10}$$

The first equality requires some matrix algebra to determine the parameters of the geometric transformation g_e and the transformation of the time variable to $t_g(t)$, while the second is a substitution from (2.6).

7. Further comments and extensions. Our present developments have been made through the intensive use of inhomogeneous linear canonical transforms. These are not the only canonical transforms, however. The method one could use for the closely related problems listed below do not differ significantly from those solved above. As Leach [13], [14] has shown, in classical mechanics one can handle other time-dependent analytic potentials in a recursive fashion. The only other one-dimensional potentials for which we can now offer a straightforward solution are those which contain $A(t)\mu\mathbb{Q}^{-2}$ centrifugal or centripetal terms, $\mu \in \mathcal{R}$, to the exclusion of the linear terms $D\mathbb{Q}+E\mathbb{P}$. For these we have radial [20] and hyperbolic [24] canonical transforms. The integral kernel (2.5b) (for $\varepsilon = \eta = 0$) will now consist of Gaussians times cylinder functions, while the similarity solutions (4.1)–(4.3) will involve Laguerre, Bessel and Whittaker functions. The gist of the method, the 2×2 matrix algebra, will be the same for the evolution operators determined through (3.4), separating coordinates (4.4), multipliers (4.5) and constants of motion (6.5). They are given by the same formulae with $\varepsilon = \eta = 0$, and should be compared with the results in [3] and [8].

Complex canonical transforms were used in § 5 for the study of Gaussians and coherent states, but not insisted upon. The same formulae apply for complex values of the time-dependent Hamiltonian formalism in the diffusion or Fokker-Planck systems. The added complication is that one must impose on the evolution operator that it remain within the semigroup of bounded operators. This condition is given in terms of demanding [19] that if $\alpha \neq 0$, $\operatorname{Im}(\alpha/\beta) \geqq 0$ while if $\alpha = 0$, $\operatorname{Im}\beta = 0$. The study of the conditions this imposes on the coefficients of $\mathbb{H}(t)$ in (1.1) (allowing temporary inverse-diffusion behavior, for instance) would be of interest.

The extension of our results to N-dimensional systems generalizing (1.1) needs the $\mathrm{w}_N\mathrm{sp}(2N, R)$ algebra and the corresponding group [19] (recall that $\mathrm{sp}(2, R) \simeq \mathrm{sl}(2, R)$). The orbit structure of $\mathrm{w}_N\mathrm{sp}(2N, R)$ has not yet been subject to a complete analysis, and neither can we apply the methods of § 4 to nonsubgroup separating coordinates [17]. The corresponding dynamical algebra, moreover, will be the full $\mathrm{w}_N\mathrm{sp}(2N, R)$, properly containing the commonly listed [11, 12] $\mathrm{su}(N, 1)$ or Schrödinger $\mathrm{w}_N \wedge (\mathrm{sl}(2, R)+\mathrm{so}(N))$ algebras. The $\mathrm{w}_N\mathrm{sp}(2N, R)$ are not all reducible to first-order differential operators in space and time [18].

Finally, regarding applications, Lewis and Riesenfeld [16] devoted much attention to the calculation of the S-matrix elements of a system which is asymptotically time-independent in the remote past and future. See also [7]. Indeed, the matrix elements of the time evolution operator $\mathbb{U}(t_1 \to t_2) = \exp(i\mathbb{K}(t_2)) \exp(-i\mathbb{K}(t_1))$ between asymptotic Hamiltonian eigenstates is a purely group-theoretic problem which requires the asymptotic properties of the irreducible representation matrix elements of the WSL $(2, R)$ group element g associated to $\mathbb{U}(t_1 \to t_2)$ between appropriate bases (4.1). In this regard, the state of the art seems to stand as follows. For the f-orbit eigenstates

(4.1d), the S-matrix elements are given by the Green's function corresponding to the group element fgf^{-1}, $f=\{-\mathbf{\Omega}, \mathbf{0}, 0\}$. For the h-, r- (and f-) orbits, the SL (2, R) part of the S-matrix can be given in terms of the results of [2], and for the Airy function l-orbit in terms of [23]. The W-part is known for the f- and h-orbits [21, §. IID] but not yet for the other two, and neither has their composition been performed explicitly.

Note added in proof. The author is indebted to Dr. V. I. Man'ko for kindly pointing out the work of the Lebedev Institute group, which has studied coherent states in time-dependent systems and has applied the results to electromagnetic interactions and molecular physics. This work was slighted, as the first paragraph of this article mentions it only through References [3] and [11]. Among the explicit references which should be added are [26]–[32].

REFERENCES

[1] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform, Part I*, Comm. Pure Appl. Math., 14 (1961), pp. 187–214.

[2] C. P. Boyer and K. B. Wolf, *Finite SL (2, R) representation matrices of the D_k^+ series for all subgroup reductions*, Rev. Mex. Física, 25 (1976), pp. 31–45.

[3] P. Camiz, A. Geradi, C. Marchioro, E. Presutti and E. Scacciatelli, *Exact solution of a time-dependent quantum harmonic oscillator with a singular perturbation*, J. Math. Phys., 12 (1971), pp. 2040–2043.

