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# On Timoshenko thin elastic inclusions inside elastic bodies

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## Abstract

The paper concerns the analysis of equilibrium problems for 2D elastic bodies with thin inclusions modeled in the framework of Timoshenko beams. The first focus is on the well-posedness of the model problem in a variational setting. Then delaminations of the inclusions are considered, forming a crack between the elastic body and the inclusion. Nonlinear boundary conditions at the crack faces are considered to prevent a mutual penetration between the faces. The corresponding variational formulations together with weak and strong solutions are discussed. The model contains various physical parameters characterizing the mechanical properties of the inclusion, such as flexural and shear stiffness. The paper provides an asymptotic analysis of such parameters. It is proved that in the limit cases corresponding to infinite and zero rigidity, we obtain rigid inclusions and cracks with the non-penetration conditions, respectively. Finally, exemplary networks of Timoshenko beams are considered as inclusions as well.

## Keywords

Thin elastic inclusion, Timoshenko beams, crack, delamination, non-penetration boundary condition

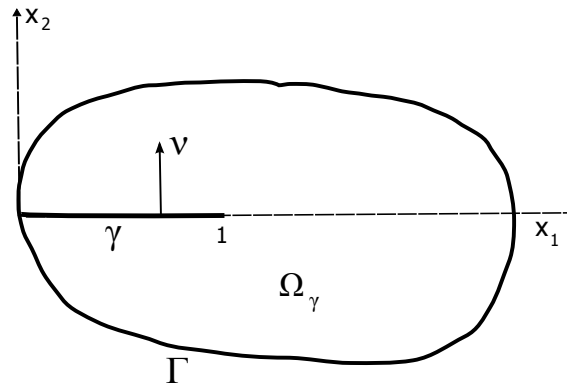
## 1. Introduction

Damage and failure of deformable structures largely depend on the non-homogeneity of the bodies. The commonly used idea of strengthening structures is realized via the exploitation of different inclusions. The inclusions can be divided into thin and thick ones. The terminology “thin inclusion” is used in the case when a dimension of the inclusion is less than a dimension of the body. On the other hand, among thin inclusions we can distinguish rigid and elastic ones. Cracks also can be viewed as thin inclusions with a zero rigidity, while thin rigid or elastic inclusions may result from cracks filled with material. In view of this, elastic bodies containing rigid and/or elastic thin inclusions are considered in problems with imperfections or damage and composites, where reinforcement plays a role. Whereas this is commonplace in continuum mechanics and its applications, bio-medical applications are also now considered [1]. A mathematical treatment of thin elastic inclusions embedded into elastic material has been provided by a number of authors. Here we refer to Bessoud et al. [2, 3], Pasternak [4], Savula et al. [5], and Vynnytska and Savula [6] for recent articles treating thin 2D elastic inclusions embedded into 3-D elastic material. In the works cited, and to the best knowledge of the authors, no delamination of such

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**Figure 1.** Elastic body  $\Omega_\gamma$  with thin elastic inclusion  $\gamma$ .

inclusions have been studied in the context of composite materials of the kind described here. The aim of this work is to partly fill that gap and initiate similar considerations for more complex composites.

As for cracks, it is known that the classical crack models are characterized by linear boundary conditions at the crack faces [7–9]. These linear models allow the opposite crack faces to penetrate each other, which may lead to inconsistency with applications. During the last 20 years a crack theory with non-penetration conditions at the crack faces has been analyzed very actively. This theory is characterized by inequality type boundary conditions leading to a free boundary approach to the modeling. Khludnev and Kovtunenکو's book [10] contains numerous results on crack models with non-penetration conditions for different constitutive laws. The elastic behavior of bodies with cracks and inequality type boundary conditions is analyzed in the monograph by Khludnev [11] (see also [12–15]). In particular, differentiability of energy functionals with respect to the crack perturbation is investigated. Finding the derivatives of the energy functionals with respect to the crack length is important from the standpoint of the Griffith rupture criterion.

To analyze composite materials one has to consider mathematical models of elastic bodies with elastic and rigid inclusions as well cracks. In such a case, new types of boundary value problems and boundary conditions appear. In particular, nonlocal boundary conditions appear suitable from the mechanical standpoint. Rigid inclusions may be delaminated, hence the crack approach with non-penetration conditions is to be applied. Existence theorems and qualitative properties of solutions in equilibrium problems for elastic bodies with rigid inclusions can be found in the literature [16–24].

In a recent paper [25], a model for an elastic body with a delaminated thin inclusion was proposed. The thin inclusion was modeled by a Kirchhoff–Love beam incorporated in the elastic body. A solution existence was proved, and passages to limits with respect to a rigidity parameter was investigated.

In the present paper, we propose a new model of a thin elastic inclusion inside of elastic body on the basis of the Timoshenko approach. The inclusion is assumed to be delaminated, and, therefore, a crack appears. To exclude a mutual penetration between the crack faces, non-linear boundary conditions of inequality type are considered at the cracks. Different problem formulations are proposed relating to weak and strong solutions which are proved to be equivalent under sufficient regularity conditions. We prove existence and uniqueness of solutions and analyze limit cases describing a passage to infinity and zero of the rigidity parameter associated with the inclusion. Both isotropic and anisotropic cases are investigated. In particular, different models of thin rigid inclusions and crack models with the non-penetration conditions are obtained in the limits. Finally, we also provide a first model for a network of Timoshenko beams embedded into a 2D elastic body.

## 2. Inclusion without delamination

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with Lipschitz boundary  $\Gamma$  such that  $\bar{\gamma} \cap \Gamma = (0, 0)$ ,  $\gamma = (0, 1) \times \{0\}$ ,  $\gamma \subset \Omega$ . Denote by  $\nu = (0, 1)$  a unit normal vector to  $\gamma$ ,  $\tau = (1, 0)$ , and set  $\Omega_\gamma = \Omega \setminus \bar{\gamma}$ , see Figure 1. Assume that the angle between  $\Gamma$  and  $\gamma$  is nonzero at the point  $(0, 0)$ .

In what follows the domain  $\Omega_\gamma$  represents a region filled with an elastic material, and  $\gamma$  is an elastic inclusion with specified properties. In particular, we consider  $\gamma$  as a Timoshenko beam incorporated in the elastic body. By the assumptions,  $\bar{\gamma}$  crosses the external boundary  $\Gamma$  at the given point.

