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ORIGINAL ARTICLE

On topological structures of virtual fuzzy parametrized fuzzy soft sets

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Abstract
With the generalization of the concept of set, more comprehensive structures could be constructed in topological spaces. In this way, it is easier to express many relationships on existing mathematical models in a more comprehensive way. In this paper, the topological structure of virtual fuzzy parametrized fuzzy soft sets is analyzed by considering the virtual fuzzy parametrized fuzzy soft set theory, which is a hybrid set model that offers very practical approaches in expressing the membership degrees of decision makers, which has been introduced to the literature in recent years. Thus, it is aimed to contribute to the development of virtual fuzzy parametrized fuzzy soft set theory. To construct a topological structure on virtual fuzzy parametrized fuzzy soft sets, the concepts of point, quasi-coincident and mapping are first defined for this set theory and some of its characteristic properties are investigated. Then, virtual fuzzy parametrized fuzzy soft topological spaces are defined and concepts such as open, closed, closure, Q-neighborhood, interior, base, continuous, cover and compact are given. In addition, some related properties of these concepts are analyzed. Finally, many examples are given to make the paper easier to understand.

Keywords Virtual fuzzy parametrized fuzzy soft set · Virtual fuzzy parametrized fuzzy soft mapping · Virtual fuzzy parametrized fuzzy soft topology

Mathematics Subject Classification 03E72 · 11B05 · 54A05

Introduction

Vagueness and uncertainty are important characteristics which have to be dealt with during a data analysis to increase the robustness of the results. However, it is in general not so straightforward to decompose the vagueness of the data. Therefore, many mathematical approaches, which are based on the analysis of certain data, might be inadequate to capture this component. Many theories have been introduced to handle with the vagueness involved in data. To name a few, we can think of the theory of fuzzy sets (briefly FSs) [41], the theory of rough sets [31] and the theory of intuitionistic FSs [5]. Among these theories, Zadeh's FS theory [41] is the most popular one. Although these theories have brought several novelties into the classical theories, they all have some kind of drawbacks. In 1999, Molodtsov [29] introduced the soft set (briefly SS) theory and he further stated that this new theory is exempt from the difficulties seen in other theories, since it has sufficient parametrization tools. The fact that SSs offer a better approach than other mathematical models enabled this theory to be applied in various fields such as smoothness of functions, perron integrations, game theory and so on. Moreover, researchers who have studied this theory have applied this mathematical model to topological spaces [1–4,6,19,34], decision-making problems [12–14,16–18,21,42], and also ring and group theories [24,25,38].

Especially recently, there has been an increase in the number of studies that deal with FSs and SSs, which are two successful mathematical models in combating uncertainty. The first combination of these sets are fuzzy soft sets given by Maji and et al [28]. For these sets, a membership degree of the objects in the universe set is mentioned, so that a better approach to uncertainty is presented. As with SSs, this theory has been applied to various aspects such as decision making [7,8,20,32,33,40], algebraic structures [35,39] and topology [22,23,26,27,30,36,37]. Another combination of FSs and SSs are fuzzy parametrized soft sets given by Çağman et al. [10]. In this set theory, unlike [28], a membership degree of the elements in the parameter set is mentioned. Handling these two cases together, Çağman et al.

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[9] introduced the fuzzy parametrized fuzzy soft sets expressing the membership degrees of the elements in both object and parameter sets to the literature. However, this mathematical model cannot express uncertainty very well. Because determining the membership degrees depends on the decision maker and it is a very difficult task to express this in the range of [0, 1]. Another important reason is that how many mistakes the decision maker can make in determining the membership degrees of parameters and objects in the most accurate way is another uncertainty. In order to overcome these problems, Dalkılıç [11] allowed the decision maker to express a lower limit and an upper limit for each membership value and thus proposed virtual fuzzy parametrized fuzzy soft sets (briefly VFP-fuzzy soft sets or VFPFSS), a new hybrid set type of FSs and SSs with the combination of three different fuzzy parametrized fuzzy soft sets. The purpose of this paper is to analyze the topological structure of virtual fuzzy parametrized fuzzy soft sets, which facilitates the job of the decision maker in expressing the membership degrees in an uncertainty problem more accurately. In this way, it is aimed to express many relationships on existing mathematical models in a better way.

The presentation of the rest of this paper is structured as follows: In the second section, the framework of VFPFSSs are introduced. In the third section, some concepts required to construct a topological structure based on VFPFSSs have been defined and some related properties are given. In the third section, VFP-fuzzy soft topological spaces are analyzed. Moreover, some concepts of VFP-fuzzy soft topological spaces such as VFP-fuzzy soft open, VFP-fuzzy soft closed, VFP-fuzzy soft closure, VFP-fuzzy soft Q-neighborhood, VFP-fuzzy soft interior, base, VFP-fuzzy soft continuous, cover, VFP-fuzzy soft compact and pear have been given and some related properties have been analyzed. In addition, some examples are given to better understand the defined concepts. The final section consists of the conclusion of the paper.

Preliminaries

In this section, some definitions and results for set theories associated with this paper are reminded. Detailed explanations especially for VFPFSS among the reminded set theories can be found in [11].

