ON TIME-SEQUENTIAL POINT ESTIMATION OF THE MEAN OF AN EXPONENTIAL DISTRIBUTION

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ABSTRACT

In the context of life testing, an asymptotically risk-efficient time-sequential procedure for estimating the mean of an exponential distribution is considered and its various properties are studied.

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1. INTRODUCTION

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) non-negative random variables (r.v.) with the distribution function (d.f.) $F_\theta(x) = 1 - e^{-x/\theta}$, $x \in [0, \infty)$, where $\theta(> 0)$ is an unknown parameter. For $n(> 1)$, items under *life testing*, the *failures* $X_{n,1}, \ldots, X_{n,n}$ are the *order statistics* corresponding to $X_1, \ldots, X_n$ and, from cost and time considerations, one may curtail experimentation at the k^{th} failure $X_{n,k}$ and estimate θ by

(1.1) $\hat{\theta}_{nk} = k^{-1}V_{nk}$, where $V_{nk} = \sum_{i=1}^{k} X_{n,i} + (n-k)X_{n,k}$ for $1 \leq k \leq n$.

Note that V_{nk} is the *total life under test upto the k^{th} failure*, $EV_{nk} = k\theta$ and $E(V_{nk} - k\theta)^2 = k\theta^2$, for $k = 1, \ldots, n$. Thus, if $a_1(> 0)$ and $a_2(> 0)$ be respectively the cost of recruitment (per individual) and of follow-up (per unit of test-life), then one may conceive of the *loss* incurred in estimating θ by $\hat{\theta}_{nk}$ as

(1.2) $$L_{nk} = a_0(\hat{\theta}_{nk} - \theta)^2 + a_1 n + a_2 V_{nk} \quad (1 \leq k \leq n),$$

where the weights $a_0(> 0)$, a_1 and a_2 are all known. Thus, the *risk* in estimating θ by $\hat{\theta}_{nk}$ is

(1.3) $$R_{nk}(\underset{\sim}{a}, \theta) = EL_{nk} = k^{-1}a_0\theta^2 + a_1 n + a_2 k\theta \quad (\underset{\sim}{a} = (a_0, a_1, a_2)'),$$

and, naturally, one would seek to minimize (1.3) by a proper choice of k. However, as θ is unknown, no single value of k minimizes $R_{nk}(\underset{\sim}{a}, \theta)$ for all $\theta(> 0)$, and hence, a *time-sequential* procedure for choosing such a value of k is desirable.

Motivated by the works of Robbins (1959), Starr and Woodroofe (1972) and Ghosh and Mukhopadhyay (1979) [all dealing with the classical sequential point estimation case], in Section 2, we formulate a time-sequential procedure for our problem and under asymptotic setup (similar to their cases) study its various properties. The derivation of the main results are postponed to concluding section.

2. TIME-SEQUENTIAL POINT ESTIMATION OF θ

Note that by (1.3),

(2.1) $\quad R_{nk}(\underset{\sim}{a}, \theta) - R_{nk+1}(\underset{\sim}{a}, \theta) \gtreqless 0$ according as $k(k+1) \lesseqgtr \theta a_0/a_2$.

Thus, if $n(n-1) < a_0\theta/a_2$, then $R_{nk}(\underset{\sim}{a}, \theta)$ is $\downarrow$ in $k(1 \le k \le n)$, and hence, $k = n$ is an optimal choice. On the other hand, if $n(n-1) \ge \theta a_0/a_2$, then there exists an optimal $k_n(= k_n(\underset{\sim}{a}, \theta))$ for which $k_n < n$ and $R_{nk}(\underset{\sim}{a}, \theta)$ is minimized for $k = k_n$. Since $\hat{\theta}_{nk} = k^{-1}V_{nk}$ is an unbiased estimator of θ, motivated by the above, we consider the following *stopping number*

$$(2.2) \quad N_n = N_n(\underset{\sim}{a}) = \begin{cases} \text{smallest } k(1 \le k \le n-1) \text{ for which } V_{nk} \le k^2(k+1)a_2/a_0, \\ n \text{ if } V_{nk} > k^2(k+1)a_2/a_0, \text{ for every } 1 \le k \le n-1. \end{cases}$$