An equilibrium problem for the body  $\Omega_\gamma$  and the inclusion  $\gamma$  is formulated as follows. For given external forces  $f = (f_1, f_2) \in L^2(\Omega)^2$  acting on the body we want to find a displacement field  $u = (u_1, u_2)$ , a stress tensor  $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$ , defined in  $\Omega_\gamma$ , and thin inclusion displacements  $v, w$  and a rotation angle  $\varphi$  defined on  $\gamma$  such that

$$-\operatorname{div} \sigma = f, \quad \sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{1}$$

$$-w_{xx} = [\sigma_\tau] \quad \text{on } \gamma, \tag{2}$$

$$-\varphi_{xx} + v_x + \varphi = 0 \quad \text{on } \gamma, \tag{3}$$

$$-v_{xx} - \varphi_x = [\sigma_v] \quad \text{on } \gamma, \tag{4}$$

$$u = 0 \quad \text{on } \Gamma; \quad v = w = \varphi = 0 \quad \text{as } x = 0, \tag{5}$$

$$\varphi + v_x = w_x = \varphi_x = 0 \quad \text{as } x = 1, \tag{6}$$

$$[u] = 0, \quad v = u_2, \quad w = u_1 \quad \text{on } \gamma. \tag{7}$$

Here  $[\phi] = \phi^+ - \phi^-$  is a jump of a function  $\phi$  on  $\gamma$ , where  $\phi^\pm$  are the traces of  $\phi$  on the crack faces  $\gamma^\pm$ . The signs  $\pm$  correspond to positive and negative directions of  $v$ ;  $w_x = \frac{dw}{dx}$ ,  $x = x_1$ ,  $(x_1, x_2) \in \Omega$ ;  $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$  is the strain tensor,  $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ ,  $i, j = 1, 2$ ;  $\sigma v = (\sigma_{1j}v_j, \sigma_{2j}v_j)$ ,  $\sigma_v = \sigma_{ij}v_j v_i$ ,  $\sigma_\tau = \sigma v \cdot \tau$ .

By  $A = \{a_{ijkl}\}, i, j, k, l = 1, 2$ , we denote a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega),$$

$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi_{ji} = \xi_{ij}, \quad c_0 = \text{const} > 0.$$

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices.

Functions defined on  $\gamma$  we identify with functions of the variable  $x$ .

Relations (1) are the equilibrium equations for the elastic body and Hooke's law; (2)–(4) are Timoshenko equilibrium equations for the inclusion  $\gamma$  with respect to vertical and tangential displacements  $v, w$  (along the axes  $x_2, x_1$ , respectively) and for rotation angle  $\varphi$ . The right-hand sides  $[\sigma_\tau], [\sigma_v]$  describe forces acting on  $\gamma$  from the surrounding elastic media. According to (7), the displacements of the elastic body coincide at  $\gamma$  with the inclusion displacements. Since there is no delamination the jump  $[u]$  is zero on  $\gamma$ .

Now we provide a variational formulation of the problem (1)–(7). To this end, we introduce the Sobolev space

$$V = \{(u, v, w, \varphi) \mid u \in H_0^1(\Omega)^2, (v, w, \varphi) \in H^{1,0}(\gamma)^3, \\ v = u_v, w = u_\tau \text{ on } \gamma\},$$

where

$$H^{1,0}(\gamma) = \{\phi \in H^1(\gamma), \phi = 0 \text{ as } x = 0\}, \quad u_v = uv, \quad u_\tau = u\tau,$$

and the energy functional

$$\pi(u, v, w, \varphi) = \frac{1}{2} \int_\Omega \sigma(u)\varepsilon(u) - \int_\Omega fu + \frac{1}{2} \int_\gamma F(v, w, \varphi).$$

Here  $\sigma(u) = \sigma$  is defined from the second relation of (1), i.e.  $\sigma(u) = A\varepsilon(u)$ ,  $F(v, w, \varphi) = w_x^2 + \varphi_x^2 + (v_x + \varphi)^2$ , and for simplicity we write  $\sigma(u)\varepsilon(u) = \sigma_{ij}(u)\varepsilon_{ij}(u)$ ,  $fu = f_i u_i$ .

Consider the minimization problem:

$$\text{Find } (u, v, w, \varphi) \in V \text{ such that } \pi(u, v, w, \varphi) = \inf_V \pi.$$

This problem has a unique solution satisfying the identity

$$(u, v, w, \varphi) \in V, \quad (8)$$

$$\int_{\Omega} \sigma(u) \varepsilon(\bar{u}) - \int_{\Omega} f \bar{u} \quad (9)$$

$$+ \int_{\gamma} (w_x \bar{w}_x + \varphi_x \bar{\varphi}_x + (v_x + \varphi)(\bar{v}_x + \bar{\varphi})) = 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in V.$$

To check a solvability of the problem (8), (9) it suffices to establish a coercivity of the functional  $\pi$  on the space  $V$  since its weak lower semicontinuity is obvious. We have for  $\alpha > 0$ ,

$$2v_x \varphi \geq -(1 + \alpha)\varphi^2 - \frac{1}{(1 + \alpha)}v_x^2.$$

Hence

$$F(v, w, \varphi) \geq w_x^2 + \varphi_x^2 - \alpha\varphi^2 + \frac{\alpha}{(1 + \alpha)}v_x^2. \quad (10)$$

Due to Korn's inequality, we obtain

$$\pi(u, v, w, \varphi) \geq c_0 \|u\|_{1,\Omega}^2 - c_1 \|u\|_{1,\Omega} + \frac{1}{2} \int_{\gamma} F(v, w, \varphi), \quad (11)$$

where  $\|\cdot\|_{1,\Omega}$  is the norm in  $H_0^1(\Omega)^2$ . Thus, by (10), from (11) for a small  $\alpha > 0$  it follows

$$\pi(u, v, w, \varphi) \rightarrow +\infty, \quad \|(u, v, w, \varphi)\|_V \rightarrow \infty,$$

what is needed.

In what follows we check an equivalence of (1)–(7) and (8), (9) for smooth solutions.

**Theorem 2.1** *Problem formulations (1)–(7) and (8), (9) are equivalent provided that solutions are quite smooth.*

**Proof** Let (1)–(7) be fulfilled. Take  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in V$  and multiply the first equation of (1) by  $\bar{u}$ , and (2)–(4) by  $\bar{v}$ ,  $\bar{w}$ ,  $\bar{\varphi}$ . Integrating over  $\Omega_{\gamma}$  and  $\gamma$ , respectively, we get

$$\begin{aligned} & \int_{\Omega_{\gamma}} (-\operatorname{div} \sigma - f) \bar{u} + \int_{\gamma} (-w_{xx} - [\sigma_{\tau}]) \bar{w} + \int_{\gamma} (-\varphi_{xx} + v_x + \varphi) \bar{\varphi} \\ & + \int_{\gamma} (-v_{xx} - \varphi_x - [\sigma_v]) \bar{v} = 0. \end{aligned}$$

Hence, by the boundary conditions (5)–(7),

$$\begin{aligned} & \int_{\Omega_{\gamma}} (\sigma(u) \varepsilon(\bar{u}) - f \bar{u}) + \int_{\gamma} [\sigma v] \bar{u} \quad (12) \\ & + \int_{\gamma} (w_x \bar{w}_x - [\sigma_{\tau}] \bar{w}) + \int_{\gamma} \{v_x \bar{v}_x + \varphi \bar{v}_x - [\sigma_v] \bar{v}\} \\ & + \int_{\gamma} \{\varphi_x \bar{\varphi}_x + (v_x + \varphi) \bar{\varphi}\} = 0. \end{aligned}$$

We have  $[\sigma \nu] \bar{u} = [\sigma_\nu] \bar{u}_\nu + [\sigma_\tau] \bar{u}_\tau$  on  $\gamma$ . Accounting  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in V$ , from (12) it follows the identity (9). In so doing we change the integration domain  $\Omega_\gamma$  by  $\Omega$  since  $[u] = 0$  on  $\gamma$ .