Throughout this paper, $R = \{r_1, r_2, \ldots\}$ is an initial universe, 2^R is the power set of R and $P = \{p_1, p_2, \ldots\}$ is a set of parameters. In this case, the lower virtual parameter set and the upper virtual parameter set are expressed as $\underline{\mathcal{P}} = \{p_1^{\underline{\alpha_1}}, p_2^{\underline{\alpha_2}}, \ldots\}$ and $\overline{\mathcal{P}} = \{p_1^{\overline{\alpha_1}}, p_2^{\overline{\alpha_2}}, \ldots\}$,respectively.

Definition 2.1 [41] A FS X over R is a set defined by a function μ_X representing a mapping $\mu_X : R \to [0, 1]$. μ_X is called the membership function of X and the value $\mu_X(r)$ is called the grade of membership of r in X. Thus, a FS X over R can be represented as follows:

$$X = \{\langle \mu_X(r)/r \rangle : r \in R\}.$$

Then,

(i) A fuzzy point in R, whose value is a $(0 < \eta \leq 1)$ at the support $r \in R$; is denoted by r_η.
(ii) A fuzzy point $r_\eta \in X$, where X is FS in R iff $\eta \leq \mu_X(r)$.
(iii) A is called empty FS if $\mu_X(r) = 0$ for all $r \in R$, denoted by $\overline{0}$. If $\mu_X(r) = 1$ for all $r \in R$, X is denoted by $\overline{1}$.

State that the set of all the FSs over R will be denoted by $2^{F(R)}$.

Definition 2.2 [29] A pair (F, P) is called a SS over U, where F is a mapping given by $F : P \to 2^R$.

In other words, a SS over U is a parameterized family of subsets of U. For $p \in P$, $F(p)$ may be considered as the set of p-elements of the SS (F, P), or as the set of p-approximate elements of the SS, i.e.

$$(F, P) = \left\{(p, F(p)) : p \in P,\ \ F : P \to 2^R\right\}.$$

Definition 2.3 [11] Let $\underline{X}, X, \overline{X}$ be a FS over $\underline{P}, P, \overline{P}$, respectively. A VFPFSS Γ_X^V on R is defined as follows:

$$\Gamma_X^V = \underline{\Gamma_X} \cup \Gamma_X \cup \overline{\Gamma_X}$$

such that

$$\underline{\Gamma_X} = \left\{\left(\frac{\mu_{\underline{X}}(p^{\underline{\alpha}})}{p^{\underline{\alpha}}}, \underline{\gamma_X}\left(p^{\underline{\alpha}}\right)\right) : p^{\underline{\alpha}} \in \underline{P}, \underline{\gamma_X}\left(p^{\underline{\alpha}}\right) \in 2^{F(R)};\ \mu_X(p), \mu_{\underline{X}}(p^{\underline{\alpha}}) \in [0, 1], 0 \leq \underline{\alpha} < \mu_X(p)\right\},$$

$$\Gamma_X = \left\{\left(\frac{\mu_X(p)}{p}, \gamma_X(p)\right) : p \in P, \gamma_X(p) \in 2^{F(R)}, \mu_X \in [0, 1]\right\},$$

$$\overline{\Gamma_X} = \left\{\left(\frac{\mu_{\overline{X}}(p^{\overline{\alpha}})}{p^{\overline{\alpha}}}, \overline{\gamma_X}\left(p^{\overline{\alpha}}\right)\right) : p^{\overline{\alpha}} \in \overline{P}, \overline{\gamma_X}\left(p^{\overline{\alpha}}\right) \in 2^{F(R)};\ \mu_X(p), \mu_{\overline{X}}(p^{\overline{\alpha}}) \in [0, 1], 0 \leq \overline{\alpha} \leq 1 - \mu_X(p)\right\}$$

where the functions $\underline{\gamma_X} : \underline{P} \to 2^{F(R)}, \gamma_X : P \to 2^{F(R)}, \overline{\gamma_X} : \overline{P} \to 2^{F(R)}$ are called lower approximate function, approximate function, upper approximate function, respectively, and the function $\mu_X : P \to [0, 1]$ is called membership function of the X, such that "$\underline{\gamma_X}(p^{\underline{\alpha}}) = \overline{0}$ if $\mu_{\underline{X}}(p^{\underline{\alpha}}) = 0$", "$\gamma_X(p) =$

$\overline{0}$ if $\mu_X(p) = 0$" and "$\overline{\gamma_X}\left(p^{\overline{\alpha}}\right) = \overline{0}$ if $\mu_{\overline{X}}(p^{\overline{\alpha}}) = 0$". Here $\mu_{\underline{X}}(p^{\underline{\alpha}}) = \mu_X(p) - \underline{\alpha}$ and $\mu_{\overline{X}}(p^{\overline{\alpha}}) = \mu_X(p) + \overline{\alpha}$.

From now on, VFPFSS(R, P) denotes the family of all VFPFSSs over R with P as the set of parameters.

Definition 2.4 [11] Let $\Gamma_X^V \in$ VFPFSS(R, P). Then,

(i) Γ_X^V is called the empty-VFPFSS if $\mu_{\underline{X}}(p^{\underline{\alpha}}) = 0$ and $\underline{\gamma_X}(p^{\underline{\alpha}}) = \overline{0}$; $\forall p^{\underline{\alpha}} \in \underline{P}$, denoted by $\Gamma_{\emptyset}^V$.
(ii) Γ_X^V is called the X-universal-VFPFSS if $\mu_{\overline{X}}(p^{\overline{\alpha}}) = 1$ and $\overline{\gamma_X}(p^{\overline{\alpha}}) = \overline{1}$; $\forall p^{\overline{\alpha}} \in \overline{P}$, denoted by $\Gamma_{\tilde{X}}^V$. If $X = P$, then X-universal-VFPFSS is called universal-VFPFSS, denoted by $\Gamma_{\tilde{P}}^V$.

