ISSN 1560-3547, Regular and Chaotic Dynamics, 2020, Vol. 25, No. 6, pp. 716–728. © Pleiades Publishing, Ltd., 2020.

On Topological Classification of Gradient-like Flows on an n-sphere in the Sense of Topological Conjugacy

Vladislav E. Kruglov[1*], Dmitry S. Malyshev[1], Olga V. Pochinka[1***], and Danila D. Shubin[1****]**

[1] *National Research University Higher School of Economics, ul. Bolshaya Pecherskaya 25/12, 603155 Nizhny Novgorod, Russia*

Received March 14, 2020; revised October 22, 2020; accepted November 12, 2020

Abstract—In this paper, we study gradient-like flows without heteroclinic intersections on an n-sphere up to topological conjugacy. We prove that such a flow is completely defined by a bicolor tree corresponding to a skeleton formed by codimension one separatrices. Moreover, we show that such a tree is a complete invariant for these flows with respect to the topological equivalence also. This result implies that for these flows with the same (up to a change of coordinates) partitions into trajectories, the partitions for elements, composing isotopies connecting time-one shifts of these flows with the identity map, also coincide. This phenomenon strongly contrasts with the situation for flows with periodic orbits and connections, where one class of equivalence contains continuum classes of conjugacy. In addition, we realize every connected bicolor tree by a gradient-like flow without heteroclinic intersections on the n-sphere. In addition, we present a linear-time algorithm on the number of vertices for distinguishing these trees.

MSC2010 numbers: 37D15, 37C15

DOI: 10.1134/S1560354720060143

Keywords: gradient-like flow, topological classification, topological conjugacy, n-sphere, linear-time algorithm

1. INTRODUCTION

It is well known that the Morse functions exist on any manifolds and, hence, there exist gradient flows. Generically, they are structurally stable and the dynamics of such systems is a basis for the class of *gradient-like flows*, i.e., flows whose nonwandering set consists of a finite number of hyperbolic fixed points, whose invariant manifolds are transversally intersected.

Gradient flows are used for modeling regular processes in different natural sciences (see, for example, [18]). In particular, these flows model solar corona reconnection processes (see, for example, [7]). This is the reason why it is important to be able to compare, regardless of the nature of their origin, the dynamics of such models and to compare, depending on research goals, both the qualitative behavior of a system (partition into trajectories) and time-based motion along trajectories. In dynamical systems theory, the relation preserving the partition into trajectories up to a homeomorphism is called *topological equivalence* and the relation additionally preserving the time of motion along trajectories is called *topological conjugacy*. Revealing invariants that uniquely determine the equivalence class for a system is called *topological classification*.

The finiteness of the set of nonwandering orbits of a gradient-like flow leads to the idea of reducing the problem of topological classification to a combinatorial problem. This was first done by E. Leontovich and A. Mayer [11, 12] for the classification of flows on a two-dimensional sphere with

*E-mail: kruglovslava21@mail.ru
**E-mail: dsmalyshev@rambler.ru
***E-mail: olga-pochinka@yandex.ru
****E-mail: schub.danil@yandex.ru

a finite set of singular orbits. Later, it was developed in the studies of M. Peixoto [17], A. Oshemkov, V. Sharko [15], S. Pilyugin [19], A. Prishlyak [20], where a similar problem was solved for Morse–Smale flows on closed manifolds in dimensions 2, 3 and higher.

All these works were devoted to the topological classification of gradient-like systems with respect to topological equivalence. In [9], it is shown that classes of equivalence and conjugacy coincide for gradient-like flows on surfaces and, hence, all classification results on topological equivalence are also true for topological conjugacy of such flows. In this paper, we give a similar result for the class G of gradient-like flows without heteroclinic intersections on an n-sphere, where $n \geqslant 3$.

Namely, we prove that a flow from G is completely defined by a bicolor tree corresponding to a skeleton formed by codimension one separatrices. Moreover, we show that such a tree is a complete invariant for these flows with respect to the topological equivalence also. This result implies that for the flows of interest with the same (up to a change of coordinates) partitions into trajectories, the partitions for elements composing isotopies connecting time-one shifts of these flows with the identity map also coincide. This phenomenon strongly contrasts with the situation for flows with periodic orbits and connections, where one class of equivalence contains continuum classes of conjugacy. In addition, we realize every connected bicolor tree by a gradient-like flow without heteroclinic intersections on an n-sphere. In addition, we present a linear-time algorithm on the size of input data for distinguishing these trees.

2. FORMULATION OF THE RESULTS

2.1. Dynamics of Flows of Class G

Let us begin with some definitions.

Definition 1. By a hyperbolic fixed point we mean any fixed point whose eigenvalues have nonzero real part.

Definition 2. By class G we mean the class of gradient-like flows without heteroclinic intersections on an n-sphere, where $n \geqslant 3$.

Recall that by an *n-ball* or *n-disk* we mean a manifold with a boundary homeomorphic to a standard ball $\mathbb{B}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 \leqslant 1\}$. By an *open n-ball* or *n-disk* we mean a manifold homeomorphic to the interior of $\mathbb{B}^n$. By a *sphere* we mean a manifold S^n homeomorphic to $\partial\mathbb{B}^n$.

Let us consider the class G of gradient-like flows on an n-dimensional sphere $S^n, n \geqslant 3$, without heteroclinic intersections, that is, flows whose nonwandering set consists of a finite number of hyperbolic fixed points such that the invariant manifolds of saddle points have no intersections.

Let $f^t \in G$. We will denote the stable and unstable invariant manifolds of a fixed point p by W_p^u and W_p^s, respectively.

Proposition 1 ([19, Lemma 2.2]). *1. There is no fixed point $p \in S$ of the flow $f^t \in G$ such that $2 \leqslant \dim W_p^u \leqslant n-2$;*

2. For any fixed point p such that $W_p^u = n-1$ the ω-limit set of W_p^u consists of a single stable fixed point.

According to this proposition, the dimensions of the invariant saddle manifolds of f^t have to be only $(n-1)$ and 1. Let Ω_{f^t} denote the nonwandering set of f^t, and let

$$\Omega^i_{f^t} = \{p \in \Omega_{f^t} \mid \dim W_p^u = i\}.$$

By [22, Theorem 2.3],

$$S^n = \bigcup_{p \in \Omega_{f^t}} W_p^u = \bigcup_{p \in \Omega_{f^t}} W_p^s.$$

It follows once again from Proposition 1 that for any saddle point σ of a flow f^t the closure of its invariant manifold W_σ^δ with dimension $(n-1)$ contains, except the manifold itself, exactly one fixed point. That point is a sink if $\delta = u$ and a source if $\delta = s$. Then the set $cl\, W^{\delta}_{\sigma}$ is a sphere with dimension $(n-1)$. By [3] and [2] this sphere is cylindrically embedded[1)]. Denote by m_{f^t} the number of saddle points of a flow f^t. Then the union

$$\mathcal{W}_{f^t} = \bigcup_{p\in\Omega^1_{f^t}} cl\, W^s_p \cup \bigcup_{q\in\Omega^{n-1}_{f^t}} cl\, W^u_q$$

of closures of all invariant manifolds of dimension $(n-1)$ divides a sphere S^n into $k_{f^t} = m_{f^t}+1$ connected components. Denote such components by $D_1,\dots,D_{k_{f^t}}$, and let

$$\mathcal{D}_{f^t} = \bigcup_{i=1}^{k_{f^t}} D_i.$$

2.2. Description of a Graph and a Bicolor Tree

Recall several definitions from graph theory.