Conversely, let (8), (9) be fulfilled. We take test functions of the form  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) = (\phi, 0, 0, 0)$ ,  $\phi \in C_0^\infty(\Omega_\gamma)^2$ . It gives the equilibrium equation from (1). Next, from (9) it follows

$$-\int_\gamma [\sigma \nu] \bar{u} + \int_\gamma (w_x \bar{w}_x + \varphi_x \bar{\varphi}_x + (v_x + \varphi)(\bar{v}_x + \bar{\varphi})) = 0.$$

Consequently

$$\begin{aligned} & -\int_\gamma ([\sigma_\nu] \bar{u}_2 + [\sigma_\tau] \bar{u}_1) \\ & -\int_\gamma (w_{xx} \bar{w} + (\varphi_{xx} - v_x - \varphi) \bar{\varphi} + (v_{xx} + \varphi_x) \bar{v}) \\ & + w_x \bar{w}|_0^1 + \varphi_x \bar{\varphi}|_0^1 + (v_x + \varphi) \bar{v}|_0^1 = 0. \end{aligned} \tag{13}$$

Taking here  $\bar{\varphi} = \bar{w} = \bar{v} = 0$  as  $x = 1$ , by  $\bar{u}_2 = \bar{v}$ ,  $\bar{u}_1 = \bar{w}$  at  $\gamma$ , we get (2)–(4). Then, from (13) the boundary conditions (6) follow.

Hence, the equivalence of (1)–(7) and (8), (9) is proved.

### 3. Delaminated elastic inclusion

Assume that a delamination of the elastic inclusion takes place at  $\gamma^+$ , thus we have a crack between the elastic body and the thin inclusion. In our model, inequality type boundary conditions will be considered to prevent a mutual penetration between the crack faces. Displacements of the inclusion should coincide with the displacements of the elastic body at  $\gamma^-$ . The problem formulation is as follows. We have to find a displacement field  $u = (u_1, u_2)$ , a stress tensor  $\sigma = \{\sigma_{ij}\}$ ,  $i, j = 1, 2$ , defined in  $\Omega_\gamma$ , and thin inclusion displacements  $v, w$  and a rotation angle  $\varphi$  defined on  $\gamma$  such that

$$-\operatorname{div} \sigma = f, \quad \sigma - A \varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{14}$$

$$-w_{xx} = [\sigma_\tau] \quad \text{on } \gamma, \tag{15}$$

$$-\varphi_{xx} + v_x + \varphi = 0 \quad \text{on } \gamma, \tag{16}$$

$$-v_{xx} - \varphi_x = [\sigma_\nu] \quad \text{on } \gamma, \tag{17}$$

$$u = 0 \quad \text{on } \Gamma; \quad v = w = \varphi = 0 \quad \text{as } x = 0, \tag{18}$$

$$\varphi + v_x = w_x = \varphi_x = 0 \quad \text{as } x = 1, \tag{19}$$

$$[u] \nu \geq 0, \quad v = u_2^-, \quad w = u_1^- \quad \text{on } \gamma, \tag{20}$$

$$\sigma_\nu^+ \leq 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+ \cdot [u] \nu = 0 \quad \text{on } \gamma. \tag{21}$$

Note that equations (14)–(17) coincide with (1)–(4). The inequality in (20) provides a mutual non-penetration between the crack faces. The second and the third relations of (20) show that the inclusion displacements coincide with the vertical and tangential displacements of the elastic body at  $\gamma^-$ . Moreover, due to (21) the normal stresses are non-positive on  $\gamma^+$  and tangential forces are zero. The last relation of (21) states that  $\sigma_\nu^+$  or  $[u] \nu$  are equal to zero. In particular, if  $\sigma_\nu^+ = 0$  we have  $\sigma \nu = 0$  on  $\gamma^+$ .

First, we provide a variational formulation of the problem (14)–(21). Introduce a set of admissible displacements

$$\begin{aligned} K = \{ & (u, v, w, \varphi) \mid u \in H_\Gamma^1(\Omega_\gamma)^2, (v, w, \varphi) \in H^{1,0}(\gamma)^3; \\ & [u_\nu] \geq 0, \quad v = u_\nu^-, \quad w = u_\tau^- \text{ on } \gamma \} \end{aligned}$$

and the energy functional

$$\pi_1(u, v, w, \varphi) = \frac{1}{2} \int_{\Omega_\gamma} \sigma(u) \varepsilon(u) - \int_{\Omega_\gamma} f u + \frac{1}{2} \int_{\gamma} F(v, w, \varphi), \quad (22)$$

where the Sobolev space  $H_\Gamma^1(\Omega_\gamma)$  is defined as

$$H_\Gamma^1(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma\}.$$

There exists a unique solution of the problem:

$$\text{Find } (u, v, w, \varphi) \in K \text{ such that } \pi_1(u, v, w, \varphi) = \inf_K \pi_1.$$

This solution satisfies the variational inequality

$$(u, v, w, \varphi) \in K, \quad (23)$$

$$\begin{aligned} & \int_{\Omega_\gamma} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \\ & + \int_{\gamma} \{w_x(\bar{w}_x - w_x) + \varphi_x(\bar{\varphi}_x - \varphi_x)\} \\ & + \int_{\gamma} (v_x + \varphi)(\bar{v}_x + \bar{\varphi} - v_x - \varphi) \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K. \end{aligned} \quad (24)$$

The coercivity of the functional  $\pi_1$  can be proved as that in Section 2, hence the problem (23), (24) indeed has a solution.

Let us check that (14)–(21) and (23), (24) are equivalent for smooth solutions. Assume that (14)–(21) hold. Take  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K$  and multiply the first equation of (14) by  $\bar{u} - u$ , and (15)–(17) by  $\bar{w} - w$ ,  $\bar{\varphi} - \varphi$ ,  $\bar{v} - v$ , respectively. Integrating over  $\Omega_\gamma$  and  $\gamma$ , respectively, we have

$$\begin{aligned} & \int_{\Omega_\gamma} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) + \int_{\gamma} [\sigma v(\bar{u} - u)] + \\ & + \int_{\gamma} \{w_x(\bar{w}_x - w_x) + \varphi_x(\bar{\varphi}_x - \varphi_x) + (v_x + \varphi)(\bar{v}_x + \bar{\varphi} - v_x - \varphi)\} \\ & - \int_{\gamma} \{[\sigma_\tau](\bar{w} - w) + [\sigma_v](\bar{v} - v)\} - \\ & - w_x(\bar{w} - w)|_0^1 - \varphi_x(\bar{\varphi} - \varphi)|_0^1 - v_x(\bar{v} - v)|_0^1 - \varphi(\bar{v} - v)|_0^1 = 0. \end{aligned} \quad (25)$$

To prove the variational inequality (24), it suffices to state that in (25) the following inequality holds

$$\int_{\gamma} [\sigma v(\bar{u} - u)] - \int_{\gamma} \{[\sigma_\tau](\bar{w} - w) + [\sigma_v](\bar{v} - v)\} \leq 0.$$

Let us check this. By the second condition of (21), we have

$$[\sigma_\tau](\bar{u}_\tau - u_\tau) - [\sigma_\tau](\bar{w} - w) = 0.$$

On the other hand, by  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K$  and (21)

$$[\sigma_v](\bar{u}_v - u_v) - [\sigma_v](\bar{v} - v) = \sigma_v^+([\bar{u}]v - [u]v) \leq 0,$$

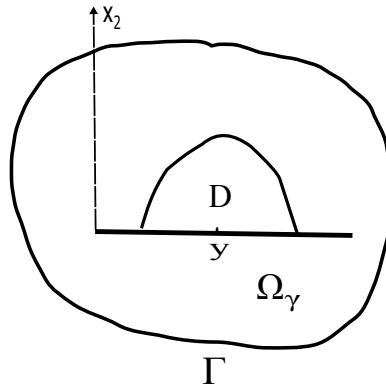


Figure 2. Neighborhood  $D$  near the point  $y$ .

what is needed. Consequently, the variational inequality (23), (24) follows from (14)–(21).