Definition 2.5 [11] Let $\Gamma_X^V, \Gamma_Y^V \in$ VFPFSS(R, P). Then,

(i) Γ_X^V is called a subset of Γ_Y^V if

- $\mu_{\underline{X}}(p^{\underline{\alpha}}) \le \mu_{\underline{Y}}(p^{\underline{\beta}})$ and $\underline{\gamma_X}(p^{\underline{\alpha}}) \subseteq \underline{\gamma_Y}(p^{\underline{\beta}})$; $\forall p^{\underline{\alpha}}, p^{\underline{\beta}} \in \underline{P}$,
- $\mu_X(p) \le \mu_Y(p)$ and $\gamma_X(p) \subseteq \gamma_Y(p)$; $\forall p \in P$,
- $\mu_{\overline{X}}(p^{\overline{\alpha}}) \le \mu_{\overline{Y}}(p^{\overline{\beta}})$ and $\overline{\gamma_X}(p^{\overline{\alpha}}) \subseteq \overline{\gamma_Y}(p^{\overline{\beta}})$; $\forall p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$

and we write $\Gamma_X^V \tilde{\subseteq} \Gamma_Y^V$.

(ii) Γ_X^V and Γ_Y^V are said to be equal, denoted by $\Gamma_X^V = \Gamma_Y^V$ if $\Gamma_X^V \tilde{\subseteq} \Gamma_Y^V$ and $\Gamma_Y^V \tilde{\subseteq} \Gamma_X^V$.
(iii) The complement of Γ_X^V, denoted by $\left[\Gamma_X^V\right]^c$, is the VFPFSS, defined by

- $\mu_{\underline{X}}^c(p^{\underline{\alpha}}) = 1 - \mu_{\underline{X}}(p^{\underline{\alpha}})$ and $\underline{\gamma_X^c}(p^{\underline{\alpha}}) = \overline{1} - \underline{\gamma_X}(p^{\underline{\alpha}})$; $\forall p^{\underline{\alpha}} \in \underline{P}$,
- $\mu_X^c(p) = 1 - \mu_X(p)$ and $\gamma_X^c(p) = \overline{1} - \gamma_X(p)$; $\forall p \in P$,
- $\mu_{\overline{X}}^c(p^{\overline{\alpha}}) = 1 - \mu_{\overline{X}}(p^{\overline{\alpha}})$ and $\overline{\gamma_X^c}(p^{\overline{\alpha}}) = \overline{1} - \overline{\gamma_X}(p^{\overline{\alpha}})$; $\forall p^{\overline{\alpha}} \in \overline{P}$. Clearly, $\left[\left[\Gamma_X^V\right]^c\right]^c = \Gamma_X^V$, $\left[\Gamma_{\tilde{P}}^V\right]^c = \gamma_{\emptyset}^V$ and $\left[\Gamma_{\emptyset}^V\right]^c = \Gamma_{\tilde{P}}^V$.

(iv) The union of Γ_X^V and Γ_X^V, denoted by $\Gamma_X^V \tilde{\cup} \Gamma_Y^V$, is the VFPFSS, defined by the membership and approximate functions

- $\mu_{\underline{X \cup Y}}(p^{\underline{\delta}}) = \max\left\{\mu_{\underline{X}}(p^{\underline{\alpha}}), \mu_{\underline{Y}}(p^{\underline{\beta}})\right\}$ and $\underline{\gamma_{X \cup Y}}(p^{\underline{\delta}}) = \underline{\gamma_X}(p^{\underline{\alpha}}) \vee \underline{\gamma_Y}(p^{\underline{\beta}})$; $\forall p^{\underline{\alpha}}, p^{\underline{\beta}}, p^{\underline{\delta}} \in \underline{P}$,
- $\mu_{X \cup Y}(p) = \max\{\mu_X(p), \mu_Y(p)\}$ and $\gamma_{X \cup Y}(p) = \gamma_X(p) \vee \gamma_Y(p)$; $\forall p \in P$,
- $\mu_{\overline{X \cup Y}}(p^{\overline{\delta}}) = \max\left\{\mu_{\overline{X}}(p^{\overline{\alpha}}), \mu_{\overline{Y}}(p^{\overline{\beta}})\right\}$ and $\overline{\gamma_{X \cup Y}}(p^{\overline{\delta}}) = \overline{\gamma_X}(p^{\overline{\alpha}}) \vee \overline{\gamma_Y}(p^{\overline{\beta}})$; $\forall p^{\overline{\alpha}}, p^{\overline{\beta}}, p^{\overline{\delta}} \in \overline{P}$, respectively.