The corresponding *stopping time* is X_{n,N_n} and the point estimator of θ is $\hat{\theta}_{nN_n}$. Then, the *risk* corresponding to $\hat{\theta}_{nN_n}$ is

$$(2.3) \qquad R_n^*(\underset{\sim}{a}, \theta) = a_0 E(\hat{\theta}_{nN_n} - \theta)^2 + a_1 n + a_2 EV_{nN_n} .$$

We may recall that by definition,

$$(2.4) \quad k_n = k_n(\underset{\sim}{a}, \theta) = \begin{cases} \text{smallest } k(1 \le k \le n-1) \text{ for which } k(k+1) \ge \theta a_0/a_2. \\ n \text{ if } n(n-1) < \theta a_0/a_2 . \end{cases}$$

Let then

$$(2.5) \qquad R_n^0(\underset{\sim}{a}, \theta) = R_{nk_n}(\underset{\sim}{a}, \theta) .$$

Our primary interest centers around the behavior of (a) N_n/k_n and

(b) $R_n^*(\underset{\sim}{a}, \theta)/R_n^0(\underset{\sim}{a}, \theta)$ when we impose some asymptotic considerations on $\underset{\sim}{a}$ and n.

In the classical sequential point estimation theory [c.f. Robbins (1959) and others], $a_2 = 0$, $L_n = a_0(\hat{\theta}_{nn} - \theta)^2 + a_1 n$ and the problem is to choose n in such a way that the corresponding risk is minimized. In this context, one lets $a_1 \to 0$ and, in this asymptotic sense, one obtains some optimal results. In our case, however, for a given n, the stopping number N_n depends on a_0 and a_2, but not on a_1, and we let $a_2/a_0 \to 0$ or, simply, $a_2 \to 0$, keeping a_0 fixed. Note that our main interest lies in the case where k_n in (2.4) is $< n$ and in this case, $a_2 n^2 > a_2 n(n-1) \geq \theta a_0 > 0$. We assume that the sample size $n = n(a_2)$ depends on a_2 in such a way that

(2.6) $$\lim_{a_2 \to 0} a_2[n(a_2)]^2 = a^*: \quad 0 < a^* < \infty .$$

We may note that by (1.3), $R_{n'k}(\underset{\sim}{a}, \theta) = R_{nk}(\underset{\sim}{a}, \theta) + a_1(n' - n) \geq R_{nk}(\underset{\sim}{a}, \theta)$, $\forall\, n' \geq n$, and hence, there is no point in increasing $n(a_2)$ indefinitely even when we allow $a_2 \to 0$, so that the restriction that a^* in (2.6) is $< \infty$ is of no loss of generality. Secondly, we note that for $\{n\}$ satisfying (2.6), by (2.4),

(2.7) $$\lim_{n\to\infty} k_n/n = \gamma = (\theta a_0/a_2)^{\frac{1}{2}} \text{ and we assume that } 0 < \gamma < 1.$$

In terms of (2.6), (2.7) demands that $a^* > \theta a_0$. Finally, as in the classical sequential point estimation case, we assume that $a_1 \to 0$. More explicitly, we let

(2.8) $$a_1 = \rho a_2, \text{ where } \rho > 0, \text{ and allow } a_2 \to 0.$$

Then, we have the following

<u>Theorem 1.</u> *Under* (2.6) *and* (2.7),

(2.9) $$N_n/k_n \to 1 \quad \textit{almost surely (a.s.) as} \quad a_2 \to 0.$$

Moreover, for every real x $(-\infty < x < \infty)$, under (2.6) and (2.7),

(2.10) $$\lim_{n\to\infty} P\{2(N_n - k_n)/((n\gamma)^{\frac{1}{2}}\theta) \le x\} = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt.$$

<u>Theorem 2.</u> *Under* (2.6), (2.7) *and* (2.8),

(2.11) $$\lim_{a_2\to 0} R_n^*(\underset{\sim}{a}, \theta)/R_n^0(\underset{\sim}{a}, \theta) = 1.$$

We may remark that by (2.2), $N_n = N_n(\underset{\sim}{a})$ is $\downarrow$ in a_2 and for any given n, there exists an $a_2(n)\,(>0)$, such that $N_n = n$, $\forall\ 0 < a_2 \le a_2(n)$. Also, $N_n \le n$, with probability 1, so that (2.7) and (2.9) insure that $EN_n/k_n \to 1$ as $n \to \infty$. Further, (2.11) holds even when in (2.8), $\rho = 0$. If $a_1/a_2 \to \infty$ as $a_2 \to 0$, then R_n^* or R_n^0 are both dominated by $a_1 n$, and hence, (2.11) holds trivially.