Now we prove the converse. Let (23), (24) be fulfilled. First, it is easy to derive the equilibrium equation (14) from (23), (24). Indeed, we substitute test functions  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) = (u, v, w, \varphi) \pm (\phi, 0, 0, 0)$  in (24),  $\phi \in C_0^\infty(\Omega_\gamma)^2$ . It gives the first equation of (14).

Let us prove the relations (21). Substitute test functions  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) = (u, v, w, \varphi) + (\tilde{u}, 0, 0, 0)$  in (24),  $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ ,  $\tilde{u}_v^+ \geq 0$  on  $\gamma$ ,  $\text{supp } \tilde{u} \subset \bar{D}$ , see Figure 2. In this case the following inequality follows

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u}) - \int_{\Omega_\gamma} f\tilde{u} \geq 0.$$

Thus

$$\int_{\gamma^+} (\sigma_v \tilde{u}_v + \sigma_\tau \tilde{u}_\tau) \leq 0,$$

i.e. the first two relations of (21) follow.

We next substitute test functions  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) = (u, v, w, \varphi) \pm (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\varphi})$  in (24),  $[\tilde{u}]v = 0$ ,  $\tilde{u}_v^- = \tilde{v}$ ,  $\tilde{u}_\tau^- = \tilde{w}$  on  $\gamma$ ,  $\tilde{\varphi} = 0$  as  $x = 0$ . This gives

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u}) - \int_{\Omega_\gamma} f\tilde{u} + \int_{\gamma} (w_x \tilde{w}_x + \varphi_x \tilde{\varphi}_x + (v_x + \varphi)(\tilde{v}_x + \tilde{\varphi})) = 0.$$

Hence

$$\begin{aligned} - \int_{\gamma} [\sigma v \cdot \tilde{u}] - \int_{\gamma} \{w_{xx} \tilde{w} + (\varphi_{xx} - v_x - \varphi) \tilde{\varphi} + (v_{xx} + \varphi_x) \tilde{v}\} + \\ + w_x \tilde{w}|_0^1 + \varphi_x \tilde{\varphi}|_0^1 + (v_x + \varphi) \tilde{v}|_0^1 = 0. \end{aligned} \tag{26}$$

Assuming  $\tilde{v} = \tilde{w} = \tilde{\varphi} = 0$  as  $x = 1$ , from (26) one gets (15)–(17). Also, from (26) it follows (19).

It remains to prove the last equality of (21). Assume that at any point  $y \in \gamma$  we have  $[u(y)]v > 0$ . In this case we substitute  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) = (u, v, w, \varphi) + (\lambda \tilde{u}, 0, 0, 0)$  in (24),  $\text{supp } \tilde{u} \subset \bar{D}$ ,  $\lambda \in \mathbb{R}$  is small, and  $\tilde{u}$  is smooth function (see Figure 2). This provides

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u}) - \int_{\Omega_\gamma} f\tilde{u} = 0,$$



hence

$$\int_{\gamma} \sigma_v^+ \tilde{u}_v^+ = 0.$$

Due to arbitrariness of  $\tilde{u}_v^+$  we obtain  $\sigma_v^+ = 0$  near  $y$ . Conversely, if  $\sigma_v(y) < 0$  we easily get  $[u(y)]v = 0$ , and again the last relation of (21) follows.

Thus we have proved the following statement.

**Theorem 3.1** *Problem formulations (14)–(21) and (23), (24) are equivalent provided that solutions are quite smooth.*

#### 4. Rigidity convergence to infinity

In practice, a solution of the problem like (14)–(21) should depend on the rigidity parameter of the thin inclusion. In the model (14)–(21) this parameter was taken to be equal to 1. In this section we introduce a parameter  $\lambda > 0$  into the model and analyze its passage to infinity. To this end, the energy functional is considered,

$$\pi_{\lambda}(u, v, w, \varphi) = \frac{1}{2} \int_{\Omega_{\gamma}} \sigma(u) \varepsilon(u) - \int_{\Omega_{\gamma}} f u + \frac{\lambda}{2} \int_{\gamma} F(v, w, \varphi).$$

There exists a unique solution of the minimization problem:

$$\text{Find } (u^{\lambda}, v^{\lambda}, w^{\lambda}, \varphi^{\lambda}) \in K \text{ such that } \pi_{\lambda}(u^{\lambda}, v^{\lambda}, w^{\lambda}, \varphi^{\lambda}) = \inf_K \pi_{\lambda}.$$

A solution of this problem exists and satisfies the variational inequality

$$(u^{\lambda}, v^{\lambda}, w^{\lambda}, \varphi^{\lambda}) \in K, \tag{27}$$

$$\int_{\Omega_{\gamma}} \sigma(u^{\lambda}) \varepsilon(\bar{u} - u^{\lambda}) - \int_{\Omega_{\gamma}} f(\bar{u} - u^{\lambda}) \tag{28}$$

$$+ \lambda \int_{\gamma} \{w_x^{\lambda}(\bar{w}_x - w_x^{\lambda}) + \varphi_x^{\lambda}(\bar{\varphi}_x - \varphi_x^{\lambda})$$

$$+ (v_x^{\lambda} + \varphi^{\lambda})(\bar{v}_x + \bar{\varphi} - v_x^{\lambda} - \varphi^{\lambda})\} \geq 0 \quad \forall(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K.$$

We can also provide an equivalent differential formulation of the problem (27), (28). Indeed, it is necessary to find a displacement field  $u^{\lambda} = (u_1^{\lambda}, u_2^{\lambda})$ , a stress tensor  $\sigma^{\lambda} = \{\sigma_{ij}^{\lambda}\}, i, j = 1, 2$ , defined in  $\Omega_{\gamma}$ , and thin inclusion displacements  $v^{\lambda}, w^{\lambda}$  and a rotation angle  $\varphi^{\lambda}$  defined on  $\gamma$ , such that

$$-\text{div } \sigma^{\lambda} = f, \quad \sigma - A \varepsilon(u^{\lambda}) = 0 \quad \text{in } \Omega_{\gamma},$$

$$-\lambda w_{xx}^{\lambda} = [\sigma_{\tau}^{\lambda}] \quad \text{on } \gamma,$$

$$-\lambda \varphi_{xx}^{\lambda} + \lambda v_x^{\lambda} + \lambda \varphi^{\lambda} = 0 \quad \text{on } \gamma,$$

$$-\lambda v_{xx}^{\lambda} - \lambda \varphi_x^{\lambda} = [\sigma_v^{\lambda}] \quad \text{on } \gamma,$$

$$u^{\lambda} = 0 \quad \text{on } \Gamma; \quad v^{\lambda} = w^{\lambda} = \varphi^{\lambda} = 0 \quad \text{as } x = 0,$$

$$\varphi^{\lambda} + v_x^{\lambda} = w_x^{\lambda} = \varphi_x^{\lambda} = 0 \quad \text{as } x = 1,$$

$$[u^{\lambda}]v \geq 0, \quad v^{\lambda} = u_2^{\lambda-}, \quad w^{\lambda} = u_1^{\lambda-} \quad \text{on } \gamma,$$

$$\sigma_v^{\lambda+} \leq 0, \quad \sigma_{\tau}^{\lambda+} = 0, \quad \sigma_v^{\lambda+} \cdot [u^{\lambda}]v = 0 \quad \text{on } \gamma.$$

Now we are aiming to justify a passage to the limit as  $\lambda \rightarrow \infty$  in (27), (28).