A *graph* is a pair (V,E), where V is a set of *vertices* and E is a set of *pairs of vertices*, which are called *edges*. If E contains ordered pairs, then the graph is called a *directed* one. A *k-edge-coloring* of a graph is an assignment of k colors to its edges.

Two vertices are called *adjacent* if they are *connected* by an edge (i. e., they constitute the edge), and the edge is *incident to* each of the vertices. A *loop* is an edge whose end vertices coincide. A *simple graph* is an undirected graph without loops.

The number of edges incident to a vertex is called the *degree* of the vertex.

A set $\{v_1,(v_1,v_2),v_2,\dots,v_{k-1},(v_{k-1},v_k),v_k\}$ is called a *path* of length k. A graph is called *connected* if every two of its vertices are joined by a path.

A tree is a simple connected graph in which any two vertices are connected by exactly one path.

Any tree has at least two *pendant vertices*, that is, vertices of degree 1.

Any tree becomes a *directed planted tree* if an arbitrary pendant vertex r of this tree is selected as *a root*. In other words, a planted tree is a tree in which one vertex r has been designated as the root and every edge is directed away from the root.

If v is a vertex in a planted tree other than the root, *the parent of* v is the unique vertex w such that there is a directed edge (w,v). If w is the parent of v, then v is called *a child of* w.

By definition, the rooted vertex r has a level 0. *The level d of any other vertex v* in such a planted tree is the number of edges in the unique path between the vertex v and the root r. *The depth of a tree D* is the maximum level of any vertex there.

An ordered planted tree is a directed planted tree where the children of each vertex are ordered.

2.3. The Classification Result

Definition 3. A bicolor graph of a flow $f^t\in G$ is a graph Γ_{f^t} such that:

1) the set $\Gamma^0_{f^t}$ of vertices of Γ_{f^t} bijectively corresponds to $\mathcal{D}_{f^t}$ by a bijection

$$\xi_0\colon \Gamma^0_{f^t}\to\mathcal{D}_{f^t},$$

the set $\Gamma^1_{f^t}$ of edges of Γ_{f^t} bijectively corresponds to the set $\mathcal{W}_{f^t}$ by a bijection

$$\xi_1\colon \Gamma^1_{f^t}\to\mathcal{W}_{f^t};$$

2) two vertices v_i, v_j are connected by an edge $e_{i,j}$ iff domains $D_i=\xi_0(v_i)$, $D_j=\xi_0(v_j)$ have a common boundary;

3) an edge $e_{i,j}$ has a color u (resp. s) if it corresponds to a manifold $W^s_p\subset\mathcal{W}_{f^t}$ (resp. $W^u_p\subset\mathcal{W}_{f^t}$) (see Fig. 1).

[1)] A sphere $S^{n-1}\subset M^n$ is called *cylindrically embedded in* M^n if there exists a topological embedding $h\colon \mathbb{S}^{n-1}\times[-1;+1]\to M^n$, such that $h(\mathbb{S}^{n-1}\times\{0\}) = S^{n-1}$.

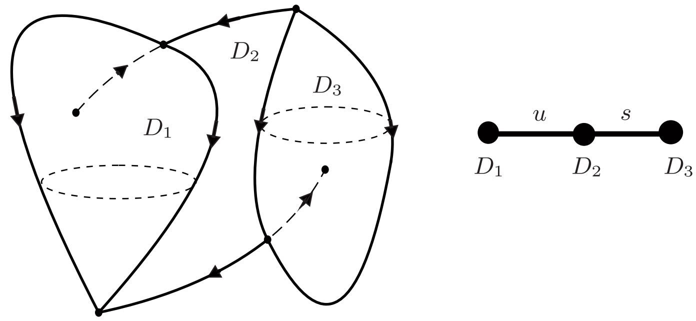

Fig. 1. Example of a flow and its bicolor graph.

Definition 4. Two graphs Γ_{f^t} and $\Gamma_{f'^t}$ of some flows f^t, f'^t are called *isomorphic* if there exists an isomorphism $\eta\colon \Gamma_{f^t} \to \Gamma_{f'^t}$ mapping the vertices and edges of Γ_{f^t} into the vertices and edges of $\Gamma_{f'^t}$ preserving colors.

It follows from [5] that the flows $f^t, f'^t \in G$ are topologically equivalent iff their graphs Γ_{f^t} and $\Gamma_{f'^t}$ are isomorphic. Indeed, for any flow $f^t \in G$ its bicolor graph is a tree, i. e., a connected graph without cycles.

Theorem 1. *Two flows $f^t, f'^t \in G$ are topologically conjugate iff their graphs Γ_{f^t} and $\Gamma_{f'^t}$ are isomorphic.*

Really we prove a stronger result in the following propositions.

Proposition 2. *If the graphs Γ_{f^t}, $\Gamma_{f'^t}$ of the flows $f^t, f'^t \in G$ are isomorphic, then the flows f^t, f'^t are topologically conjugate.*

Notice that, if f^t and f'^t are topologically conjugate, then they are topologically equivalent.

Proposition 3. *If the flows $f^t, f'^t \in G$ are topologically equivalent, then their graphs Γ_{f^t}, $\Gamma_{f'^t}$ are isomorphic.*

Besides, for any flow $f^t \in G$ its bicolor graph belongs to a specific type.

Proposition 4. *For each flow $f^t \in G$ its bicolor graph is a tree, i. e., a connected graph without cycles.*

2.4. The Realization Theorem and the Linear-Time Algorithm

Then the realization theorem has the following form.

Theorem 2. *For every bicolor tree Γ there is a flow $f^t \in G$ whose graph Γ_{f^t} is isomorphic to graph*[2)] Γ.