3. <u>PROOFS OF THEOREMS 1 AND 2</u>

Let us denote by $V_{n0} = 0$ and

(3.1) $$Z_{nk} = V_{nk} - V_{nk-1} = (n-k+1)(X_{n,k} - X_{n,k-1}), \quad 1 \le k \le n$$

(where $X_{n,0} = 0$). Then $Z_{n1}, \ldots, Z_{nn}$ are i.i.d.r.v. each having the d.f. $F_\theta(x) = 1 - e^{-x/\theta}$. Also, note that for every $n(\ge 1)$,

(3.2) $$V_{nk} \text{ is } \uparrow \text{ in } k: \quad 0 \le k \le n.$$

Further, note that for every $\eta > 0$,

(3.3) $$\begin{aligned} &P\{X_{m,1} < m^{-1-\eta}, \text{ for some } m \ge n\} \\ &\le \textstyle\sum_{k=0}^{\infty} P\{X_{n2^k+j,1} < (n2^k+j)^{-1-\eta} \text{ for some } 0 \le j \le n2^k\} \\ &\le \textstyle\sum_{k\ge 0} P\{X_{2^{k+1}n,1} < (2^k n)^{-1-\eta}\} = \sum_{k\ge 0}\{1 - e^{-2(2^k n)^{-\eta}}\} \\ &\le \textstyle\sum_{k\ge 0}\{2(n2^k)^{-\eta}\} = 2n^{-\eta}(1-2^{-\eta})^{-1} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, by (3.1) and (3.3), for every $\eta > 0$,

(3.4) $$V_{n1} > n^{-\eta} \text{ a.s., as } n \to \infty.$$

Let us now choose a positive number λ such that

(3.5) $$\frac{1}{2} < \lambda < \frac{2}{3}, \quad \text{i.e.,} \quad \xi = \frac{2}{3} - \lambda > 0 .$$

Then, under (2.6) and (2.7), by using (2.4) and (3.2),

(3.6) $$P\{N_m \le m^\lambda \quad \text{for some} \quad m \ge n\}$$
$$= P\{\bigcup_{m\ge n} [V_{mk} \le k^2(k+1)a_2/a_0, \quad \text{for some} \quad k \le m^\lambda]\}$$
$$\le P\{\bigcup_{m\ge n} [V_{m1} \le (a_2 m^2/a_0)[m^{2\lambda}(m^\lambda + 1)/m^2]]\} ,$$

where $a_2 m^2/a_0 \to a^*/a_0 (> 0)$ while by (3.5), $m^{2\lambda}(m^\lambda + 1)/m^2 \sim m^{-\xi}$, so that by (3.4), the right-hand side (rhs) of (3.6) converges to 0 as $n \to \infty$. Let us now denote by

(3.7) $$k_{n\varepsilon}^{(1)} = \max\{k: \quad k(k+1) \le (1-\varepsilon)k_n(k_n + 1)\}, \quad 0 < \varepsilon < 1,$$

where k_n is defined by (2.4). Also, we choose n so large that $n^\lambda \le k_{n\varepsilon}$. Then

(3.8) $$P\{N_m \le k_{m\varepsilon} \quad \text{for some} \quad m \ge n\}$$
$$\le P\{N_m \le m^\lambda \quad \text{for some} \quad m \ge n\} +$$
$$P\{\bigcup_{m\ge n} [V_{mk} < k^2(k+1)a_2/a_0 \quad \text{for some} \quad k: \quad m^\lambda \le k \le k_{m\varepsilon}]\} .$$

By (3.6) the first term on the rhs of (3.8) converges to 0 as $n \to \infty$, while by (3.7), the second term is bounded by