From (28) it follows

$$\int_{\Omega_\gamma} \sigma(u^\lambda) \varepsilon(u^\lambda) - \int_{\Omega_\gamma} f u^\lambda + \lambda \int_\gamma F(v^\lambda, w^\lambda, \varphi^\lambda) = 0,$$

and, as in Section 2, we obtain the following inequality for  $\alpha > 0$ ,

$$\begin{aligned} \lambda F(v^\lambda, w^\lambda, \varphi^\lambda) &\geq \\ &\geq \lambda \{(w_x^\lambda)^2 + (\varphi_x^\lambda)^2 - \alpha(\varphi^\lambda)^2 + \frac{\alpha}{(1 + \alpha)}(v_x^\lambda)^2\}. \end{aligned} \tag{29}$$

Consequently, by Korn’s inequality, for a small  $\alpha > 0$ , we derive uniformly in  $\lambda$

$$\|u^\lambda\|_{H^1_\Gamma(\Omega_\gamma)^2}^2 \leq c, \quad \lambda \|(v^\lambda, w^\lambda, \varphi^\lambda)\|_{H^{1,0}(\gamma)^3}^2 \leq c. \tag{30}$$

Now define a set of admissible displacements suitable for a limit problem,

$$K_r = \{u \in H^1_\Gamma(\Omega_\gamma)^2 \mid u^- = 0, [u_\nu] \geq 0 \text{ on } \gamma\}.$$

Taking into account the estimates (30) we can assume that as  $\lambda \rightarrow \infty$

$$u^\lambda \rightarrow u \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2, \tag{31}$$

$$(v^\lambda, w^\lambda, \varphi^\lambda) \rightarrow (0, 0, 0) \quad \text{strongly in } H^{1,0}(\gamma)^3. \tag{32}$$

Hence, a passage to the limit in (27), (28) as  $\lambda \rightarrow \infty$  can be fulfilled. It gives

$$u \in K_r, \tag{33}$$

$$\int_{\Omega_\gamma} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \geq 0 \quad \forall \bar{u} \in K_r. \tag{34}$$

Thus the following statement is proved.

**Theorem 4.1** *As  $\lambda \rightarrow \infty$ , the solutions of the problem (27), (28) converge in the sense (31), (32) to the solution of (33), (34).*

Along with the variational formulation (33), (34) a differential formulation of this problem can be provided: find functions  $u = (u_1, u_2)$ ,  $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$ , such that

$$-\text{div } \sigma = f \quad \text{in } \Omega_\gamma, \tag{35}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{36}$$

$$u = 0 \quad \text{on } \Gamma, \tag{37}$$

$$[u]_\nu \geq 0, u^- = 0 \quad \text{on } \gamma, \tag{38}$$

$$\sigma_\nu^+ \leq 0, \sigma_\tau^+ = 0, \sigma_\nu^+ \cdot [u]_\nu = 0 \quad \text{on } \gamma. \tag{39}$$

In what follows we check that formulations (33), (34) and (35)–(39) are equivalent for smooth solutions.

**Theorem 4.2** *Problem formulations (35)–(39) and (33), (34) are equivalent provided that solutions are quite smooth.*

**Proof** Assume that (33), (34) hold. We take test functions in (34) of the form  $\bar{u} = u \pm \varphi$ ,  $\varphi \in C_0^\infty(\Omega_\gamma)^2$ . It provides the equilibrium equation (35). All relations (39) also can be derived from (34). Indeed, assume that

$\psi = (\psi_1, \psi_2)$ ,  $\text{supp } \psi \subset \bar{D}$ ,  $\psi_2 \geq 0$  on  $\gamma^+$  (see Figure 2). In this case  $\bar{u} = u + \psi \in K_r$ . Hence, from (34) it follows

$$\int_{\Omega_\gamma} \sigma(u) \varepsilon(\psi) - \int_{\Omega_\gamma} f \psi \geq 0.$$

Consequently

$$\int_{\gamma^+} (\sigma_\nu \psi_\nu + \sigma_\tau \psi_\tau) \leq 0,$$

and we derive the two first relations of (39). Next, assume that at a given point  $y \in \gamma$  an inequality  $[u(y)]_\nu > 0$  holds. In this case a test function  $\bar{u} = u \pm \delta \psi$ ,  $\text{supp } \psi \subset \bar{D}$ , can be substituted in (34), where  $\psi$  is a smooth function and  $\delta \in \mathbb{R}$  is small. We obtain

$$\int_{\Omega_\gamma} \sigma(u) \varepsilon(\psi) - \int_{\Omega_\gamma} f \psi = 0,$$

i.e.

$$\int_{\gamma^+} \sigma_\nu \psi_\nu = 0.$$

This identity (due to arbitrariness of  $\psi$ ) provides the last relation of (39). On the other hand, if  $\sigma_\nu^+(y) < 0$  we again get  $[u(y)]_\nu = 0$ . Hence (39) is proved.

Conversely, let us prove that (33), (34) can be derived from (35)–(39). To this end, multiply (35) by  $\bar{u} - u$  and integrate over  $\Omega_\gamma$ ,  $\bar{u} \in K_r$ . This gives

$$\int_{\Omega_\gamma} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) + \int_{\gamma} [\sigma_\nu(\bar{u} - u)] = 0 \quad \forall \bar{u} \in K_r.$$

To prove the variational inequality (35) it suffices to state the inequality

$$\int_{\gamma} [\sigma_\nu(\bar{u} - u)] \leq 0 \quad \forall \bar{u} \in K_r.$$

But this relation easily follows from (38), (39). So we have proved an equivalence of (33), (34) and (35)–(39).

## 5. Rigidity convergence to zero

In this section we analyze a convergence to zero of the rigidity parameter  $\lambda$  of the elastic inclusion. Again, consider the problem like (27), (28). We have to find a solution of the problem

$$(u^\lambda, v^\lambda, w^\lambda, \varphi^\lambda) \in K, \quad (40)$$

$$\int_{\Omega_\gamma} \sigma(u^\lambda) \varepsilon(\bar{u} - u^\lambda) - \int_{\Omega_\gamma} f(\bar{u} - u^\lambda) \quad (41)$$

$$+ \lambda \int_{\gamma} \{w_x^\lambda(\bar{w}_x - w_x^\lambda) + \varphi_x^\lambda(\bar{\varphi}_x - \varphi_x^\lambda)$$

$$+ (v_x^\lambda + \varphi^\lambda)(\bar{v}_x + \bar{\varphi} - v_x^\lambda - \varphi^\lambda)\} \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K.$$