An algorithm is called *effective* if its execution time is bounded by some polynomial on the length of input information. We will present the fastest possible algorithm for recognizing the isomorphism of graphs of the flows considered.

Theorem 3. *The isomorphism problem for graphs of flows from G can be solved in linear time on the number of their vertices.*

3. SOLUTION OF THE CLASSIFICATION PROBLEM

In this part, we will prove Theorem 1, providing a list of necessary facts and proofs of Propositions 2–4. It was proved by us in the Russian-language paper [10], but here we give its proof in English.

[2)] Notice that flows of the class under consideration, under the assumption that they have a unique sink, were classified and realized in [4] by means of a directed graph

3.1. Necessary Facts

Proposition 5 ([16, Ch. 2, Theorem 4.10]; [9, Lemma 1]; [21, Ch. 4, Theorem 7.1]). *Let $F\colon M^n \to M^n$ be a C^r-vector field with a hyperbolic equilibrium point p, and let $DF_p\colon \mathbb{R}^n \to \mathbb{R}^n$ be the differential of F. Then there exists a neighborhood U_p of the point p, where the flow f^t, generated by F, is topologically conjugate to the flow ϕ^t, generated by DF_p.*

Proposition 6 ([21, Ch. 4, Theorem 7.1]). *Let A and B be two $n \times n$ real matrices, such that all the eigenvalues of A and B have nonzero real parts and the dimension of the direct sum of all the eigenspaces with negative (and, obviously, positive too) real part is the same for A and B. Then two flows, generated by the vector fields $\dot{x} = Ax$ and $\dot{x} = Bx$, are topologically conjugate.*

Recall that a set $\mathcal{A}$ is called an *attractor* of f^t if there exists a closed neighborhood $U \subset S^n$ of the set $\mathcal{A}$ such that $f^t(U) \subset \mathrm{int}(U)$ and $\mathcal{A} = \bigcup\limits_{t\geqslant 0} f^t(U)$. The set $\mathcal{R}$ is called a *repeller* of f^t if $\mathcal{R}$ is an attractor for f^{-t}.

Let $f^t \in G$. Suppose

$$\mathcal{A}_{f^t} = \bigcup_{\sigma_j \in \Omega^1_{f^t}} W^u_{\sigma_j} \cup \Omega^0_{f^t}, \quad \mathcal{R}_{f^t} = \bigcup_{\sigma_j \in \Omega^{n-1}_{f^t}} W^s_{\sigma_j} \cup \Omega^n_{f^t}, \quad \mathcal{V}_{f^t} = S^n \backslash (\mathcal{A}_{f^t} \cup \mathcal{R}_{f^t}).$$

By [6, Theorem 2.2.2], $\mathcal{A}^t_f$ and $\mathcal{R}_{f^t}$ are connected sets, besides, $\mathcal{A}_{f^t}$ is an attractor, $\mathcal{R}_{f^t}$ is a repeller, and $\mathcal{V}_{f^t}$ consists of wandering trajectories going from $\mathcal{R}_{f^t}$ to $\mathcal{A}_{f^t}$.

Following [13], by the *self-indexing energy function of the flow* f^t we mean the function $\varphi\colon S^n \to [0, n]$ with the following properties:

1) φ is a Morse function, i. e., a C^2-function with nondegenerate critical points;
2) the set of critical points of the function φ coincides with Ω_{f^t};
3) $\varphi(f^t(x)) < \varphi(x)$ for each $x \notin \Omega_{f^t}$ and each $t > 0$;
4) $\varphi(p) = \dim W^u_p$;

According to [13], any flow f^t of class G has its self-indexing energy function $\varphi\colon S^n \to [0, n]$. Suppose

$$\Sigma = \varphi^{-1}\left(\frac{n}{2}\right).$$

By construction, the hypersurface Σ crosses every trajectory of $f^t \subset \mathcal{V}_{f^t}$ at a single point (see Fig. 3, where a fragment of Σ is depicted) and is diffeomorphic to a sphere of dimension $n-1$ (see, for example, [5]).

3.2. Proof of Proposition 2

Let flows f^t, f'^t belong to class G and their graphs Γ_{f^t}, $\Gamma_{f'^t}$ be isomorphic by means of an isomorphism $\xi = (\xi_0, \xi_1)$. Let us prove that these flows are topologically conjugate.

Step 1. Recall that $D_1, \ldots, D_{k_{f^t}}$ are connected components of the set $\mathcal{D}_{f^t} = S^n \backslash (\bigcup\limits_{p \in \Omega^1_{f^t}} \mathrm{cl}\, W^s_p \cup \bigcup\limits_{q \in \Omega^{n-1}_{f^t}} \mathrm{cl}\, W^u_q)$. Then for each set D_i, $i \in \{1, \ldots, k_{f^t}\}$ there exists a set $\Omega_i \subset \left(\Omega^{n-1}_{f^t} \cup \Omega^1_{f^t}\right)$ such that $\partial D_i = \bigcup\limits_{\sigma \in \Omega_i} \mathrm{cl}\, W^\delta_\sigma$, where $\delta = u$ if $\sigma \in \Omega^{n-1}_{f^t}$ and $\delta = s$ if $\sigma \in \Omega^1_{f^t}$.

Without loss of generality let us assume that the connected components of the set $\mathcal{D}_{f'^t}$ are numbered so that $D'_i = \xi'_0 \xi_0^{-1}(D_i)$ and $\partial D'_i = \bigcup\limits_{\sigma' \in \Omega'_i} \mathrm{cl}\, W^\delta_{\sigma'}$, where $\Omega'_i = \{\sigma' \mid \mathrm{cl}\, W^\delta_{\sigma'} = \xi'_1 \xi_1^{-1}(\mathrm{cl}\, W^\delta_\sigma), \sigma \in \Omega_i\}$.

By Propositions 5 and 6 there exist neighborhoods U_σ and $U_{\sigma'}$ of saddle points σ, σ' such that $f^t|_{\mathrm{cl}\,U_\sigma}$, $f'^t|_{\mathrm{cl}\,U_{\sigma'}}$ are topologically conjugate by means of a homeomorphism $h_\sigma\colon \mathrm{cl}\,U_\sigma \to \mathrm{cl}\,U_{\sigma'}$, i.e., $h_\sigma \circ f^t|_{U_\sigma} = f'^t|_{U_{\sigma'}} \circ h_\sigma$ for each t not carrying points outside U_σ, $U_{\sigma'}$, respectively.

Without loss of generality we assume that the neighborhood U_σ is chosen so that any trajectory crossing its boundary crosses U_σ exactly by a single connected component (one can always do so by virtue of the conjugacy of a flow with a linear one in a neighborhood of σ, see Fig. 2).