(3.9) $$\sum_{m\ge n} P\{(V_{mk} - k\theta)/k\theta < -\eta, \quad \text{for some} \quad k: \quad m^\lambda \le k \le k_{m\varepsilon}\} ,$$

where $\eta (> 0)$ depends on $\varepsilon (> 0)$ in (3.7). By (3.1), for every $n (\ge 1)$, $\{(V_{nk} - k\theta)/\theta, \quad 0 \le k \le n\}$ is a martingale, so that $\{(V_{nk} - k\theta)^4/\theta^4, \quad 0 \le k \le n\}$

is a sub-martingale, and hence, by the Chow (1961) extension of the Hájek-Rényi inequality,

(3.10) $$P\{(V_{mk}-k\theta)/k\theta < -\eta, \text{ for some } m^{\lambda} \le k \le k_{m\varepsilon}\}$$
$$\le P\{(V_{mk}-k\theta)^4/k^4\theta^4 > \eta^4 \text{ for some } k:\ m^{\lambda} \le k \le k_{m\varepsilon}\}$$
$$\le \sum_{k=[m^{\lambda}]}^{k_{m\varepsilon}} \left\{\eta^{-4}E(V_{mk}-k\theta)^4/\theta^4\right\}\left\{k^{-4}-(k+1)^{-4}\right\}$$
$$\le \eta^{-4}\sum_{k\ge[m^{\lambda}]}[O(k^{-3})] = \eta^{-4}\cdot O(m^{-2\lambda}),$$

so that by (3.5) and (3.10), the second term on the rhs of (3.8) converges to 0 as $n\to\infty$. Thus, for every $\varepsilon>0$,

(3.11) $$N_n/k_n > 1-\varepsilon \quad \text{a.s., as } n\to\infty .$$

In a similar way, it follows that for every $\varepsilon>0$,

(3.12) $$N_n/k_n < 1+\varepsilon \quad \text{a.s., as } n\to\infty ,$$

and (2.9) follows from (3.11) and (3.12).

To prove (2.10), we note that for every (fixed) $u\in(-\infty,\infty)$,

(3.13) $$P\{N_n \ge k_n + u\sqrt{n}\} = P\{V_{nk} > k^2(k+1)a_2/a_0,\ \forall\ k \le k_n + u\sqrt{n}\},$$

and we choose n so large that $k_n + u\sqrt{n} > k_{n\varepsilon}$, where $k_{n\varepsilon}$ is defined by (3.7) and k_n by (2.4). Then, by using (3.11), the rhs of (3.13) can be written as

(3.14) $$P\{V_{nk} > k^2(k+1)a_2/a_0,\ \forall\ k_{n\varepsilon} \le k \le k_n + u\sqrt{n}\} + o(1)$$
$$= P\left\{\frac{V_{nk}-k\theta}{\theta\sqrt{n}} > \frac{k}{\sqrt{n}}\left[\frac{k(k+1)}{k_n(k_n+1)} - 1\right],\ \forall\ k_{n\varepsilon} \le k \le k_n + u\sqrt{n}\right\} + o(1).$$

Let us now consider a sequence $\{W_n = \{W_n(t),\ t\in[0,1]\},\ n\ge 1\}$ of stochastic processes, where we let $W_n(t) = W_n(\frac{k}{n})$, for $\frac{k}{n} \le t < \frac{k+1}{n}$, $0\le k\le n-1$ and $W_n(k/n) = (V_{nk}-k\theta)/\theta\sqrt{n}$, $k=0,1,\dots,n$. Then by virtue of (3.1), the classical Donsker Theorem applies and we have

(3.15) $$W_n \xrightarrow{\mathcal{D}} W = \{W(t),\ t\in[0,1]\},$$

where W is a standard Wiener process on $[0, 1]$. As a corollary to (3.15), we have that for every $\varepsilon' > 0$ and $\eta' > 0$ there exist a δ: $0 < \delta < 1$ and an n_0 such that

(3.16) $P\{\sup\{|W_n(t) - W_n(s)|: 0 \le s < t \le s + \delta \le 1\} > \varepsilon'\} < \eta'$, $\forall\, n \ge n_0$.