Our aim is to pass to the limit in (40), (41) as  $\lambda \rightarrow 0$ . First note that (41) implies

$$\int_{\Omega_\gamma} \sigma(u^\lambda) \varepsilon(u^\lambda) - \int_{\Omega_\gamma} f u^\lambda + \lambda \int_\gamma F(v^\lambda, w^\lambda, \varphi^\lambda) = 0. \tag{42}$$

We can use the inequality (29) for a small  $\alpha > 0$ , and hence, from (42) it follows uniformly in  $\lambda$

$$\|u^\lambda\|_{H^1_\Gamma(\Omega_\gamma)^2}^2 \leq c, \quad \lambda \|(v^\lambda, w^\lambda, \varphi^\lambda)\|_{H^{1,0}(\gamma)^3}^2 \leq c, \tag{43}$$

with constants  $c$  independent of  $\lambda$ . Choosing a subsequence, if necessary, we assume that as  $\lambda \rightarrow 0$

$$u^\lambda \rightarrow u \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2, \tag{44}$$

$$\sqrt{\lambda} v^\lambda \rightarrow \tilde{v}, \sqrt{\lambda} w^\lambda \rightarrow \tilde{w}, \sqrt{\lambda} \varphi^\lambda \rightarrow \tilde{\varphi} \quad \text{weakly in } H^{1,0}(\gamma). \tag{45}$$

By (44), (45), we can pass to the limit in (40), (41) as  $\lambda \rightarrow 0$ . To this end, introduce a set of admissible displacements,

$$K_0 = \{u \in H^1_\Gamma(\Omega_\gamma)^2 \mid [u_\nu] \geq 0 \text{ on } \gamma\}.$$

We choose  $\bar{u} \in K_0$  such that  $\bar{u}$  is smooth at  $\gamma^-$ , and define the function  $\bar{v} = \bar{u}_2^-$ ,  $\bar{w} = \bar{u}_1^-$  on  $\gamma$ . Then  $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K$ , where  $\bar{\varphi} \in H^{1,0}(\gamma)$  is an arbitrary function. After a substitution of this test function in (41) and passage to the limit as  $\lambda \rightarrow 0$  we get

$$\int_{\Omega_\gamma} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \geq 0. \tag{46}$$

The inequality (46) holds for all functions  $\bar{u} \in K_0$  such that  $\bar{u}$  is quite smooth at  $\gamma^-$ . Hence it will be valid for all  $\bar{u} \in K_0$ . A suitable result for the density can be found in Khudnev and Negri [25]. Hence the limit function  $u$  from (46) satisfies the variational inequality

$$u \in K_0, \tag{47}$$

$$\int_{\Omega_\gamma} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \geq 0 \quad \forall \bar{u} \in K_0. \tag{48}$$

Thus we have proved that the limit problem for (40), (41) as  $\lambda \rightarrow 0$  coincides with the well-known boundary value problem describing an equilibrium of the elastic body with the crack  $\gamma$ . An equivalent differential formulation of the problem (47), (48) is as follows. We have to find functions  $u = (u_1, u_2)$ ,  $\sigma = \{\sigma_{ij}\}$ ,  $i, j = 1, 2$ , defined in  $\Omega_\gamma$  such that

$$\begin{aligned} -\operatorname{div} \sigma &= f && \text{in } \Omega_\gamma, \\ \sigma - A\varepsilon(u) &= 0 && \text{in } \Omega_\gamma, \\ u &= 0 && \text{on } \Gamma, \\ [u_\nu] &\geq 0, \sigma_\nu^\pm \leq 0, [\sigma_\nu] = 0, \sigma_\tau^\pm = 0, \sigma_\nu[u_\nu] = 0 && \text{on } \gamma. \end{aligned}$$

This model has been extensively analyzed in the books by Khudnev and Kovtunenکو [10, 11].

Consequently, the following result has been proved.

**Theorem 5.1** *As  $\lambda \rightarrow 0$ , the solutions of the problem (40), (41) converge in the sense (44), (45) to the solution of (47), (48).*

### 6. Anisotropic elastic inclusion

We can consider a model corresponding to the anisotropic thin inclusion inside the elastic body. In this case we have two positive parameters  $\lambda, \mu$ . The problem formulation is as follows. We have to find a displacement field  $u = (u_1, u_2)$ , a stress tensor  $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$ , defined in  $\Omega_\gamma$ , and thin inclusion displacements  $v, w$  and a rotation angle  $\varphi$  defined on  $\gamma$  such that

$$\begin{aligned}
 -\operatorname{div} \sigma &= f, \quad \sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, & (49) \\
 -\lambda w_{xx} &= [\sigma_\tau] \quad \text{on } \gamma, & (50) \\
 -\lambda \varphi_{xx} + \mu v_x + \mu \varphi &= 0 \quad \text{on } \gamma, & (51) \\
 -\mu v_{xx} - \mu \varphi_x &= [\sigma_v] \quad \text{on } \gamma, & (52) \\
 u = 0 \quad \text{on } \Gamma; \quad v = w = \varphi &= 0 \quad \text{as } x = 0, & (53) \\
 \varphi + v_x = w_x = \varphi_x &= 0 \quad \text{as } x = 1, & (54) \\
 [u]v \geq 0, \quad v = u_2^-, \quad w = u_1^- &\quad \text{on } \gamma, & (55) \\
 \sigma_v^+ \leq 0, \quad \sigma_\tau^+ = 0, \quad \sigma_v^+ \cdot [u]v &= 0 \quad \text{on } \gamma. & (56)
 \end{aligned}$$

For fixed  $\lambda > 0, \mu > 0$  we can prove a solution existence of this problem. It is interesting to pass to the limit as  $\lambda \rightarrow \infty$  for a fixed  $\mu$ , and  $\mu \rightarrow 0$  for a fixed  $\lambda$ .

Case (i): Let  $\mu$  be fixed. We put  $\mu = 1$ . Then the suitable solution of the problem (49)–(56) satisfies the variational inequality

$$(u^\lambda, v^\lambda, w^\lambda, \varphi^\lambda) \in K, \tag{57}$$

$$\begin{aligned}
 &\int_{\Omega_\gamma} \sigma(u^\lambda) \varepsilon(\bar{u} - u^\lambda) - \int_{\Omega_\gamma} f(\bar{u} - u^\lambda) \\
 &+ \int_\gamma \{ \lambda w_x^\lambda (\bar{w}_x - w_x^\lambda) + \lambda \varphi_x^\lambda (\bar{\varphi}_x - \varphi_x^\lambda) \\
 &+ (v_x^\lambda + \varphi^\lambda)(\bar{v}_x + \bar{\varphi} - v_x^\lambda - \varphi^\lambda) \} \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K.
 \end{aligned} \tag{58}$$

Below we analyze a passage to the limit in (57), (58) as  $\lambda \rightarrow \infty$ . From (57), (58) it follows

$$\begin{aligned}
 &\int_{\Omega_\gamma} \sigma(u^\lambda) \varepsilon(u^\lambda) - \int_{\Omega_\gamma} f u^\lambda \\
 &+ \int_\gamma \{ \lambda (w_x^\lambda)^2 + \lambda (\varphi_x^\lambda)^2 + (v_x^\lambda + \varphi^\lambda)^2 \} = 0.
 \end{aligned}$$