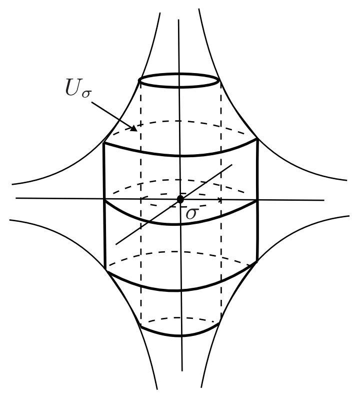

Fig. 2. Neighborhood of a saddle point.

For a point z let $\mathcal{O}_z$ ($\mathcal{O}'_z$) denote the trajectory of the flow f^t (f'^t) going through z. Let

$$V_\sigma = \bigcup_{z\in \mathrm{cl}\,U_\sigma} \mathcal{O}_z, \quad V_{\sigma'} = \bigcup_{z\in \mathrm{cl}\,U_{\sigma'}} \mathcal{O}'_z.$$

Extend h_σ to a homeomorphism $h_{V_\sigma}\colon V_\sigma \to V_{\sigma'}$ by the following rule:

$$h_{V_\sigma}(z) = f'^{-t_z}(h_\sigma(f^{t_z}(z))),$$

where $z \in V_\sigma$ and $t_z \in \mathbb{R}$ is such that $f^{t_z}(z) \in \mathrm{cl}\,U_\sigma$. It can be checked directly that the constructed map is a homeomorphism and does not depend on the way t_z is chosen. Suppose

$$V = \bigcup_{\sigma\in\Omega^{n-1}_{f^t}\cup\Omega^1_{f^t}} V_\sigma,\ V' = \bigcup_{\sigma'\in\Omega^{n-1}_{f'^t}\cup\Omega^1_{f'^t}} V_{\sigma'}$$

and denote by $h_V : V \to V'$ a homeomorphism constructed of the maps h_{V_σ}.

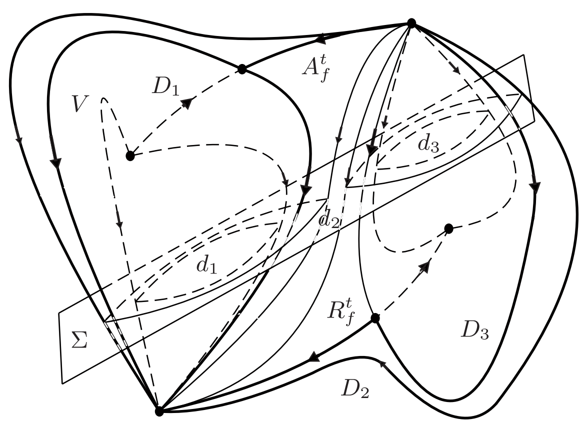

Fig. 3. Basic constructions from Statement 2.

Step 2. Let $\Sigma\,(\Sigma')$ be a level hypersurface for the energy function of $f^t\,(f'^t)$. For $\sigma \in \Omega^{n-1}_{f^t} \cup \Omega^1_{f^t}$ $\sigma' = h_V(\sigma) \in \Omega^{n-1}_{f'^t} \cup \Omega^1_{f'^t}$ suppose $K_\sigma = \Sigma \cap V_\sigma$, $K'_{\sigma'} = \Sigma' \cap V_{\sigma'}$ and $\tilde{K}'_{\sigma'} = h_V(K_\sigma)$. By construction, each of the three sets is homeomorphic to $S^{n-2} \times [0,1]$, and the relation $f'^{T_{\sigma'}(z)}(z) \in \tilde{K}'_{\sigma'}, z \in K'_{\sigma'}$ defines a real function $T_{\sigma'}$, which can be continuously extended to some neighborhood $U(K'_{\sigma'})$ of an annulus $K'_{\sigma'}$ so that $T_{\sigma'}(\partial U(K'_{\sigma'})) = 0$.

Without loss of generality assume that the annuli $U(K'_{\sigma'})$ do not cross each other for different saddle points. Then on the sphere Σ' a continuous function $T_{\Sigma'}$ coinciding with $T_{\sigma'}$ on $U(K'_{\sigma'})$ and equal to 0 outside the annuli is defined correctly. So, the sphere

$$\tilde{\Sigma}' = \bigcup_{z \in \Sigma'} f'^{T_{\Sigma'}(z)}(z)$$

is a section for the trajectories of the flow f'^t of the set $\mathcal{V}_{f'^t}$. Suppose $U = \mathcal{V}_{f^t} \setminus V$, $U' = \mathcal{V}_{f'^t} \setminus V'$, $\Sigma_V = \Sigma \cap V$, $\Sigma_U = \Sigma \cap U$, $\tilde{\Sigma}'_{V'} = \tilde{\Sigma}' \cap V'$ and $\tilde{\Sigma}'_{U'} = \tilde{\Sigma}' \cap U'$. Extend a homeomorphism $h_{\Sigma_V} : \Sigma_V \to \tilde{\Sigma}'_{V'}$ to a homeomorphism $h_\Sigma : \Sigma \to \tilde{\Sigma}'$.

To do this, we note that each connected component d_i of the set Σ_U lies in a connected component D_i of the set $\mathcal{D}_{f^t}$ and is homeomorphic to $(n-2)$-sphere with holes, whose number is equal to the number of boundary components of D_i (see Fig. 3). Denote by $\tilde{d}'_i$ a connected component of the set $\tilde{\Sigma}'_{U'}$ belonging to D'_i. From the isomorphism of the graphs it follows that $\partial\tilde{d}'_i = h_{\Sigma_V}(\partial d_i)$. By virtue of [14], the homeomorphism $h_{\Sigma_V}|_{\partial d_i}$ may be extended to homeomorphism $h_{d_i} : d_i \to \tilde{d}'_i$.

Denote by $h_{\Sigma_U} : \Sigma_U \to \tilde{\Sigma}'_{U'}$ a homeomorphism constructed of the maps h_{d_i}. Then the map h_Σ coinciding with h_{Σ_U} on Σ_U and with h_{Σ_V} on Σ_V is the sought-for homeomorphism.

Finally, let us define a homeomorphism $h_U \colon U \to U'$ by the formula

$$h_U(z) = f'^{-t_z}(h_\Sigma(f^{t_z}(z))),$$

where $z \in U$ and $t_z \in \mathbb{R}$ is such that $f^{t_z}(z) \in \tilde{\Sigma}'$. It can by directly checked that the map $h \colon U \cup V \to U' \cup V'$ coinciding with h_U on U and with h_V on V is a homeomorphism and may be continuously extended to the sought-for conjugating homeomorphism.

