On Time-Fractional Relativistic Diffusion Equations

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Abstract

The purpose of this paper is to consider the Cauchy problem for the time-fractional (both sub- and super- diffusive) relativistic diffusion equation. Based on the viewpoint of theory of pseudo-differential operators, we regard the equation as a pseudo differential equation, and we act the equation via the space-time transform. The solution is expressed as the convolution of the Green's function (heat kernel) and the initial data. The Green's functions are determined by their Fourier transforms, and are written in terms of Mittag-Leffler functions.

1 Introduction

The purpose of this paper is to consider the Cauchy problem for the following time-fractional relativistic diffusion equation (TFRDE, for brevity),

$$\frac{\partial^\beta}{\partial t^\beta}u(t,x) = H_{\alpha,m}u(t,x), \quad u(0,x) = u_0(x),\ t \geq 0, \quad x \in \mathbb{R}^n. \tag{1.1}$$

In the above, the fractional temporal derivative $\frac{\partial^\beta}{\partial t^\beta}$ is in the Caputo-Djrbashian sense (see the end of this section); while the spatial differential operator

$$H_{\alpha,m} := m - (m^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}}$$

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is a relativistic diffusion operator with the spatial-fractional parameter $\alpha \in (0, 2)$ and the normalized mass parameter $m > 0$.

The operator $H_{\alpha,m}$ appears in vast literature of mathematics and physics. The prominent case is $\alpha = 1$, which $-H_{1,m}$ represents the free energy of the relativistic Schrödinger operator with a relativistic particle of mass m; see the seminal paper of Carmona *et al.* [2] for mathematical discussions. For study on general $\alpha \in (0, 2)$, one may refer to Ryznar [10], Baeumer *et al.* [1], and the references therein. The operator $H_{\alpha,m}$ has also played an essential role in the theory of computer vision; see a special volume edited by Kimmel *et al.* [6], in which $H_{\alpha,m}$ is employed to connect the PDEs and the computer vision theory. In this paper, we consider (1.1) from the viewpoint of theory of pseudo-differential operators; see, for example, the book of Wong [14]. We regard (1.1) as a pseudo differential equation, and we act the equation via the space-time transform; see the statement and the proof of Proposition 1. We will also present in Proposition 3 a corresponding stochastic processes viewpoint, when $\beta \leq 1$. We should remark that the viewpoint of pseudo-differential operators is very inspiring to catch the essence of the solution of (1.1), and also works for the whole time-fractional range $\beta \in (0, 2)$. We state our results in Section 2, and all the proofs are given in Section 3. In the final Section 4, we give remarks on the two-scale property of relativistic Green's functions (heat kernels).

Here, we review briefly the fractional time-derivatives as follow; see, for example, Djrbashian [4] for details.

$$\frac{d^\beta f}{dt^\beta}(t) := \begin{cases} f^{(m)}(t) & \text{if } \beta = m \in \mathbb{N} \\ \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} d\tau & \text{if } \beta \in (m-1, m), \end{cases} \tag{1.2}$$

where $f^{(m)}(t)$ denotes the ordinary (non-fractional) derivative of order m of a causal function $f(t)$ (i.e., f is vanishing for $t < 0$). In this note, we mainly use $0 < \beta < 1$ and $1 < \beta < 2$; which are referred respectively as the *sub-diffusive* and the *super-diffusive*, since the $\beta = 1$ is well-known as the diffusive.

For the expression of the Green's function in the time-fractional index β, we need the following Mittag-Leffler (ML) functions; see, for example, [4, Chapter 1] for the details. The ML function is defined by, for $k = 1, 2$,

$$E_{\beta,k}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\beta i + k)}, \quad z \in \mathbb{C}. \tag{1.3}$$

We remark that a ML function is an entire function on the complex plane and its asymptotic behaviors, when $\beta \in (0, 2), \quad \beta \neq 1$, have the inverse power law as follows:

$$|E_{\beta,k}(z)| \ \sim \ O(\frac{1}{|z|}), \quad |z| \ \to \ \infty \text{ with } |\arg(-z)| < \pi(1 - \frac{\beta}{2}), \tag{1.4}$$

where arg: $\mathbb{C} \to (-\pi, \pi)$ and the notation $f(z) \sim O(g(z))$ means that $f(z)/g(z)$ remains bounded as z approaches the indicated limit point; for (1.4), see the classic book by Erdélyi *et al.* [5] (pp. 206-212, especially p. 206 (7) and p. 210 (21)).