To make use of (3.15) and (3.16) in (3.14), we note that for $k = k_n + [u\sqrt{n}]$ $n^{-\frac{1}{2}}k[k(k+1)/k_n(k_n+1) - 1] \to 2u$. Thus, the rhs of (3.14) can be expressed as

(3.17) $$P\left\{\frac{V_{nk} - k\theta}{\sqrt{n}\,\theta} > \frac{k}{\sqrt{n}\theta}\left[\frac{k(k+1)}{k_n(k_n+1)} - 1\right],\ \forall\ k_{n\varepsilon} \le k \le k_n + u\sqrt{n},\quad W_n(\gamma) > \frac{2u - \varepsilon}{\theta}\right\} +$$

$$P\left\{\frac{V_{nk} - k\theta}{\theta\sqrt{n}} > \frac{k}{\theta\sqrt{n}}\left[\frac{k(k+1)}{k_n(k_n+1)} - 1\right],\ \forall\ k_{n\varepsilon} \le k \le k_n + u\sqrt{n},\ W_n(\gamma) \le \frac{2u - \varepsilon}{\theta}\right\} + o(1),$$

where $\varepsilon > 0$. The second term is bounded by $P\{W_n(n^{-1}k_n + un^{-\frac{1}{2}}) - W_n(\gamma) > \varepsilon/\theta\}$ and, by (3.16), it converges to 0 as $n_2 \to \infty$ (or $a_2 \to 0$). Similarly, the first term is convergent-equivalent to

(3.18) $$P\{W_n(\gamma) > (2u - \varepsilon)/\theta\} \to P\{W(\gamma) \geq (2u - \varepsilon)/\theta\}$$

$$= P\{W(1) > (2u - \varepsilon)/\theta\sqrt{\gamma}\} = (2\pi)^{-\frac{1}{2}} \int_{(2u-\varepsilon)/\theta\sqrt{\gamma}}^{\infty} \exp(-\tfrac{1}{2}t^2)dt.$$

Thus, (2.10) follows from (3.13), (3.14), (3.17) and (3.18) by letting $u = \theta\sqrt{\gamma}\,x/2$ and $\varepsilon \to 0$. This completes the proof of Theorem 1.

To prove Theorem 2, we first note that under (2.6), (2.7) and (2.8),

(3.19) $(a^*/a_2)^{\frac{1}{2}}R_n^0(\underset{\sim}{a}, \theta) \to a_0\theta^2/\gamma + \rho a^* + a^*\gamma\theta$ as $a_2 \to 0$.

Also, recalling that $n^{-1}N_n \le 1$, with probability 1, we have by (2.9),

(3.20) $$\lim_{n\to\infty} E(n^{-1}N_n)^m = \gamma^m (< 1),\ \forall\ m = 1, 2, \ldots.$$

Further, by (2.2), (3.1), and the fact that Z_{nk} is ≥ 0, $\forall\ k \ge 1$, we

(3.21) $N_n(N_n - 1)^2 a_2/a_0 < V_{nN_n-1} < V_{nN_n} \le N_n^2(N_n + 1)a_0^{-1}a_2 I_{(N_n<n)} + V_{nn} I_{(N_n=n)}$.

Note that by (2.6) and (2.7),

(3.22) $$P\{N_n = n\} = P\{V_{nk} > k^2(k+1)a_2/a_0,\ \ \forall\ 1 \le k \le n-1\}$$
$$\le P\{V_{nn-1} > (n-1)^2 na_2/a_0\} = P\{V_{nn-1} - (n-1)\theta > (n-1)[n(n-1)a_2/a_0 - \theta]\}$$
$$\le \theta^2/(n-1)[n(n-1)a_2/a_0 - \theta]^2 \sim \theta^2/[(n-1)\{a^*/a_0 - \theta\}^2] = 0(n^{-1}).$$

Also, $E(V_{nn}^2) = n(n+1)\theta^2$, so that by the Schwarz inequality and (3.22),

(3.23) $$|E\{V_{nn}I_{(N_n=n)}\}| \le \theta^2\sqrt{n(n+1)}\{0(n^{-\frac{1}{2}})\} = 0(n^{\frac{1}{2}}) = 0(a_2^{-\frac{1}{4}}),\quad \text{by } (2.6).$$

Further, by (3.21)

(3.24) $$a_2|V_{nN_n} - a_0^{-1}a_2N_n^3| \le \frac{a_2^2}{a_0}\, 2N_n^2 I_{(N_n<n)} + a_2|V_{nn} - a_0^{-1}a_2n^3|I_{(N_n=n)},$$

where by (2.6), $a_0^{-1}a_2^2n^3 \to a_0^{-1}\sqrt{a_2}\ (a^*)^{3/2}$, while $a_2^2N_n^2I_{(N_n<n)} \le a_2^2n^2 \to a_2a^*$.
Hence, by (3.22), (3.23) and (3.24), we have