3.3. Proof of Proposition 3

Let us prove that, if the flows $f^t, f'^t \in G$ are topologically equivalent, then their graphs $\Gamma_{f^t}, \Gamma_{f'^t}$ are isomorphic.

Let $h \colon S^n \to S^n$ be the homeomorphism sending the trajectories of f^t into trajectories of f'^t preserving the directions of trajectories.

Let $\mathrm{cl}\, W^\delta_\sigma \subset \partial(D_i)$, $i = \overline{1,k}$, $\delta = u, s$. Then there exists a vertex $v_i \in \Gamma^0_{f^t}$ and an edge $e_{i,j} \in \Gamma^1_{f^t}$ such that $(D_i, \mathrm{cl}\, W^\delta_\sigma) = \xi(v_i, e_{i,j})$. Without loss of generality suppose $D'_i = h(D_i)$. Then let us define the isomorphism $\eta \colon \Gamma_{f^t} \to \Gamma_{f'^t}$ by the formula

$$\eta = \xi'^{-1} h \xi,$$

where the isomorphism $\xi' = (\xi'_0, \xi'_1)$ is defined for f'^t similarly to the isomorphism ξ for f^t.

3.4. Proof of Proposition 4

Let us prove that for each flow $f^t \in G$ its bicolor graph Γ_{f^t} is the tree, i. e., the connected graph without cycles.

Each edge $e_{i,j}$ of Γ_{f^t} corresponds to the closure $\mathrm{cl}\, W^\delta_\sigma$ of the invariant saddle manifold of some saddle σ of dimension $n-1$, which is the $(n-1)$-sphere $S_{i,j}$, namely, embedded into S^n. Then the sphere $S_{i,j}$ divides S^n into two connected components, and the graph Γ_{f^t} may be assumed to be embedded into the ambient manifold S^n so that each vertex v_i is a point in D_i and an edge $e_{i,j} = (v_i, v_j)$ is a simple arc connecting v_i with v_j and crossing the sphere $S_{i,j}$ at a single point. So, the connectivity of the graph Γ_{f^t} directly follows from connectivity of the sphere S^n. Besides, removing any edge $e_{i,j}$ from Γ_{f^t} leads to dividing the graph into two connected components, which contradicts to the existence of cycles in the graph.

4. REALIZATION OF A BICOLOR TREE BY A FLOW

Let us prove Theorem 2.

To construct a required flow on the n-sphere $\mathbb{S}^n$ for the given bicolor tree Γ, choose a vertex r of Γ as a root and order all children to get from the tree Γ an ordered planted tree. Denote by N the number of all vertices of Γ.

To realize the bicolor tree Γ by a flow, we will use the idea of embedding of $N-1$ pairwise disjoint *Cherry boxes* B_v in a *flow-shift* $g_0^t:\mathbb{R}^n\to\mathbb{R}^n$, where the flow-shift is given by the formula

$$g_0^t(x_1,\dots,x_n)=(x_1+t,\dots,x_n)$$

and the cherry box B_v has a form

$$B_v=\{(x_1,\dots,x_n)\in\mathbb{R}^n:|x_1-\alpha_v|\leqslant\delta_v,(x_2-\beta_v)^2+x_3^2+\dots x_n^2\leqslant\delta_v^2\}$$

for some $\alpha_v,\beta_v\in\mathbb{R},\delta_v>0$ which depends on the parameters of v. The dynamics in B_v coincides with the flow-shift dynamics on the boundary of B_v and differs from that inside the box due to the appearance of a saddle and a node. We will say that the dynamics in B_v is of type u (s) if the saddle point has an $(n-1)$-dimensional unstable (stable) manifold and the node point is a source (sink) (see Fig. 4).

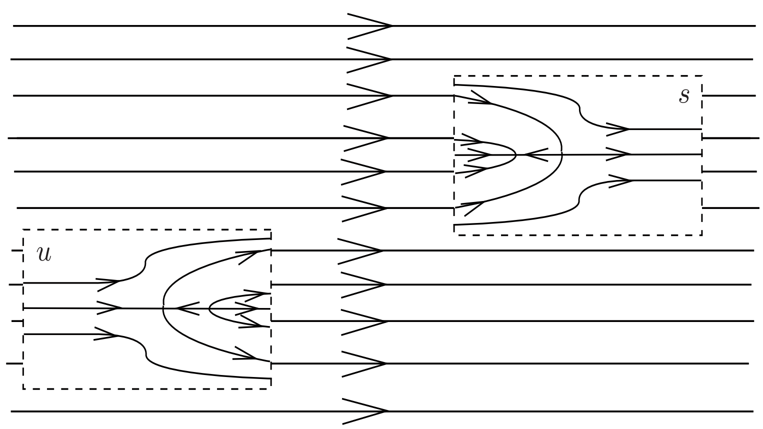

Fig. 4. An embedding of Cherry boxes of types u and s to the flow shift.

Below we give formulas for the following things:

1. Calculation of the position and size of the Cherry box B_v;
2. Definition of the flow g_v^t in B_v;
3. Embedding of the resulting dynamics in $\mathbb{S}^n$.

4.1. Calculation of the Position and Size of the Cherry Box B_v

For the vertex y which is a unique child of the root r we put

$$\alpha_y=2\rho_y\left(\frac{1}{2}+\frac{1}{2N-4}+\dots+\frac{1}{(2N-4)^{D-2}}\right),\ \ \beta_y=0,\ \ \delta_y=1,$$

where ρ_y equals 1 (-1) if the edge (r,y) has a color s (u) and D is the depth of the tree Γ. For any other vertex v with the level $d_v\geqslant 2$ the parameters of the box B_v are determined through the parameters $\alpha_w,\beta_w,\delta_w$ of its parent's box B_w, the order k_v of v as a child and a number ρ_v which equals 1 (-1) if the edge (w,v) has a color s (u) as follows:

$$\delta_v=\frac{\delta_w}{2N-4},\ \ \alpha_v=\rho_v\left(|\alpha_w|-\delta_w-\delta_v\right),\ \ \beta_v=\beta_w+\frac{\delta_w}{2}-(2k_v-1)\delta_v.$$