Thus, taking into account the inequality with a small  $\alpha > 0$ ,

$$\begin{aligned}
 &\lambda (w_x^\lambda)^2 + \lambda (\varphi_x^\lambda)^2 + (v_x^\lambda + \varphi^\lambda)^2 \geq \\
 &\geq \lambda (w_x^\lambda)^2 + \lambda (\varphi_x^\lambda)^2 - \alpha (\varphi^\lambda)^2 + \frac{\alpha}{(1 + \alpha)} (v_x^\lambda)^2,
 \end{aligned}$$

we derive uniformly for  $\lambda \geq \lambda_0$

$$\|u^\lambda\|_{H^1_\Gamma(\Omega_\gamma)^2}^2 \leq c, \quad \|(v^\lambda, \sqrt{\lambda} w^\lambda, \sqrt{\lambda} \varphi^\lambda)\|_{H^{1,0}(\gamma)^3}^2 \leq c. \tag{59}$$

Assume that as  $\lambda \rightarrow \infty$

$$\begin{aligned}
 u^\lambda &\rightarrow u \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2, \\
 (w^\lambda, \varphi^\lambda) &\rightarrow (0, 0) \quad \text{strongly in } H^{1,0}(\gamma)^2, \\
 v^\lambda &\rightarrow v \quad \text{weakly in } H^{1,0}(\gamma).
 \end{aligned} \tag{60}$$

Now, introduce a set of admissible displacements

$$K_1 = \{u \in H^1_\Gamma(\Omega_\gamma)^2, v \in H^{1,0}(\gamma) \mid [u_\nu] \geq 0, u_2^- = v, u_1^- = 0 \text{ on } \gamma\}.$$

We take  $(\bar{u}, \bar{v}) \in K_1$ . Then  $(\bar{u}, \bar{v}, 0, 0) \in K$ , and after a substitution of this test function into (58) we can pass to the limit as  $\lambda \rightarrow \infty$ . The limiting variational inequality takes the form

$$(u, v) \in K_1, \tag{61}$$

$$\begin{aligned} & \int_{\Omega_\gamma} \sigma(u) \varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \\ & + \int_\gamma v_x(\bar{v}_x - v_x) \geq 0 \quad \forall (\bar{u}, \bar{v}) \in K_1. \end{aligned} \tag{62}$$

We can provide an equivalent differential formulation of the problem (61), (62): find functions  $u = (u_1, u_2)$ ,  $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$ , defined in  $\Omega_\gamma$  and a function  $v$  defined on  $\gamma$  such that

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega_\gamma, \tag{63}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{64}$$

$$u = 0 \quad \text{on } \Gamma, \tag{65}$$

$$-v_{xx} = [\sigma_\nu] \quad \text{on } \gamma, \tag{66}$$

$$v = 0 \text{ as } x = 0; v_x = 0 \text{ as } x = 1, \tag{67}$$

$$[u]v \geq 0, u_2^- = v, u_1^- = 0 \quad \text{on } \gamma, \tag{68}$$

$$\sigma_\nu^+ \leq 0, \sigma_\tau^+ = 0, \sigma_\nu^+ \cdot [u]v = 0 \quad \text{on } \gamma. \tag{69}$$

Thus the following result has been established.

**Theorem 6.1** *As  $\lambda \rightarrow \infty$ , the solutions of the problem (57), (58) converge in the sense (60) to the solution of (61), (62).*

*Case (ii):* Consider the case when  $\lambda$  is fixed. We put  $\lambda = 1$ . The solution of the problem (49)–(56) in this case satisfies the variational inequality

$$(u^\mu, v^\mu, w^\mu, \varphi^\mu) \in K, \tag{70}$$

$$\begin{aligned} & \int_{\Omega_\gamma} \sigma(u^\mu) \varepsilon(\bar{u} - u^\mu) - \int_{\Omega_\gamma} f(\bar{u} - u^\mu) \\ & + \int_\gamma \{w_x^\mu(\bar{w}_x - w_x^\mu) + \varphi_x^\mu(\bar{\varphi}_x - \varphi_x^\mu)\} \\ & + \mu(v_x^\mu + \varphi^\mu)(\bar{v}_x + \bar{\varphi} - v_x^\mu - \varphi^\mu) \geq 0 \quad \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K. \end{aligned} \tag{71}$$

Below we analyze a passage to the limit in (70), (71) as  $\mu \rightarrow 0$ . From (70), (71) it follows

$$\begin{aligned} & \int_{\Omega_\gamma} \sigma(u^\mu) \varepsilon(u^\mu) - \int_{\Omega_\gamma} f u^\mu \\ & + \int_\gamma \{(w_x^\mu)^2 + (\varphi_x^\mu)^2 + \mu(v_x^\mu + \varphi^\mu)^2\} = 0. \end{aligned}$$

Hence, taking into account the inequality

$$\begin{aligned} & (w_x^\mu)^2 + (\varphi_x^\mu)^2 + \mu(v_x^\mu + \varphi^\mu)^2 \geq \\ & \geq (w_x^\mu)^2 + (\varphi_x^\mu)^2 - \mu\alpha(\varphi^\mu)^2 + \mu \frac{\alpha}{(1 + \alpha)} (v_x^\mu)^2, \end{aligned}$$

we derive, for a small  $\alpha > 0$ , uniformly for  $\mu \leq \mu_0$ ,

$$\|u^\mu\|_{H^1_\Gamma(\Omega_\gamma)^2}^2 \leq c, \quad \|(\sqrt{\mu}v^\mu, w^\mu, \varphi^\mu)\|_{H^{1,0}(\gamma)^3}^2 \leq c.$$

Assume that as  $\mu \rightarrow 0$

$$\begin{aligned} u^\mu &\rightarrow u \quad \text{weakly in } H^1_\Gamma(\Omega_\gamma)^2, \\ (w^\mu, \varphi^\mu) &\rightarrow (w, \varphi) \quad \text{weakly in } H^{1,0}(\gamma)^2, \\ \sqrt{\mu}v^\mu &\rightarrow \tilde{v} \quad \text{weakly in } H^{1,0}(\gamma). \end{aligned} \tag{72}$$

By the convergence (72), it is possible to pass to the limit in (70), (71) as  $\mu \rightarrow 0$ . The limiting function  $(u, w, \varphi)$  satisfies the variational inequality

$$(u, w, \varphi) \in K_2, \tag{73}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \tag{74}$$

$$+ \int_\gamma \{w_x(\bar{w}_x - w_x) + \varphi_x(\bar{\varphi}_x - \varphi_x)\} \geq 0 \quad \forall (\bar{u}, \bar{w}, \bar{\varphi}) \in K_2,$$

where

$$K_2 = \{u \in H^1_\Gamma(\Omega_\gamma)^2, (w, \varphi) \in H^{1,0}(\gamma)^2 \mid [u_\nu] \geq 0, u_1^- = w \text{ on } \gamma\}.$$

From (73), (74) it follows that the function  $\varphi$  is defined independently of  $u, w$ , and moreover,  $\varphi \equiv 0$  on  $\gamma$ . It is clear that the variational inequality (73), (74) can be modified since  $\varphi \equiv 0$ .