(3.25) $$(a^*/a_2)^{\frac{1}{2}}E\{a_2|V_{nN_n} - a_0^{-1}a_2N_n^3|\} = 0(a_2^{\frac{1}{4}}) \to 0 \quad \text{as} \quad a_2 \to 0.$$

On the other hand, by (3.20) and (2.4) - (2.7),

(3.26) $$(a^*/a_2)^{\frac{1}{2}}E(a_2^2N_n^3)/a_0 \to \gamma\theta a^* \quad \text{as} \quad a_2 \to 0.$$

Also, by (2.8),

(3.27) $$(a^*/a_2)^{\frac{1}{2}}a_1n \to \rho a^*, \quad \text{as} \quad a_2 \to 0.$$

Hence, by (2.3), (3.19), (3.26) and (3.27), for (2.11), it suffices to show that

(3.28) $$\lim_{a_2\to 0}(a^*/a_2)^{\frac{1}{2}}E(N_n^{-1}V_{nN_n} - \theta)^2 = \theta^2/\gamma.$$

Note that

(3.29) $$(a^*/a_2)^{\frac{1}{2}}E(N_n^{-1}V_{nN_n} - \theta)^2 = (a^*/a_2)^{\frac{1}{2}}k_n^{-2}E(V_{nN_n} - \theta N_n)^2 +$$
$$(a^*/a_2)^{\frac{1}{2}}k_n^{-2}E\{(V_{nN_n} - \theta N_n)^2[(k_n/N_n)^2 - 1]\}.$$

Now, for every $n(\ge 1)$, $\{V_{nk} - k\theta = \sum_{j=1}^{k}(Z_{nj} - \theta),\ \ 1 \le k \le n\}$ is a martingale,

$E(Z_{nj} - \theta)^2 = \theta^2$ and $EN_n < \infty$. Hence, by the Wald second lemma [viz. Theorem 2 of Chow, Robbins and Teicher (1965)], we have $E(V_{nN_n} - \theta N_n)^2 = \theta^2 EN_n$, so that by (2.6), (2.7) and (3.20), the first term on the rhs of (3.29) is equal to

(3.30) $$(a^*/a_2)^{\frac{1}{2}} k_n^{-2} \theta^2 EN_n \to \theta^2/\gamma \quad \text{as} \quad a_2 \to 0.$$

Thus, we need to show that the second term on the rhs of (3.29) converges to 0 as $a_2 \to 0$. Now, by the same technique as in (3.24) - (3.25), it follows that

(3.31) $$(a^*/a_2)^{\frac{1}{2}} k_n^{-2} E\{|V_{nN_n} - a_0^{-1} a_1 N_n^3]^2 |(k_n/N_n)^2 - 1|\} = 0(n^{-1}) = 0(\sqrt{a_2}) \to 0 \quad \text{as} \quad a_2 \to 0.$$

On the other hand, by (2.4), (2.6) and (2.7),

(3.32) $$\begin{aligned}(a^*/a_2)^{\frac{1}{2}} k_n^{-2} E\{[a_0^{-1} a_2 N_n^3 - N_n\theta]^2 |(k_n/N_n)^2 - 1|\} &= \sqrt{a^*}\; a_2^{3/2} a_0^{-2} E\{(N_n^2 - \theta a_0/a_2)^2 |1 - (N_n/k_n)^2|\} \\ &= \sqrt{a^*}\; a_2^{3/2} a_0^{-2} E\{(N_n^2 - k_n^2)^2 |1 - (N_n/k_n)^2|\} + 0(\sqrt{a_2}) \\ &= \sqrt{a^*}\; a_2^{3/2} a_0^{-2} k_n^{-2} E|N_n^2 - k_n^2|^3 + 0(\sqrt{a_2}) \\ &\le \sqrt{a^*}\; a_2^{3/2} a_0^{-2} 8n^3 k_n^{-2} E|N_n - k_n|^3 + 0(\sqrt{a_2})\ .\end{aligned}$$