4.2. Definition of the Flow g_v^t in B_v

Let

$$\Sigma_v = (x_1 - \alpha_v)^2 + (x_2 - \beta_v)^2 + x_3^2 + \cdots + x_n^2.$$

Define the flow $g_v^t : \mathbb{R}^n \to \mathbb{R}^n$ by the formulas

$$\begin{cases} \dot{x}_1 = \begin{cases} 1 - \dfrac{16\delta_v^2}{9}\left(\Sigma_v - \delta_v^2\right)^2, & \Sigma_v \leqslant \delta_v^2 \\ 1, & \text{otherwise} \end{cases} \\ \dot{x}_2 = \begin{cases} \dfrac{x_2 - \beta_v}{2}\left(\sin\left(\dfrac{\pi}{2}\left(\frac{4\Sigma_v}{\delta_v^2} - 3\right)\right) - 1\right), & \dfrac{\delta_v^2}{2} < \Sigma_v \leqslant \delta_v^2 \\ -(x_2 - \beta_v), & \Sigma_v \leqslant \dfrac{\delta_v^2}{2} \\ 0, & \text{otherwise} \end{cases} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_n = \begin{cases} \dfrac{x_n}{2}\left(\sin\left(\dfrac{\pi}{2}\left(\frac{4\Sigma_v}{\delta_v^2} - 3\right)\right) - 1\right), & \dfrac{\delta_v^2}{2} < \Sigma_v \leqslant \delta_v^2 \\ -x_n, & \Sigma_v \leqslant \dfrac{\delta_v^2}{2} \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

By construction, the flow g_v^t has exactly two hyperbolic fixed points: the saddle (source) point $P_v(\alpha_v + \rho_v\delta_v/2, \beta_v, 0, \ldots, 0)$ and the sink (saddle) point $Q_v(\alpha_v - \rho_v\delta_v/2, \beta_v, 0, \ldots, 0)$ for $\rho_v = 1$ ($\rho_v = -1$). Define $g_\Gamma^t : \mathbb{R}^n \to \mathbb{R}^n$ in such a way that it coincides with g_v^t in B_v and is g_0^t outside all Cherry boxes (see Fig. 5).

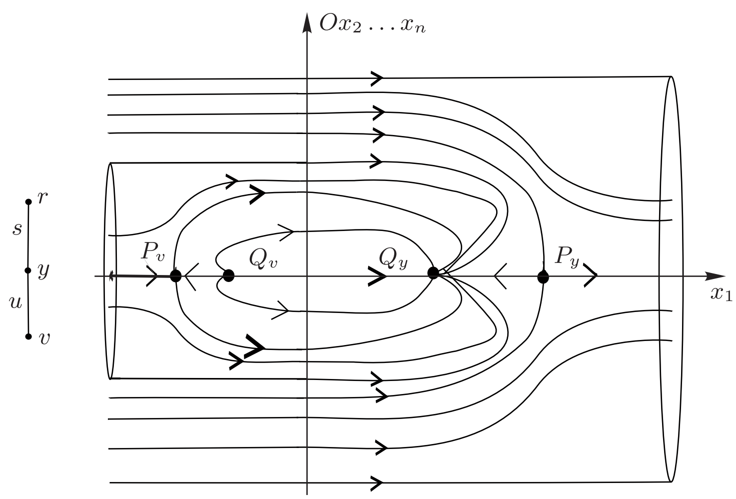

Fig. 5. An example of a tree Γ and the flow g_Γ^t.

Note that the flow g_Γ^t has no heteroclinic intersections. Indeed, by construction, the interiors of the Cherry boxes are pairwise disjoint. Moreover, in the hyperplane $x_1 = \alpha_v$ we have

$$\dot{x}_1 < 0 \ \ if \ \ (x_2 - \beta_v)^2 + x_3^2 + \cdots + x_n^2 < (\delta_v/2)^2,$$

$$\dot{x}_1 > 0 \ \ if \ \ (\delta_v/2)^2 < (x_2 - \beta_v)^2 + x_3^2 + \cdots + x_n^2 < \delta_v^2.$$

Also, in B_v we have

$$\dot{x}_2 \leqslant 0 \ \ if \ \ x_2 \geqslant \beta_v, \ \ \dot{x}_2 \geqslant 0 \ \ if \ \ x_2 \leqslant \beta_v,$$

$$\dot{x}_i \leqslant 0 \ \ if \ \ x_i \geqslant 0, \ \ \dot{x}_i \geqslant 0 \ \ if \ \ x_i \leqslant 0, i = 3, \ldots, n.$$

Thus, the invariant $(n-1)$-manifold of the saddle point from B_v outside B_v coincides with a cylinder

$$C_v = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_2 - \beta_v)^2 + x_3^2 + \cdots + x_n^2 \leqslant \nu_v^2\},$$

where $\delta_v/2 < \nu_v < \delta_v$. By construction, these cylinders are pairwise disjoint, which proves the fact.

4.3. Embedding of the Resulting Dynamics in $\mathbb{S}^n$

Let us define a flow $h^t : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$h^t(x_1, x_2, \ldots, x_n) = \left(2^t x_1, 2^t x_2, \ldots, 2^t x_n\right).$$

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geqslant 0\}$ and $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_2^2 + \cdots + x_n^2 \leqslant 1\}$. It is easy to verify that a diffeomorphism $\zeta : \mathbb{R}^n_+ \setminus O \to C$ given by the formula

$$\zeta(x_1, \ldots, x_n) = \left(\log_2 \varrho, \frac{x_2}{\varrho}, \ldots, \frac{x_n}{\varrho}\right), \ \varrho = \sqrt{x_1^2 + \cdots + x_n^2}$$

conjugates the diffeomorphisms $h^t|_{\partial\mathbb{R}^n_+ \setminus O}$ and $g^t|_{\partial C}$. This allows us to define a flow $\varphi^t \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in such a way that φ^t coincides with h^t outside $\operatorname{int}\mathbb{R}^n_+$ and coincides with $\zeta^{-1} g^t_\Gamma \zeta$ on $\mathbb{R}^n_+$.

Let us project the flow φ^t to the n-sphere by means of stereographic projection.

Denote by $\mathcal{N}(\underbrace{0, \ldots, 0}_{n}, 1)$ the north pole of the sphere $\mathbb{S}^n$. For every point $x = (x_1, \ldots, x_{n+1})$ in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ there is a unique line passing through the points $\mathcal{N}$ and x. This line intersects $\mathbb{R}^n = Ox_1 \ldots x_n$ at exactly one point $\vartheta(x)$ (see Fig. 6), which is called *the stereographic projection of the point* x. One can easily check that $\vartheta : \mathbb{S}^n \setminus \{\mathcal{N}\} \to \mathbb{R}^n$ is a diffeomorphism given by the formula

$$\vartheta(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_{n-1}}{1 - x_{n+1}}, \frac{x_n}{1 - x_{n+1}}\right).$$

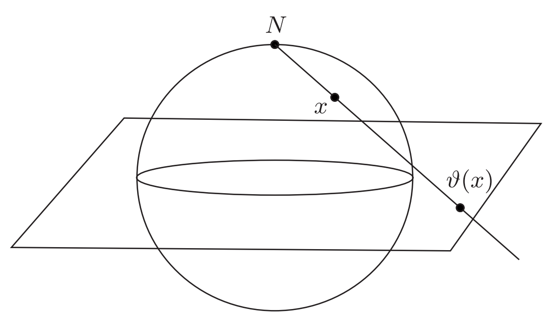

Fig. 6. The stereographic projection.