To conclude the analysis of passage to the limit as  $\mu \rightarrow 0$ , we provide an equivalent differential formulation of the problem (73), (74): find functions  $u = (u_1, u_2)$ ,  $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$ , defined in  $\Omega_\gamma$  and a function  $w$  defined on  $\gamma$  such that

$$-\text{div } \sigma = f \quad \text{in } \Omega_\gamma, \tag{75}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{76}$$

$$u = 0 \quad \text{on } \Gamma, \tag{77}$$

$$-w_{xx} = [\sigma_\tau] \quad \text{on } \gamma, \tag{78}$$

$$w = 0 \text{ as } x = 0; \quad w_x = 0 \text{ as } x = 1, \tag{79}$$

$$[u]_\nu \geq 0, \quad u_1^- = w \quad \text{on } \gamma, \tag{80}$$

$$\sigma_\nu^+ \leq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\tau^+ = 0, \quad \sigma_\nu^+ \cdot [u]_\nu = 0 \quad \text{on } \gamma. \tag{81}$$

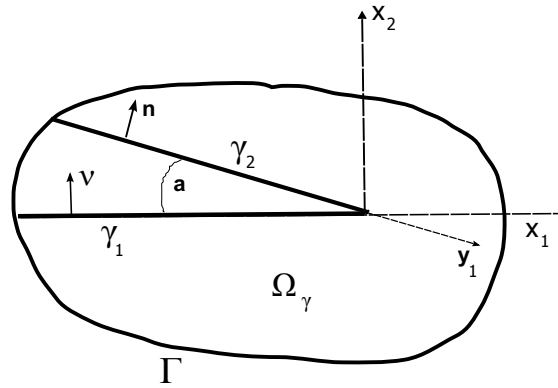
The equivalence of (73), (74) and (75)–(81) means that for smooth solutions we can derive (73), (74) from (75)–(81), and, conversely, (75)–(81) follow from (73), (74).

To conclude the section we formulate the result obtained.

**Theorem 6.2** *As  $\mu \rightarrow 0$ , the solutions of the problem (70), (71) converge in the sense (72) to the solution of (73), (74).*

### 7. Two thin inclusions

Consider a case of two thin inclusions crossing the external boundary  $\Gamma$  and having a joint point inside of the body. For simplicity no delamination is assumed. Let  $\gamma_1, \gamma_2$  be two straight lines,  $\gamma_1, \gamma_2 \subset \Omega$ , see Figure 3. Assume that  $\gamma_1 = (-1, 0) \times \{0\}$ , and  $\bar{\gamma}_1 \cap \bar{\gamma}_2 = (0, 0)$  in the coordinate system  $(x_1, x_2)$ . In our considerations,  $\gamma_1, \gamma_2$  correspond to thin elastic inclusions incorporated in the elastic body. Let  $n = (n_1, n_2)$  be a unit normal vector to  $\gamma_2$ ;  $s = (n_2, -n_1)$ . Denote by  $a$  an angle between  $\gamma_1$  and  $\gamma_2$ . By  $v^i, w^i, \varphi^i, i = 1, 2$ , we denote orthogonal



**Figure 3.** Two elastic inclusions  $\gamma_1, \gamma_2$  inside elastic body  $\Omega_\gamma$ .

(in the directions  $v, n$ ), tangential (in the directions  $\tau, s$ ) displacements and rotation angles for the inclusions  $\gamma_1, \gamma_2$ , respectively. Functions defined on  $\gamma_2$  we identify with functions of the variable  $y$ ,  $(y_1, y_2) \in \Omega_\gamma, y = y_1$ . Also assume that  $\gamma_2 = (-1, 0) \times \{0\}$  in the coordinate system  $(y_1, y_2)$ . An equilibrium problem for the elastic body  $\Omega_\gamma$  and inclusions  $\gamma_1, \gamma_2$  is formulated as follows. We have to find a displacement field  $u = (u_1, u_2)$ , a stress tensor  $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$ , defined in  $\Omega_\gamma$ , and thin inclusion displacements  $v^i, w^i$  and rotation angles  $\varphi^i, i = 1, 2$ , defined on  $\gamma^i$ , respectively, such that

$$-\operatorname{div} \sigma = f, \sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega_\gamma, \tag{82}$$

$$-w_{xx}^1 = [\sigma_\tau] \text{ on } \gamma_1; -w_{yy}^2 = [\sigma_s] \text{ on } \gamma_2, \tag{83}$$

$$-\varphi_{xx}^1 + v_x^1 + \varphi^1 = 0 \text{ on } \gamma_1; -\varphi_{yy}^2 + v_y^2 + \varphi^2 = 0 \text{ on } \gamma_2, \tag{84}$$

$$-v_{xx}^1 - \varphi_x^1 = [\sigma_v] \text{ on } \gamma_1; -v_{yy}^2 - \varphi_y^2 = [\sigma_n] \text{ on } \gamma_2, \tag{85}$$

$$u = 0 \text{ on } \Gamma; v^i = w^i = \varphi^i = 0 \text{ at } \bar{\gamma}_i \cap \Gamma, i = 1, 2, \tag{86}$$

$$[u] = 0 \text{ on } \gamma_1 \cup \gamma_2, \tag{87}$$

$$v^1 = u_v, w^1 = u_\tau \text{ on } \gamma_1; v^2 = u_n, w^2 = u_s \text{ on } \gamma_2, \tag{88}$$

$$v^2 = v^1 \cos a + w^1 \sin a, w^2 = -v^1 \sin a + w^1 \cos a, \tag{89}$$

$$\varphi^1 = \varphi^2 \text{ as } (x_1, x_2) = (0, 0),$$

$$\varphi_x^1 + \varphi_y^2 = 0 \text{ as } (x_1, x_2) = (0, 0), \tag{90}$$

$$w_x^1 + w_y^2 \cos a + (v_y^2 + \varphi^2) \sin a = 0, v_x^1 + \varphi^1 - \tag{91}$$

$$-w_y^2 \sin a + (v_y^2 + \varphi^2) \cos a = 0 \text{ as } (x_1, x_2) = (0, 0).$$

The problem (82)–(91) is solvable. It admits a variational formulation. To this end, we denote  $\eta^i = (v^i, w^i, \varphi^i), i = 1, 2$ , and consider the energy functional

$$\pi_0(u, \eta^1, \eta^2) = \frac{1}{2} \int_\Omega \sigma(u) \varepsilon(u) - \int_\Omega fu + \sum_{i=1}^{i=2} \frac{1}{2} \int_{\gamma_i} F(\eta^i).$$

Introduce the space

$$V = \{(u, \eta^1, \eta^2) \mid u \in H_0^1(\Omega)^2, \eta^i \in H^{1,0}(\gamma_i)^3, i = 1, 2; \text{conditions (88), (89) fulfilled}\}.$$

Then the minimization problem:

$$\text{Find } (u, \eta^1, \eta^2) \in V \text{ such that } \pi_0(u, \eta^1, \eta^2) = \inf_V \pi_0$$

has a solution. We omit the details.



## 8. Conclusion

In the paper, we propose a model for a 2D elastic body with a thin elastic Timoshenko inclusion and provide its rigorous mathematical analysis. It is assumed that a delamination of the inclusion may take place providing therefore a presence of a crack. Nonlinear boundary conditions at the crack faces are imposed to prevent mutual penetration between the faces. Both variational and differential problem formulations are considered, and the existence of solutions is established. Furthermore, we study the dependence of the solution on the rigidity of the inclusion. It is proved that in the limit cases corresponding to infinite and zero rigidity, we obtain a rigid inclusion and cracks with non-penetration conditions, respectively. Anisotropic behavior of the inclusion is also analyzed, and limiting cases are investigated.

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