(v) The intersection of Γ_X^V and Γ_X^V, denoted by $\Gamma_X^V \tilde{\cap} \Gamma_Y^V$, is the VFPFSS, defined by the membership and approximate functions

- $\mu_{\underline{X \cap Y}}(p^{\underline{\delta}}) = \min\left\{\mu_{\underline{X}}(p^{\underline{\alpha}}), \mu_{\underline{Y}}(p^{\underline{\beta}})\right\}$ and $\underline{\gamma_{X \cap Y}}(p^{\underline{\delta}}) = \underline{\gamma_X}(p^{\underline{\alpha}}) \wedge \underline{\gamma_Y}(p^{\underline{\beta}})$; $\forall p^{\underline{\alpha}}, p^{\underline{\beta}}, p^{\underline{\delta}} \in \underline{P}$,
- $\mu_{X \cap Y}(p) = \min\{\mu_X(p), \mu_Y(p)\}$ and $\gamma_{X \cap Y}(p) = \gamma_X(p) \wedge \gamma_Y(p)$; $\forall p \in P$,
- $\mu_{\overline{X \cap Y}}(p^{\overline{\delta}}) = \min\left\{\mu_{\overline{X}}(p^{\overline{\alpha}}), \mu_{\overline{Y}}(p^{\overline{\beta}})\right\}$ and $\overline{\gamma_{X \cap Y}}(p^{\overline{\delta}}) = \overline{\gamma_X}(p^{\overline{\alpha}}) \wedge \overline{\gamma_Y}(p^{\overline{\beta}})$; $\forall p^{\overline{\alpha}}, p^{\overline{\beta}}, p^{\overline{\delta}} \in \overline{P}$, respectively.

Proposition 2.6 [11] *Let* $\Gamma_X^V, \Gamma_Y^V, \Gamma_Z^V \in$ VFPFSS(R, P). *Then,*

((i) $\Gamma_X^V \tilde{\cup} \Gamma_{\tilde{P}}^V = \Gamma_{\tilde{P}}^V$, $\Gamma_X^V \tilde{\cup} \Gamma_{\emptyset}^V = \Gamma_X^V$.
(ii) $\Gamma_X^V \tilde{\cap} \Gamma_{\tilde{P}}^V = \Gamma_X^V$, $\Gamma_X^V \tilde{\cap} \Gamma_{\emptyset}^V = \Gamma_{\emptyset}^V$.
(iii) $\left[\Gamma_X^V \tilde{\cup} \Gamma_Y^V\right]^c = \left[\Gamma_X^V\right]^c \tilde{\cap} \left[\Gamma_Y^V\right]^c$, $\left[\Gamma_X^V \tilde{\cap} \Gamma_Y^V\right]^c = \left[\Gamma_X^V\right]^c \tilde{\cup} \left[\Gamma_Y^V\right]^c$.
(iv) $\Gamma_X^V \tilde{\cup} \left(\Gamma_Y^V \tilde{\cup} \Gamma_Z^V\right) = \left(\Gamma_X^V \tilde{\cup} \Gamma_Y^V\right) \tilde{\cup} \Gamma_Z^V$, $\Gamma_X^V \tilde{\cap} \left(\Gamma_Y^V \tilde{\cap} \Gamma_Z^V\right) = \left(\Gamma_X^V \tilde{\cap} \Gamma_Y^V\right) \tilde{\cap} \Gamma_Z^V$.
(v) $\Gamma_X^V \tilde{\cup} \left(\Gamma_Y^V \tilde{\cap} \Gamma_Z^V\right) = \left(\Gamma_X^V \tilde{\cup} \Gamma_Y^V\right) \tilde{\cap} \left(\Gamma_X^V \tilde{\cup} \Gamma_Z^V\right)$, $\Gamma_X^V \tilde{\cap} \left(\Gamma_Y^V \tilde{\cup} \Gamma_Z^V\right) = \left(\Gamma_X^V \tilde{\cap} \Gamma_Y^V\right) \tilde{\cup} \left(\Gamma_X^V \tilde{\cap} \Gamma_Z^V\right)$.

Some properties of VFPFSSs and VFP-fuzzy soft mappings

In this section, first, the concepts of union and intersection of more than two VFPFSSs, which are required in the construction of VFP-fuzzy soft topological spaces, are defined. Then, some concepts such as VFP-fuzzy soft point, VFP-fuzzy soft quasi-coincident and VFP-fuzzy soft mapping set are analyzed and some related properties are given.

Definition 3.1 Let I be an arbitrary index set and $\Gamma_{X_i}^V \in$ VFPFSS(R, P); $\forall i \in I$. Then,

(i) The union of $\Gamma_{X_i}^V \in VFPFSS(R, P)$'s, denoted by $\tilde{\bigcup}_{i \in I} \Gamma_{X_i}^V$, is the VFPFSS, defined by

- $\mu_{\underline{\cup_{i \in I} X_i}}(p^{\underline{\alpha}}) = \sup_{i \in I}\left\{\mu_{\underline{X_i}}(p^{\underline{\alpha_i}})\right\}$ and $\underline{\gamma_{\cup_{i \in I} X_i}}(p^{\underline{\alpha}}) = \bigvee_{i \in I} \underline{\gamma_{X_i}}(p^{\underline{\alpha_i}})$; $\forall p^{\underline{\alpha}}, p^{\underline{\alpha_i}} \in P$,
- $\mu_{\cup_{i \in I} X_i}(p) = \sup_{i \in I}\left\{\mu_{X_i}(p)\right\}$ and $\gamma_{\cup_{i \in I} X_i}(p) = \bigvee_{i \in I} \gamma_{X_i}(p)$; $\forall p \in P$,
- $\mu_{\overline{\cup_{i \in I} X_i}}(p^{\overline{\alpha}}) = \sup_{i \in I}\left\{\mu_{\overline{X_i}}(p^{\overline{\alpha_i}})\right\}$ and $\overline{\gamma_{\cup_{i \in I} X_i}}(p^{\overline{\alpha}}) = \bigvee_{i \in I} \overline{\gamma_{X_i}}(p^{\overline{\alpha_i}})$; $\forall p^{\overline{\alpha}}, p^{\overline{\alpha_i}} \in P$.