Thus, it suffices to show (by virtue of (2.6) - (2.7)) that

(3.33) $$\lim_{n\to\infty} k_n^{-2} E|N_n - k_n|^3 = 0 \quad [\text{as} \quad a_2^{3/2} n^3 \to (a^*)^{3/2}]\ .$$

Define λ as in (3.5). Then

(3.34) $$k_n^{-2} E|N_n - k_n|^3 = k_n^{-2} E(|N_n - k_n|^3 I_{(|N_n - k_n| \le n^\lambda)}) + k_n^{-2} E(|N_n - k_n|^3 I_{(|N_n - k_n| > n^\lambda)}),$$

where by (2.7), the first term on the rhs of (3.34) converges to 0 as $n \to \infty$. On the other hand, the second term is bounded by

(3.35) $$n^3 k_n^{-2} P\{|N_n - k_n| > n^\lambda\}\ .$$

Let $b_1 = 1/3 - \varepsilon$, $\varepsilon > 0$. Then

$$(3.36)\qquad \begin{aligned} P\{N_n \le n^{b_1}\} &= P\{V_{nk} \le k^2(k+1)a_2/a_0 \text{ for some } k \le n^{b_1}\} \\ &\le P\{V_{n1} \le \frac{a_2}{a_0} n^{2b_1}(n^{b_1}+1)\} = P\{V_{n1} \le 0(n^{-2+3b_1})\} \\ &= 0(n^{-2+3b_1}) = 0(n^{-1-3\varepsilon}). \end{aligned}$$

Also, let $k_{n\varepsilon}$ be defined as in (3.7). Then proceeding as in (3.10) but using the 8th order moment of $(V_{nk} - k\theta)$, we obtain that

$$(3.37)\qquad P\{n^{b_1} \le N_n < k_{n\varepsilon}\} = 0([n^{b_1}]^{-4}) = 0(n^{-4/3-4\varepsilon}).$$

Finally, let $k_n^* = k_n - n^\lambda$ and assume n so large that $k_n^* > k_{n\varepsilon}$. Then

$$(3.38)\qquad \begin{aligned} P\{k_{n\varepsilon} < N_n \le k_n^*\} &\le \sum_{k=k_{n\varepsilon}}^{k_n^*} P\left\{\frac{1}{k}(V_{nk} - k\theta) < \theta\left(\frac{k(k+1)}{k_n(k_n+1)} - 1\right)\right\} \\ &\le \sum_{k=k_{n\varepsilon}}^{k_n^*} k^{-2r} E(V_{nk} - k\theta)^{2r}/\theta^{2r}\left[\frac{k(k+1)}{k_n(k_n+1)} - 1\right]^{2r}, \end{aligned}$$

for any $r(>0)$. Now,

$$(3.39)\qquad E(V_{nk} - k\theta)^{2r} = 0(k^r), \quad \text{for every } r = 2, 3, 4, \ldots .$$

Also, for $k_{n\varepsilon} \le k \le k_n^*$,

$$(3.40)\qquad |k(k+1)/k_n(k_n+1) - 1| = 0\left(\frac{k_n - k}{k_n}\right),$$

so that the rhs of (3.38) is

$$(3.41)\qquad \begin{aligned} &0\left(\sum_{k=k_{n\varepsilon}}^{k_n^*} k^{-r} k_n^{2r}(k_n - k)^{-2r}\right) \\ &= 0(n^r)\left(\sum_{k=k_{n\varepsilon}}^{k_n^*} (k_n - k)^{-2r}\right) \\ &= 0(n^r)\cdot 0(n^{-\lambda(2r-1)}) = 0(n^{-(2\lambda-1)r+\lambda}). \end{aligned}$$

Since (3.38) and (3.39) hold for every positive integer r and $\lambda > \frac{1}{2}$, we may choose r so large that $(2\lambda - 1)r - \lambda > 1$, and this leads to the rhs of (3.41) as $0(n^{-1-\eta})$, for some $\eta > 0$. A similar treatment holds for the case of $N_n \ge k_n + n^\lambda$. Thus, $P\{|N_n - k_n| > n^\lambda\} = 0(n^{-1-\eta})$ for some $\eta > 0$, and this proves that (3.35) converges to 0 as $n \to \infty$.

Q.E.D.

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