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2 Main results

Since (1.1) is of linear-parabolic type, the solution must be expressed as the convolution of the Green's function (heat kernel) and the initial condition. Therefore, the form of the solution expressed below is not surprising; however the defining display for the Green's function is completely new, to our knowledge. For an $f \in L^2(\mathbb{R}^n, Leb)$, we use $\hat{f}$ to denote the Fourier (-Plancherel) transform of f. In the context henceforth, we skip the indices α, m from the Green's functions for most time; yet will resume these two indices when it is necessary.

Proposition 1. *The solution $u(t, x)$ of (1.1) is expressed:*

- $\beta = 1$*(diffusive): with the given initial $u_0(x) = f(x)$, and with the Green's function denoted by $G = G_{\alpha,m}$,*

$$u(t, x) = \int_{\mathbb{R}^n} G(t, x - y) f(y) dy.$$

- $\beta : 0 < \beta < 1$*(sub-diffusive): with the given initial* $u_0(x) = f(x)$*, and with the associated Green's function denoted by* $G_{\beta,1} = G_{\beta,1;\alpha,m}$*,*

$$u(t,x) = \int_{\mathbb{R}^n} G_{\beta,1}(t, x-y) f(y) dy.$$

- $\beta : 1 < \beta < 2$*(super-diffusive): with the two given initials* $u_0(x) = f(x)$*, and* $\frac{\partial_t u}{\partial t}|_{t=0}(x) = g(x)$*, and with the two associated Green's functions denoted by* $G_{\beta,i} = G_{\beta,i;\alpha,m}$*,* $i = 1, 2$ *(respectively associated with the two initials* $f(x), g(x)$ *),*

$$u(t,x) = \int_{\mathbb{R}^n} G_{\beta,1}(t, x-y) f(y) dy + \int_{\mathbb{R}^n} G_{\beta,2}(t, x-y) g(y) dy.$$

In the above, the initial(s) f *(and* g *in the super-diffusive) are assumed in the following subspace* D_A *of* $L^2(\mathbb{R}^n, Leb)$*.*

$$D_A := \{f \in L^2(\mathbb{R}^n, Leb) : \int_{\mathbb{R}^n} \theta(\lambda) |\hat{f}(\lambda)|^2 d\lambda < \infty\},$$

with

$$\theta(\lambda) := (m^{\frac{2}{\alpha}} + |\lambda|^2)^{\frac{\alpha}{2}} - m > 0, \quad \forall \lambda \neq 0.$$

The defining displays for the above Green's functions are, respectively, via the spatial Fourier transforms as follows: for each $t > 0$ *fixed, for* $\lambda \in \mathbb{R}^n$*, with* $< \cdot, \cdot >$ *denoting the inner product in* $\mathbb{R}^n$*,*

- $\beta = 1$,

$$\int_{\mathbb{R}^n} e^{i<\lambda,x>} G(t,x) dx = e^{-t\theta(\lambda)}. \tag{2.1}$$

- $\beta \in (0, 2), \beta \neq 1$,

$$\int_{\mathbb{R}^n} e^{i<\lambda,x>} G_{\beta,1}(t,x) dx = E_{\beta,1}(-t^\beta \theta(\lambda)). \tag{2.2}$$

- $\beta \in (1, 2)$,

$$\int_{\mathbb{R}^n} e^{i<\lambda,x>} G_{\beta,2}(t,x) dx = t \cdot E_{\beta,2}(-t^\beta \theta(\lambda)). \tag{2.3}$$

For the prominent case, that is, $\alpha = 1$ and $m > 0$, from the explicit expression of the Green's function $G_{1,m}(t,x)$ appeared in Wong [13, pp. 195-6] (we need to multiply e^{mt} to the $K_t(x)$ there to meet our present situation), we see that $G_{1,m}(t,x)$ is positive valued, and its integral over x is 1. See also Carmona *et al.* [2, pp. 123-4] from the viewpoint of stochastic processes. The following property shows that the corresponding sub-diffusive Green's function also behaves in the same manner.