(ii) The intersection of $\Gamma_{X_i}^V \in$ VFPFSS(R, P)'s, denoted by $\tilde{\bigcap}_{i \in I} \Gamma_{X_i}^V$, is the VFPFSS, defined by

- $\mu_{\underline{\cap_{i\in I} X_i}}(p^{\underline{\alpha}}) = \inf_{i\in I}\left\{\mu_{\underline{X_i}}(p^{\underline{\alpha_i}})\right\}$ and $\underline{\gamma_{\cap_{i\in I} X_i}}(p^{\underline{\alpha}}) = \bigwedge_{i\in I}\underline{\gamma_{X_i}}(p^{\underline{\alpha_i}})$; $\forall p^{\underline{\alpha}}, p^{\underline{\alpha_i}} \in P$,
- $\mu_{\cap_{i\in I} X_i}(p) = \inf_{i\in I}\left\{\mu_{X_i}(p)\right\}$ and $\gamma_{\cap_{i\in I} X_i}(p) = \bigwedge_{i\in I}\gamma_{X_i}(p)$; $\forall p \in P$,
- $\mu_{\overline{\cap_{i\in I} X_i}}(p^{\overline{\alpha}}) = \inf_{i\in I}\left\{\mu_{\overline{X_i}}(p^{\overline{\alpha_i}})\right\}$ and $\overline{\gamma_{\cap_{i\in I} X_i}}(p^{\overline{\alpha}}) = \bigwedge_{i\in I}\overline{\gamma_{X_i}}(p^{\overline{\alpha_i}})$; $\forall p^{\overline{\alpha}}, p^{\overline{\alpha_i}} \in P$.

Proposition 3.2 *Let I be an arbitrary index set and $\Gamma^V_{X_i} \in$ VFPFSS(R, P); $\forall i \in I$. Then,*

(i) $\left[\widetilde{\bigcup}_{i\in I}\Gamma^V_{X_i}\right]^c = \widetilde{\bigcap}_{i\in I}\left[\Gamma^V_{X_i}\right]^c$.

(ii) $\left[\widetilde{\bigcap}_{i\in I}\Gamma^V_{X_i}\right]^c = \widetilde{\bigcup}_{i\in I}\left[\Gamma^V_{X_i}\right]^c$.

Proof (i) Let $\Gamma^V_Y = \left[\widetilde{\bigcup}_{i\in I}\Gamma^V_{X_i}\right]^c$ and $\Gamma^V_Z = \widetilde{\bigcap}_{i\in I}\left[\Gamma^V_{X_i}\right]^c$. Then $\forall p^{\underline{\alpha}}, p^{\underline{\alpha_i}} \in \underline{P}$, $\forall p \in P$, $\forall p^{\overline{\alpha}}, p^{\overline{\alpha_i}} \in \overline{P}$;

$$\begin{aligned}
\mu_{\underline{Y}}(p^{\underline{\alpha}}) &= 1 - \mu_{\underline{\cup_{i\in I} X_i}}(p^{\underline{\alpha}}) = 1 - \sup_{i\in I}\left\{\mu_{\underline{X_i}}(p^{\underline{\alpha_i}})\right\}\\
&= \inf_{i\in I}\left\{1 - \mu_{\underline{X_i}}(p^{\underline{\alpha_i}})\right\}\\
&= \inf_{i\in I}\left\{\mu^c_{\underline{X_i}}(p^{\underline{\alpha_i}})\right\}\\
&= \mu_{\underline{Z}}(p^{\underline{\alpha}}),\\
\mu_Y(p) &= 1 - \mu_{\cup_{i\in I} X_i}(p) = 1 - \sup_{i\in I}\left\{\mu_{X_i}(p)\right\}\\
&= \inf_{i\in I}\left\{1 - \mu_{X_i}(p)\right\}\\
&= \inf_{i\in I}\left\{\mu^c_{X_i}(p)\right\} = \mu_Z(p),\\
\mu_{\overline{Y}}(p^{\overline{\alpha}}) &= 1 - \mu_{\overline{\cup_{i\in I} X_i}}(p^{\overline{\alpha}}) = 1 - \sup_{i\in I}\left\{\mu_{\overline{X_i}}(p^{\overline{\alpha_i}})\right\}\\
&= \inf_{i\in I}\left\{1 - \mu_{\overline{X_i}}(p^{\overline{\alpha_i}})\right\}\\
&= \inf_{i\in I}\left\{\mu^c_{\overline{X_i}}(p^{\overline{\alpha_i}})\right\}\\
&= \mu_{\overline{Z}}(p^{\overline{\alpha}})
\end{aligned}$$