As flow φ^t coincides with h^t in some neighborhoods of the origin O and of the infinity point, it induces on $\mathbb{S}^n$ the required flow

$$f^t(x) = \begin{cases} \vartheta^{-1}(\varphi^t(\vartheta(x))), \ x \neq \mathcal{N}; \\ \mathcal{N}, \ x = \mathcal{N}. \end{cases}$$

5. PROOF OF THEOREM 3

In this section, we construct an $O(n)$ algorithm to determine whether two n-vertex bicolor trees are isomorphic or not. To this end, we first present a linear-time algorithm from [1] to distinguish two simple trees with the same number of vertices.

The *distance between two vertices of a simple graph* Γ is the number of edges in the shortest path, connecting these vertices. The *eccentricity of a vertex of* Γ is the maximum of distances between this vertex and the other vertices of Γ. The minimum among the eccentricities of vertices of Γ is called the *radius of* Γ. The *center of* Γ is the set of its vertices, whose eccentricity equals the radius of Γ.

Suppose that Γ is a tree. Then its center consists of one vertex or two adjacent vertices, by the known Jordan theorem [8]. The tree Γ is called *central* in the former case and *bicentral* in the latter. Jordan in [8] proposed the following *trimming procedure* to find the center of a given tree. At each step, all leaves of the current tree are deleted until we obtain a tree with one or two vertices. The vertex set of this tree coincides with the center of the initial tree, by a result in [8]. Thus, the center of Γ can be found in linear time on the number of its vertices.

Given two n-vertex simple trees Γ_1 and Γ_2, one may assume that they are either both central or both bicentral. Otherwise, they are not isomorphic. The bicentral case is reduced in $O(n)$ time to the central one as follows. If $\{a_i, b_i\}$ is the center of Γ_i, then the edge a_ib_i is deleted, a new vertex c_i is added, and the new edges a_ic_i and b_ic_i are added. Clearly, the resulting trees Γ_1^* and Γ_2^* are both central (with the centers $\{c_1\}$ and $\{c_2\}$, respectively), and Γ_1^* and Γ_2^* are isomorphic if and only if Γ_1 and Γ_2 are isomorphic.

The problem of an algorithm for identifying the isomorphism of simple trees from [1] works with n-vertex central trees and constructs the so-called monotonous Dyck words, which turn out to be canonical representations of the trees. In other words, two n-vertex central trees are isomorphic iff their monotonous Dyck words coincide. Given a central tree, its *monotonous Dyck word* is a binary word defined as follows. The word 10 is assigned to all its leaves. Next, if $w_1, w_2, \ldots, w_k$ are all the words, assigned to sons of a vertex x, then they are lexicographically sorted. Assuming that w_i is lexicographically no more than w_j, for any $i < j$, we assign $1w_kw_{k-1}\ldots w_10$ to x (see Fig. 7 and Fig. 8). By a result from [1], the canonical code of the whole tree is the monotonous Dyck word corresponding to its central vertex. Obviously, it can be obtained in time linear in n.

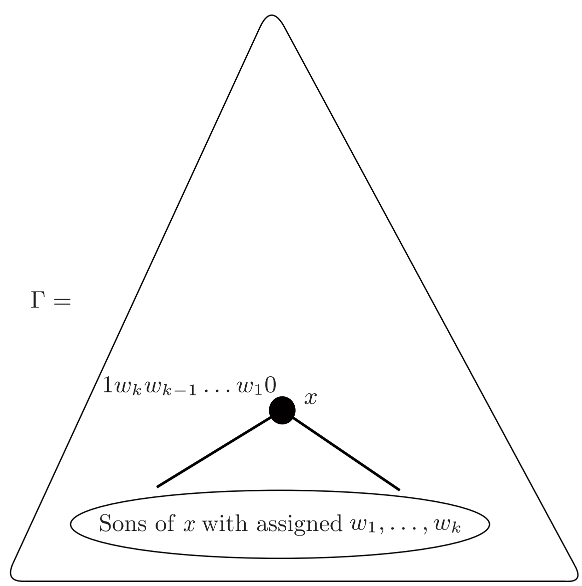

Fig. 7. A vertex x of a tree Γ and the corresponding Dyck word.

Assume that we are given n-vertex bicolor trees Γ and Γ'. We will obtain two (simple) trees $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ corresponding to Γ and Γ', respectively, such that $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ are isomorphic iff Γ and Γ' are

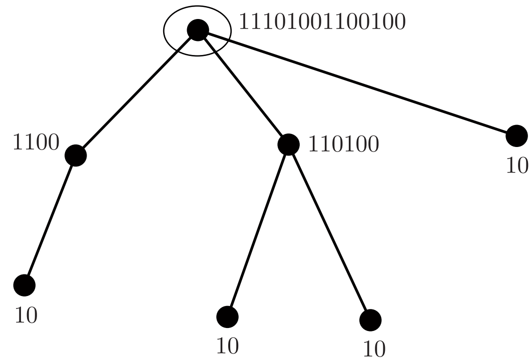

Fig. 8. A central tree with the canonical code 11101001100100.

isomorphic. To this end, we add four pendent vertices to every vertex of Γ or of Γ'. Next, for every u-edge (respectively, s-edge) $e = (a, b)$ of Γ or of Γ', we delete it, add two edges (a, c) and (c, b), and a pendent vertex adjacent to c (respectively, two pendent vertices, each adjacent to c), see Fig. 9. Let us make sure that $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ are isomorphic iff Γ and Γ' are isomorphic.

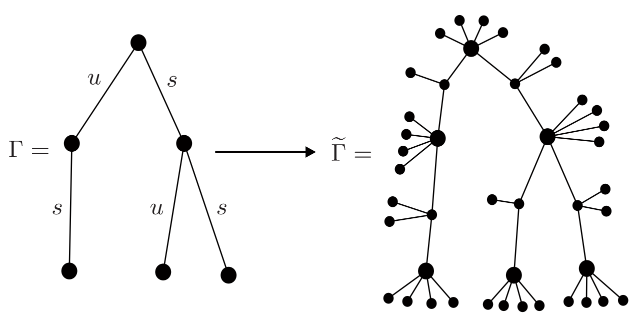

Fig. 9. A bicolor tree Γ and the corresponding tree $\tilde{\Gamma}$.