Proposition 2. *For the sub-diffusive $0 < \beta < 1$ with $\alpha = 1$ and $m > 0$; for each $t > 0$, the associated Green's function $G_{\beta,1}(t,x)$ satisfies*

$$G_{\beta,1}(t,x) > 0; \quad \int_{\mathbb{R}^n} G_{\beta,1}(t,x)dx = E_{\beta,1}(-t^{\beta}\theta(0)) = 1. \tag{2.4}$$

Remarks: 1. For the super-diffusive $\beta \in (1,2)$ and the two associated Green's functions $G_{\beta,k}, k = 1,2$, we can view Proposition 2 in the following informal way: let us rewrite the latter as $G_{\beta,k+1}, k = 0,1$, and let $\theta(\lambda)$ be the θ function defined in Proposition 1. Then, we have

$$\int_{\mathbb{R}^n} G_{\beta,k+1}(x,t)dx = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i<\lambda,x>} t^k E_{\beta,k+1}(-\theta(\lambda)t^{\beta})d\lambda \; dx. \tag{2.5}$$

Using the symbolic relation of the δ_0-function

$$\delta_0(\lambda) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i<\lambda,x>} dx,$$

we see that the display (2.5) is symbolically equal to

$$\int_{\mathbb{R}^n} \delta_0(\lambda) t^k E_{\beta,k+1}(-\theta(\lambda))d\lambda = t^k E_{\beta,k+1}(0) = t^k.$$

The last relation in the above seems to indicate that the super-diffusive *does not* have the propobilistic interpretation; such an interpretation for the diffusive and the sub-diffusive will be given in the below, Proposition 3.

2. Proposition 2 is stated and proved *only* for the case $\alpha = 1$, since we use the explicit expression of $G_{1,m}(t,x)$ to see that it is postive valued; this positivity property seems not readily seen from the Fourier transform expression of the $G_{\alpha,m}(t,x)$, for general $\alpha \in (0,2)$.

The following property a probabilistic viewpoint for the solution of (1.1) in the diffusive and the sub-diffusive cases; it is based on the (Bochner) subordination of the Brownian motion in $\mathbb{R}^n$; this probabilistic approach in, say, [3] allows to consider the sub-diffusive RDE on a bounded domain.

Proposition 3. *In the diffusive and the sub-diffusive cases, the solution $u(t,x)$ of (1.1) has a probabilistic interpretation: with the given initial data $f(x) = u_0(x)$, for the diffusive, $u(t,x) = E_x[f(X_t)]$, and for the sub-diffusive, $u(t,x) = E_x[f(Y_t)]$. The stochastic process X_t is the Brownian motion B_t on $\mathbb{R}^n$ subordinated by a relativistic $\frac{\alpha}{2}$-subordinator. The stochastic process Y_t is the X_t further subordinated by a β-subordinator. The notation $E_x[\cdot]$ means the expectation w.r.t. the process starting at x, and the foot-index x is skipped when the process starts at $x = 0$.*

3 Proofs

Proof of Proposition 1. The diffusive case is typical; indeed the display for the Green's function $G(t,x)$ is a consequence of our viewing that RDE ($\beta = 1$) as a pseudo differential equation. The proof of the sub-diffusive case can be imbedded in the super-diffusive case (with $g(x)$ identically to be zero). Therefore we proceed the proof for the latter case, $1 < \beta < 2$. The proof is based on the temporal-spatial transform arguments: take the Laplace transform $\mathcal{L}(\cdot)$ with respect to the temporal variable t, and then take the Fourier transform $\mathcal{F}(\cdot)$ with respect to the spatial variable x. For the notational consistency, we use $(\hat{x}, \hat{t})$ to denote the spatial-temporal variable in the transformed space-time. Firstly, by the convolution for Laplace transform and the integration-by-parts on the defining display of the fractional derivative (1.2) for the causal functions, we see that $u(x,t)$ must satisfy

$$\{-\frac{\partial u}{\partial t}u(x,0) - \hat{t}u(x,0) + \hat{t}^2\mathcal{L}u(x,\hat{t})\}\hat{t}^{\beta-2} = H_{\alpha,m}\mathcal{L}u(x,\hat{t}). \tag{3.1}$$

Then, we take the spatial Fourier transform $\mathcal{F}(\cdot)$ to (3.1); we then have

$$\{-\mathcal{F}\frac{\partial u}{\partial t}(\hat{x},0) - \hat{t}\mathcal{F}u(\hat{x},0) + \hat{t}^2\mathcal{F}\mathcal{L}u(\hat{x},\hat{t})\}\hat{t}^{\beta-2} = -\{(m^{\frac{2}{\alpha}} + |\hat{x}|^2)^{\frac{\alpha}{2}} - m\}\mathcal{F}\mathcal{L}u(\hat{x},\hat{t}),$$

which can be rewritten as

$$\mathcal{F}\mathcal{L}u(\hat{x},\hat{t}) = \frac{-\hat{t}^{\beta-2}}{-\{(m^{\frac{2}{\alpha}} + |\hat{x}|^2)^{\frac{\alpha}{2}} - m\} - \hat{t}^{\beta}}(\hat{t}\mathcal{F}u(\hat{x},0) + \mathcal{F}\frac{\partial u}{\partial t}(\hat{x},0)). \tag{3.2}$$