and

$$\begin{aligned}
\underline{\gamma_Y}(p^{\underline{\alpha}}) &= \overline{1} - \underline{\gamma_{\cup_{i\in I} X_i}}(p^{\underline{\alpha}})\\
&= \overline{1} - \bigvee_{i\in I}\underline{\gamma_{X_i}}(p^{\underline{\alpha_i}})\\
&= \bigwedge_{i\in I}\left(\overline{1} - \underline{\gamma_{X_i}}(p^{\underline{\alpha_i}})\right)\\
&= \bigwedge_{i\in I}\underline{\gamma^c_{X_i}}(p^{\underline{\alpha_i}})\\
&= \underline{\gamma^c_{\cap_{i\in I} X_i}}(p^{\underline{\alpha_i}})\\
&= \underline{\gamma_Z}(p^{\underline{\alpha}}),\\
\gamma_Y(p) &= \overline{1} - \gamma_{\cup_{i\in I} X_i}(p) = \overline{1} - \bigvee_{i\in I}\gamma_{X_i}(p)\\
&= \bigwedge_{i\in I}\left(\overline{1} - \gamma_{X_i}(p)\right)\\
&= \bigwedge_{i\in I}\gamma^c_{X_i}(p) = \gamma^c_{\cap_{i\in I} X_i}(p) = \gamma_Z(p),\\
\overline{\gamma_Y}(p^{\overline{\alpha}}) &= \overline{1} - \overline{\gamma_{\cup_{i\in I} X_i}}(p^{\overline{\alpha}}) = \overline{1} - \bigvee_{i\in I}\overline{\gamma_{X_i}}(p^{\overline{\alpha_i}})\\
&= \bigwedge_{i\in I}\left(\overline{1} - \overline{\gamma_{X_i}}(p^{\overline{\alpha_i}})\right)\\
&= \bigwedge_{i\in I}\overline{\gamma^c_{X_i}}(p^{\overline{\alpha_i}})\\
&= \overline{\gamma^c_{\cap_{i\in I} X_i}}(p^{\overline{\alpha_i}})\\
&= \overline{\gamma_Z}(p^{\overline{\alpha}}).
\end{aligned}$$

Therefore, the proof is completed.

(ii) It can be proved similar way (i).

Definition 3.3 Let $\Gamma^V_{X_i} \in$ VFPFSS(R, P). $\Gamma^V_{X_i}$ is called VFP-fuzzy soft point if $\underline{X}$, X, $\overline{X}$ are fuzzy points in $\underline{P}$, P, $\overline{P}$, respectively, and $\underline{\gamma_X}(p^{\underline{\alpha}})$, $\gamma_X(p)$, $\overline{\gamma_X}(p^{\overline{\alpha}})$ are fuzzy points in R for $p^{\underline{\alpha}}$, p, $p^{\overline{\alpha}} \in suppX$. If $\underline{X} = \{p^{\underline{\alpha}}\}$, $X = \{p\}$, $\overline{X} = \{p^{\overline{\alpha}}\}$, $\mu_{\underline{X}}(p^{\underline{\alpha}}) = \underline{\theta}^1$, $\mu_X(p) = \theta^1$, $\mu_{\overline{X}}(p^{\overline{\alpha}}) = \overline{\theta}^1$ and $\mu_{\underline{\gamma_X}(p^{\underline{\alpha}})}(r) = \underline{\theta}^2$, $\mu_{\gamma_X(p)}(r) = \theta^2$, $\mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r) = \overline{\theta}^2$, then we denote this VFP-fuzzy soft point by $p^{r_{\ddot{\theta}}}_{\dot{\theta}}$ for $\dot{\theta} = \left(\underline{\theta}^1, \theta^1, \overline{\theta}^1\right)$ and $\ddot{\theta} = \left(\underline{\theta}^2, \theta^2, \overline{\theta}^2\right)$.

Here, $\mu_{\underline{\gamma_X}(p^{\underline{\alpha}})}$, $\mu_{\gamma_X(p)}$, $\mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}$ are the membership functions of $\underline{\gamma_X}$, γ_X, $\overline{\gamma_X}$, respectively.

Definition 3.4 Let $p^{r_{\ddot{\theta}}}_{\dot{\theta}}, \Gamma^V_X \in$ VFPFSS(R, P). We say that $p^{r_{\ddot{\theta}}}_{\dot{\theta}}\widetilde{\in}\Gamma^V_X$ read as $p^{r_{\ddot{\theta}}}_{\dot{\theta}}$ belongs to Γ^V_X if and $\underline{\theta}^1 \leq \mu_{\underline{X}}(p^{\underline{\alpha}})$, $\theta^1 \leq \mu_X(p)$, $\overline{\theta}^1 \leq \mu_{\overline{X}}(p^{\overline{\alpha}})$ and $\underline{\theta}^2 \leq \mu_{\underline{\gamma_X}(p^{\underline{\alpha}})}(r)$, $\theta^2 \leq \mu_{\gamma_X(p)}(r)$, $\overline{\theta}^2 \leq \mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r)$.

Proposition 3.5 *Every non-empty* VFPFSS Γ^V_X *can be expresssed as the union of all the VFP-fuzzy soft points which belong to* Γ^V_X.

Proof Straighforward.

Definition 3.6 Let $\Gamma^V_X, \Gamma^V_Y \in$ VFPFSS(R, P). Γ^V_X is said to be VFP-fuzzy soft quasi-coincident with Γ^V_Y, denoted by $\Gamma^V_X \; q \; \Gamma^V_Y$, if there exists $p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$ such that $\mu_{\overline{X}}(p^{\overline{\alpha}}) + \mu_{\overline{Y}}(p^{\overline{\beta}}) > 1$ or there exists $r \in R$ such that $\mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r) + \mu_{\overline{\gamma_Y}(p^{\overline{\beta}})}(r) > 1$. If Γ^V_X is not VFP-fuzzy soft quasi-coincident with Γ^V_Y, then we write $\Gamma^V_X \; \overline{q} \; \Gamma^V_Y$.