Indeed, the nonleaf vertices of $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ of degree at most 4 correspond to the edges of Γ, Γ' and to their colors. To determine the vertices of Γ and Γ', we delete all nonleaf vertices in $\tilde{\Gamma}, \tilde{\Gamma}'$ of degree at most 4 and their neighboring vertices. Next, we delete all the leaves in the resulting graph to obtain the vertex set of a graph in $\{\Gamma, \Gamma'\}$. Since any n-vertex tree has exactly $n - 1$ edges, both graphs $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ have at most $n + 4n + 4(n - 1) < 9n$ vertices. Therefore, Theorem 3 holds.

FUNDING

The realization results were implemented as an output of the RSF project No 17-11-01041. The classification results were obtained with assistance from the Laboratory of Dynamical Systems and Applications NRU HSE of the Ministry of science and Higher Education of the RF grant ag. No 075-15-2019-1931 and the RFBR project No 20-31-90067. The algorithmic results (Theorem 2.7 and its proof) were prepared within the framework of the Basic Research Program at the National Research University “Higher School of Economics” (HSE).

CONFLICT OF INTEREST

The authors have no conflict of interest.

REFERENCES

1. Aho, A. V., Hopcroft, J., and Ullman, J. D., *The Design and Analysis of Computer Algorithms,* Reading, Mass.: Addison-Wesley, 1974.
2. Brown, M., Locally Flat Embeddings of Topological Manifolds, *Ann. of Math. (2),* 1962, vol. 75, no. 2, pp. 331–341.
3. Cantrell, J. C., Almost Locally Flat Sphere S^{n-1} in S^n, *Proc. Amer. Math. Soc.,* 1964, vol. 15, no. 4, pp. 574–578.
4. Grines, V. Z., Gurevich, E. Ya., and Medvedev, V. S., Classification of Morse – Smale Diffeomorphisms with One-Dimensional Set of Unstable Separatrices, *Proc. Steklov Inst. Math.,* 2010, vol. 270, no. 1, pp. 57–79; see also: *Tr. Mat. Inst. Steklova,* 2010, vol. 270, pp. 62–85.
5. Grines, V. Z., Gurevich, E. Ya., and Pochinka, O. V., A Combinatorial Invariant of Morse – Smale Diffeomorphisms without Heteroclinic Intersections on the Sphere S^n, $n \geq 4$, *Math. Notes,* 2019, vol. 105, no. 1, pp. 132–136; see also: *Mat. Zametki,* 2019, vol. 105, no. 1, pp. 136–141.
6. Grines, V., Medvedev, T., and Pochinka, O., *Dynamical Systems on* 2- *and* 3-*Manifolds,* Dev. Math., vol. 46, New York: Springer, 2016.
7. Grines, V., Medvedev, T., Pochinka, O., and Zhuzhoma, E., On Heteroclinic Separators of Magnetic Fields in Electrically Conducting Fluids, *Phys. D,* 2015, vol. 294, pp. 1–5.
8. Jordan, C., Sur les assemblages de lignes, *J. Reine Angew. Math.,* 1869, vol. 70, no. 2, pp. 185–190.
9. Kruglov, V., Topological Conjugacy of Gradient-Like Flows on Surfaces, *Dinamicheskie Sistemy,* 2018, vol. 8(36), no. 1, pp. 15–21.
10. Kruglov, V. E. and Pochinka, O. V., Criterion for the Topological Conjugacy of Multi-Dimensional Gradient-Like Flows with No Heteroclinic Intersections on a Sphere, *Problemy Matematicheskogo Analiza,* 2020, vol. 104, pp. 21–28 (Russian).
11. Leontovich, E. A. and Maier, A. G., On a Scheme Determining the Topological Structure of a Decomposition into Trajectories, *Dokl. Akad. Nauk SSSR,* 1955, vol. 103, no. 4, pp. 557–560 (Russian).
12. Leontovich, E. A. and Mayer, A. G., On Trajectories Determining Qualitative Structure of Sphere Partition into Trajectories, *Dokl. Akad. Nauk SSSR,* 1937, vol. 14, no. 5, pp. 251–257 (Russian).
13. Meyer, K. R., Energy Function for Morse – Smale Systems, *Am. J. Math.,* 1968, vol. 90, pp. 1031–1040.
14. Morton, H. R., The Space of Homeomorphisms of a Disc with n Holes, *Illinois J. Math.,* 1967, vol. 11, pp. 40–48.
15. Oshemkov, A. A. and Sharko, V. V., On the Classification of Morse – Smale Flows on Two-Dimensional Manifolds, *Sb. Math.,* 1998, vol. 189, nos. 7–8, pp. 1205–1250; see also: *Mat. Sb.,* 1998, vol. 189, no. 8, pp. 93–140.
16. Palis, J. Jr. and de Melo, W., *Geometric Theory of Dynamical Systems: An Introduction,* New York: Springer, 1982.
17. Peixoto, M. M., On the Classification of Flows on Two-Manifolds, in *Dynamical Systems (Salvador, 1971),* M. M. Peixoto (Ed.), New York: Acad. Press, 1973, pp. 389–419.
18. Pesin, Ya. B. and Yurchenko, A. A., Some Physical Models Described by the Reaction-Diffusion Equation, and Coupled Map Lattices, *Russian Math. Surveys,* 2004, vol. 59, no. 3, pp. 481–513; see also: *Uspekhi Mat. Nauk,* 2004, vol. 59, no. 3(357), pp. 81–114.
19. Pilyugin, S. Yu., Phase Diagrams That Determine Morse – Smale Systems without Periodic Trajectories on Spheres, *Differ. Uravn.,* 1978, vol. 14, no. 2, pp. 245–254 (Russian).
20. Prishlyak, A. O., Morse – Smale Vector Fields without Closed Trajectories on Three-Dimensional Manifolds, *Math. Notes,* 2002, vol. 71, nos. 1–2, pp. 230–235; see also: *Mat. Zametki,* 2002, vol. 71, no. 2, pp. 254–260.
21. Robinson, C., *Dynamical Systems: Stability, Symbolic Dynamics, Chaos,* 2nd ed., Stud. Adv. Math., vol. 28, Boca Raton, Fla.: CRC, 1998.
22. Smale, S., Differentiable Dynamical Systems, *Bull. Amer. Math. Soc. (NS),* 1967, vol. 73, pp. 747–817.