For $\theta = \theta(\hat{x}) := (m^{\frac{2}{\alpha}} + |\hat{x}|^2)^{\frac{\alpha}{2}} - m > 0$, we expand the $\frac{-\hat{t}^{\beta-1}}{-\theta-\hat{t}^{\beta}}$ and the $\frac{-\hat{t}^{\beta-2}}{-\theta-\hat{t}^{\beta}}$ in powers of $\hat{t}^{-1}$ by geometric series and represent the inverse power of $\hat{t}$ by the integral representation for the Γ-function (see, for example, Podlubny [9, (1.80)]), then we can have

$$\frac{-\hat{t}^{\beta-1}}{-\theta-\hat{t}^{\beta}} = \mathcal{L}(E_{\beta,1}(-\theta\odot^{\beta}))(\hat{t})$$

and

$$\frac{-\hat{t}^{\beta-2}}{-\theta-\hat{t}^{\beta}} = \mathcal{L}(\odot E_{\beta,2}(-\theta\odot^{\beta}))(\hat{t}),$$

in which $\odot$ wants to mean the variable in t to be under the $\mathcal{L}$. Then, we take the inverse Laplace transform to (3.2) to obtain

$$\mathcal{F}u(\hat{x},t) = E_{\beta,1}(-\theta t^{\beta}) + tE_{\beta,2}(-\theta t^{\beta}).$$

As a final step, we take the the inverse Fourier transform to the above, and then use the convolution to obtain the solution; the displays for the Green's functions are consequently derived. □

Proof of Proposition 2. From Wyss and Wyss [15], noting that we are now in the sub-diffusive, we can express the associated Green's function as

$$G_{\beta,1}(t,x) = \int_0^{\infty} f_{\beta}(z)G(t^{\beta}z,x)dz,\ x\in\mathbb{R}^n,\ t>0; \tag{3.3}$$

in which, $G(t,x)$ is the Green's function determined by the diffusive case, with $\alpha = 1$ and $m > 0$ (see Proposition 1), and $f_{\beta}(z)$ is a nonnegative-valued function of $z \geq 0$ represented in terms of so-called H-function (see, for example, Schneider [12, p. 284])

$$f_{\beta}(z) = H_{11}^{10}\left(z|\begin{array}{c}(1-\beta,\beta)\\(0,1)\end{array}\right),\ z\geq 0; \tag{3.4}$$

indeed f_β is with Laplace transform

$$\int_0^\infty e^{-sz} f_\beta(z)dz = E_{\beta,1}(-s),\ \ s \geq 0. \tag{3.5}$$

By (3.3), and the positivity of $G(t,x)$ as we remark in Section 2, we see that $G_{\beta,1}(t,x) > 0$. Moreover,

$$\begin{aligned} \int_{\mathbb{R}^n} G_{\beta,1}(t,x)\ dx &= \int_{\mathbb{R}^n} \int_0^\infty f_\beta(z) G(t^\beta z, x)\ dzdx \\ &= \int_0^\infty f_\beta(z) \int_{\mathbb{R}^n} G(t^\beta z, x)\ dxdz \\ &= \int_0^\infty f_\beta(z) \cdot e^{-t^\beta z\theta(0)}\ dz = \ E_{\beta,1}(-t^\beta\theta(0)) = 1. \end{aligned} \tag{3.6}$$

In the above, we use Tonelli Theorem (it is legitimate to use, since the integrand is known to be nonnegative) to change the order of integrations, and we also use the (3.5) and the defining display of $G(t,x)$ in Proposition 1. Note that $\theta(0) = 0$, and that $E_{\beta,1}(0) = 1$. □

Proof of Proposition 3. The proof is adapted from two references, Ryznar [10] and Chen *et al.* [3]. In [10], the relativistic relativistic $(\alpha/2)$-subordinator is introduced, as a Lévy process T_t with increasing sample paths and with the Laplace function determined by

$$E\Big[e^{-uT_t}\Big] = e^{-t\{(m^{\frac{2}{\alpha}}+u)^{\frac{\alpha}{2}}-m\}},\ \ u > 0.$$