Definition 3.7 Let $\Gamma^V_X, p^{r_{\ddot{\theta}}}_{\dot{\theta}} \in$ VFPFSS(R, P). $p^{r_{\ddot{\theta}}}_{\dot{\theta}}$ is said to be VFP-fuzzy soft quasi-coincident with Γ^V_X, denoted by $p^{r_{\ddot{\theta}}}_{\dot{\theta}} \; q \; \Gamma^V_X$, if $\overline{\theta}^1 + \mu_{\overline{X}}(p^{\overline{\alpha}}) > 1$ or $\overline{\theta}^2 + \mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r) > 1$. If $p^{r_{\ddot{\theta}}}_{\dot{\theta}}$ is not VFP-fuzzy soft quasi-coincident with Γ^V_X, then we write $p^{r_{\ddot{\theta}}}_{\dot{\theta}} \; \overline{q} \; \Gamma^V_X$.

Proposition 3.8 *Let* $\Gamma_X^V, \Gamma_Y^V \in$ VFPFSS(R, P). *Then,*

(i) $\Gamma_X^V \widetilde{\subseteq} \Gamma_Y^V \Rightarrow \Gamma_X^V \overline{q} \left[\Gamma_Y^V\right]^c$.
((ii)) $\Gamma_X^V \overline{q} \left[\Gamma_X^V\right]^c$.
(iii) $\Gamma_X^V q \Gamma_Y^V \Leftrightarrow$ *there exists an* $p_{\dot\theta}^{r_{\ddot\theta}} \widetilde{\in} \Gamma_X^V$ *such that* $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_Y^V$.
(iv) $\Gamma_X^V \widetilde{\subseteq} \Gamma_Y^V \Rightarrow$ *if* $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_X^V$, *then* $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_Y^V$*;* $\forall p_{\dot\theta}^{r_{\ddot\theta}} \in$ VFPFSS(R, P).
(v) $\Gamma_X^V q \Gamma_Y^V \Rightarrow \Gamma_X^V \widetilde{\cap} \Gamma_Y^V \neq \Gamma_\emptyset^V$.
(vi) $p_{\dot\theta}^{r_{\ddot\theta}} \widetilde{\in} \left[\Gamma_X^V\right]^c \Leftrightarrow p_{\dot\theta}^{r_{\ddot\theta}}\ \overline{q}\ \Gamma_X^V$*;* $\forall p_{\dot\theta}^{r_{\ddot\theta}} \in$ VFPFSS(R, P).

Proof (i)

$$\begin{aligned}
\Gamma_X^V \widetilde{\subseteq} \Gamma_Y^V &\Rightarrow \mu_{\overline{X}}\left(p^{\overline{\alpha}}\right) \leq \mu_{\overline{Y}}\left(p^{\overline{\beta}}\right) \text{ and } \overline{\gamma_X}\left(p^{\overline{\alpha}}\right) \\
&\subseteq \overline{\gamma_Y}\left(p^{\overline{\beta}}\right); \forall p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P} \\
&\Rightarrow \mu_{\overline{X}}\left(p^{\overline{\alpha}}\right) \leq \mu_{\overline{Y}}\left(p^{\overline{\beta}}\right) \text{ and } \mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) \\
&\leq \mu_{\overline{\gamma_Y\left(p^{\overline{\beta}}\right)}}(r); \forall p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}, r \in R \\
&\Rightarrow \mu_{\overline{X}}\left(p^{\overline{\alpha}}\right) - \mu_{\overline{Y}}\left(p^{\overline{\beta}}\right) \leq 0 \text{ and } \mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) \\
&-\mu_{\overline{\gamma_Y\left(p^{\overline{\beta}}\right)}}(r) \leq 0; \forall p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}, r \in R \\
&\Rightarrow \mu_{\overline{X}}\left(p^{\overline{\alpha}}\right) + 1 - \mu_{\overline{Y}}\left(p^{\overline{\beta}}\right) \leq 1 \text{ and } \mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) \\
&+1 - \mu_{\overline{\gamma_Y\left(p^{\overline{\beta}}\right)}}(r) \leq 1; \forall p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}, r \in R \\
&\Rightarrow \Gamma_X^V \overline{q} \left[\Gamma_Y^V\right]^c.
\end{aligned}$$