[4] C. J. Eliezer and A. Gray, *A note on the time-dependent harmonic oscillator*, this Journal, 30 (1976), pp. 463–468.

[5] N. J. Günther, *Dynamical symmetries of the Lewis-Riesenfeld oscillator*, Proceedings of the VII Integrative Conference on Group Theory and Mathematical Physics, Sept. 11–16, 1978, University of Texas, Austin TX.

[6] N. J. Günther and P. G. L. Leach, *Generalized invariants for the time-dependent harmonic oscillator*, J. Math. Phys., 18 (1977), pp. 572–576.

[7] J. Howland, *Stationary scattering theory for time-dependent hamiltonians*, Math. Ann., 207 (1974), pp. 315–335.

[8] D. C. Khandegar and S. V. Lawande, *Exact propagators for a time-dependent harmonic oscillator with or without a singular perturbation*, J. Math. Phys., 16 (1975), pp. 384–388.

[9] P. G. L. Leach, *On the theory of time-dependent linear canonical transformations as applied to Hamiltonians of the harmonic oscillator type*, J. Math. Phys., 18 (1977), pp. 1608–1611.

[10] ———, *Invariants and wavefunctions for some time-dependent harmonic oscillator-type Hamiltonians*, J. Math. Phys., 18 (1977), pp. 1902–1907.

[11] ———, *Quadratic Hamiltonians, quadratic invariants and the symmetry group* SU (n) J. Math. Phys., 19 (1978), pp. 446–451.

[12] ———, *On a generalization for the Lewis invariant for the time-dependent harmonic oscillator*, this Journal, 34 (1978), pp. 496–503.

[13] ———, *On nonlinear transformations for time-dependent polynomial hamiltonians*, J. Math. Phys., 20 (1979), pp. 86–95.

[14] ———, *Towards an invariant for the time-dependent anharmonic oscillator*, J. Math. Phys., 20 (1979), pp. 96–100.

[15] H. R. Lewis, *Class of exact invariants for classical and quantum time-dependent oscillators*, J. Math. Phys., 9 (1968), pp. 1976–1986.

[16] H. R. Lewis and W. B. Riesenfeld, *An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field*, J. Math. Phys., 10 (1969), pp. 1458–1473.

[17] W. Miller Jr., *Symmetry groups and separation of variables*, Encyclopedia of Mathematics Vol. 4, B.-C. Rota, ed., Addison Wesley, Reading, MA., 1977.

[18] S. Steinberg and K. B. Wolf, *Groups of integral transforms generated by Lie algebras of second-and-higher order differential operators*, Il Nuovo Cimento, 53A (1979), pp. 149–177.

[19] K. B. Wolf, *Canonical transforms. I. Complex linear transforms*, J. Math. Phys., 15 (1974), pp. 1295–1301.

[20] ———, *Canonical transforms. II. Complex radial transforms*, J. Math. Phys., 15 (1974), pp. 2102–2111.

[21] ———, *The Heisenberg-Weyl ring in quantum mechanics*, in Group Theory and its Applications, Vol. III, E. M. Loebl, ed., Academic Press, New York, 1975.

[22] ———, *Canonical transforms, separation of variables and similarity solutions for a class of parabolic differential equations*, J. Math. Phys., 17 (1976), pp. 602–613.

[23] ———, *The Airy function as a non-subgroup basis for the oscillator representation* $D^+_{1/4} \dot{+} D^+_{3/4}$ *of* SL (2, R). Rev. Mex. Física, 26 (1979), pp. 1–11.

[24] ———, *Canonical transforms IV. Hyperbolic transforms: continuous series of* SL (2, R) *representations*, J. Math. Phys., 21 (1980), pp. 680–688.

[25] ———, *Integral transforms in science and engineering, in* Mathematical Concepts and Methods in Science and Engineering, Vol. 11, A. Miele, ed., Plenum., New York, 1979.

[26] I. A. MALKIN AND V. I. MAN'KO, *Coherent states and excitation of N-dimensional non-stationary forced oscillator*, Phys. Lett., 32A (1970), pp. 243–244.

[27] I. A. MALKIN, V. I. MAN'KO AND D. A. TRIFONOV, *Coherent states and transition probabilities in a time-dependent electromagnetic field*, Phys. Rev. D., 2 (1970), pp. 1371–1385.

[28] ———, *Linear adiabatic invariants and coherent states*, J. Math. Phys., 14 (1973), pp. 573–582.

[29] E. V. IVANOVA, I. A. MALKIN AND V. I. MAN'KO, *Coherent states and radiation of non-stationary quadratic systems*, Phys. Lett., 50A (1974), pp. 23–24.

[30] V. V. DODONOV, I. A. MALKIN AND V. I. MAN'KO, *Integrals of the motion, Green functions, and coherent states of dynamical systems*, Int. J. Theoret. Phys., 24 (1975), pp. 37–54.

[31] E. V. DOKTOROV, I. A. MALKIN AND V. I. MAN'KO, *Dynamical symmetry of vibronic transitions in polyatomic molecules and the Franck–Condon principle*, J. Molec. Spectr., 64 (1977), pp. 302–326.

[32] I. A. MALKIN AND B. I. MAN'KO, *Dynamical symmetries and coherent states of quantum systems*, Nauka, Moscow, 1979 (In Russian).