Assume that T_t and the Brownian motion B_t in $\mathbb{R}^n$ are totally independent, then the subordinated process $X_t = B_{T_t}$ is a Lévy process $\mathbb{R}^n$, for which the characteristic function is given by,

$$E\Big[e^{i<\lambda, X_t>}\Big] = e^{-t\{(m^{\frac{2}{\alpha}}+|\lambda|^2)^{\frac{\alpha}{2}}-m\}},\ \ \lambda \in \mathbb{R}^n.$$

This means that the transition density function of the process X_t is exactly the Green's function of the (1.1). Now, for the sub-diffusive case $\beta < 1$, the general theory for sub-diffusive processes mentioned, for example, in [3] and the references therein, asserts the second part of the proposition. Indeed, let S_t be a subordinator for which Laplace function is determined by

$$E\Big[e^{-uS_t}\Big] = e^{-tu^\beta},\ \ u > 0.$$

Assume that the processes S_t is totally independent from a Markov process X_t for which the infinitesimal generator is denoted by $\mathcal{A}$, then, by [3, Section 3], the sub-fractional diffusive equation, regarded as an evolution equation,

$$\frac{\partial^\beta u}{\partial t^\beta} = \mathcal{A}u, \quad u(0) = f,$$

has the solution $u(t,x) = E_x[f(Y_t)]$, where the process $Y_t := X_{E_t}$, and E_t is the right-inverse of S_t, i.e. $E_t = \inf\{s : S_s > t\}$. The generator of the Lévy process X_t is $H_{\alpha,m}$, which can be seen from the characteristic function expression of X_t given above and the theory of Lévy processes in, say, the book of Sato [11]. This gives the assertion. □

Remark: We should remark that Proposition 3 *is not* fully equivalent to Proposition 1, since our domain of the action D_A in Proposition 1 is not, in general, known to be the same as the domain of the Markov generator $\mathcal{A}$.

4 Remark on Green's functions

Here, we make the remarks on how the viewpoint of pseudo-differential operators gives us the insight on the RDEs. The Fourier expression $\widehat{G}_{\alpha,m}$ of the Green's function $G_{\alpha,m}(t,x)$ in Proposition 1 (we add foot-index α, m here to clarity their roles in the below) gives us the following two-scale property:

When $T \to \infty$,

$$\widehat{G}_{\alpha,m}(Tt, T^{-\frac{1}{2}}\lambda) = \exp\Big\{Tt(m - (m^{\frac{2}{\alpha}} + T^{-1}|\lambda|^2)^{\frac{\alpha}{2}})\Big\} \to \exp\Big\{-t\frac{\alpha}{2}m^{1-\frac{2}{\alpha}}|\lambda|^2\Big\}.$$

When $\varepsilon \to 0$,

$$\widehat{G}_{\alpha,m}(\varepsilon t, \varepsilon^{-\frac{1}{\alpha}}\lambda) = e^{\varepsilon t \mathfrak{m}} e^{-\varepsilon t(\mathfrak{m}^{\frac{2}{\alpha}} + \varepsilon^{-\frac{2}{\alpha}}|\lambda|^2)^{\frac{\alpha}{2}}} \to e^{-t|\lambda|^\alpha}.$$

The first one is observed by the limiting on the first two terms in the concerned fractional-binomial expansion, and the second one is observed by the direct limiting.

The above two-scale property asserts that: the large-scale is dominated by the mass index m, and the small-scale is dominated by the spatial-fractional index α. This two-scale property is key to consider the multi-scaling property of RDEs, as shown in [7, 8].

As for the time-fractional case, the Fourier expression $\widehat{G}_{\beta,1}$ of the first associated Green's function $G_{\beta,1}$ in Proposition 1 also exhibits the two-scale via its Fourier transform: $\widehat{G}_{\beta,1}(Tt, T^{-\frac{\beta}{2}}\lambda)$, for large T, and $\widehat{G}_{\beta,1}(\varepsilon t, \varepsilon^{-\frac{\beta}{\alpha}}\lambda)$, for small ε. The displays will then use the ML function $E_{\beta,1}$ to replace the exponential function.

However, the the Fourier expression $\widehat{G}_{\beta,2}$ of the second associated Green's function $G_{\beta,2}$, appearing only in the super-diffusive, has a aggregating time factor t in its expression, which should be quite different from the above observations.

Finally, we mention that, though the explicit expression for the prominent $G_{1,m}(t,x)$ has appeared in Carmona *et al.* [2, pp. 123-4] and Wong [13, pp. 195-6], it seems that there is no such explicit expression for $G_{\alpha,m}(t,x)$ for general $\alpha \in (0,2)$.

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