(ii) Let $\Gamma_X^V q \left[\Gamma_X^V\right]^c$. Then there exists $p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$ and $r \in R$ such that $\mu_{\overline{X}}(p^{\overline{\alpha}}) + 1 - \mu_{\overline{Y}}(p^{\overline{\beta}}) > 1$ or $\mu_{\overline{\gamma_X(p^{\overline{\alpha}})}} + 1 - \mu_{\overline{\gamma_Y(p^{\overline{\beta}})}} > 1$, i.e., the contradiction is obtained.
(iii) If $\Gamma_X^V q \Gamma_Y^V$, then there exist an $p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$ and $r \in R$ such that $\mu_{\overline{X}}(p^{\overline{\alpha}}) + \mu_{\overline{Y}}(p^{\overline{\beta}}) > 1$ or $\mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) + \mu_{\overline{\gamma_Y(p^{\overline{\beta}})}}(r) > 1$. Let $\mu_{\overline{X}}(p^{\overline{\alpha}}) = \dot\theta$ and $\mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) = \ddot\theta$. Thus we have $p_{\dot\theta}^{r_{\ddot\theta}} \widetilde{\in} \Gamma_X^V$ and $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_Y^V$.
Now, let $p_{\dot\theta}^{r_{\ddot\theta}} \widetilde{\in} \Gamma_X^V$ and $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_Y^V$. Then $\dot\theta \leq \mu_{\overline{X}}(p^{\overline{\alpha}})$ and $\ddot\theta \leq \mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r)$. Also, since $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_Y^V$, then $\dot\theta + \mu_{\overline{Y}}(p^{\overline{\beta}}) > 1$ or $\ddot\theta \mu_{\overline{\gamma_Y(p^{\overline{\beta}})}}(r) > 1$. Thus the proof is complete.
(iv) Let $p_{\dot\theta}^{r_{\ddot\theta}}, \Gamma_X^V \in$ VFPFSS(R, P). Since $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_X^V$, then $\dot\theta + \mu_{\overline{X}}(p^{\overline{\alpha}}) > 1$ or $\ddot\theta + \mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) > 1$. Also, since $\Gamma_X^V \widetilde{\subseteq} \Gamma_Y^V$, then $\dot\theta + \mu_{\overline{Y}}(p^{\overline{\beta}}) > 1$ or $\ddot\theta + \mu_{\overline{\gamma_Y(p^{\overline{\beta}})}}(r) > 1$. Thus, we have $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_Y^V$.
(v) Since $\Gamma_X^V q \Gamma_Y^V$, then there exists an $p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$ and $r \in R$ such that $\mu_{\overline{X}}(p^{\overline{\alpha}}) + \mu_{\overline{Y}}(p^{\overline{\beta}}) > 1$ or $\mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) + \mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) > 1$. If

- $\mu_{\overline{X}}(p^{\overline{\alpha}}) + \mu_{\overline{Y}}(p^{\overline{\beta}}) > 1$, then $X \wedge Y \neq \overline{0}$,
- $\mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) + \mu_{\overline{\gamma_X(p^{\overline{\alpha}})}}(r) > 1$, then $\overline{\gamma_X}(p^{\overline{\alpha}}) \wedge \overline{\gamma_X}(p^{\overline{\alpha}}) \neq \overline{0}$. Hence $\Gamma_X^V \widetilde{\cap} \Gamma_Y^V \neq \Gamma_\emptyset^V$.

(vi) It is obvious from (i).

Proposition 3.9 *Let* $\left\{\Gamma_{X_i}^V : i \in I\right\}$ *be a family of* VFPFSSs *in* VFPFSS(R, P) *where* I *is an index set. Then,* $p_{\dot\theta}^{r_{\ddot\theta}}$ *a is quasi-coincident with* $\widetilde{\bigcup}_{i \in I} \Gamma_{X_i}^V$ *if and only if there exists some* $\Gamma_{X_i}^V \in \left\{\Gamma_{X_i}^V : i \in I\right\}$ *such that* $p_{\dot\theta}^{r_{\ddot\theta}}\ q\ \Gamma_{X_i}^V$.

Proof Straighforward.

Definition 3.10 Let VFPFSS(R, P) and VFPFSS(M, N) be families of all VFPFSSs over R and M, respectively. Let $\underline{\rho} : \underline{P} \to \underline{N}$, $\rho : P \to N$, $\overline{\rho} : \overline{P} \to \overline{N}$ and $\varrho : R \to M$. Then, a VFP-fuzzy soft mapping $\gamma_{\varrho,\rho}$: VFPFSS$(R, P) \to$ $VFPFSS(M, N)$ is defined as:

(i) for $\Gamma_X^V \in$ VFPFSS(R, P), then the image of Γ_X^V under the $\gamma_{\varrho,\rho}$ is the VFPFSS Λ_S^V over M defined by; $\forall n^{\underline{\beta}} \in \underline{N}, \forall n \in N, \forall n^{\overline{\beta}} \in \overline{N}$,

$$\mu_{\underline{\lambda_S}(n^{\underline{\beta}})}(m) = \begin{cases} \bigvee_{r \in \varrho^{-1}(m)} \mu_{\underline{w}}(r), & \text{if } \varrho^{-1}(m) \neq \emptyset \text{ and} \\ \quad p^{\underline{\alpha}} \in \underline{\rho}^{-1}(n^{\underline{\beta}}) \cap limsupp\underline{\widetilde{X}} \neq \emptyset, & \\ \overline{0}, & \text{otherwise.} \end{cases},$$

$$\mu_{\lambda_S(n)}(m) = \begin{cases} \bigvee_{r \in \varrho^{-1}(m)} \mu_w(r), & \text{if} \varrho^{-1}(m) \neq \emptyset \text{ and} \\ \quad \rho^{-1}(n) \cap \text{limsupp}\widetilde{X}\gamma_X(p) \neq \emptyset, & \\ \overline{0}, & \text{otherwise.} \end{cases},$$

$$\mu_{\overline{\lambda_S}(n^{\overline{\beta}})}(m) = \begin{cases} \bigvee_{r \in \varrho^{-1}(m)} \mu_{\overline{w}}(r), & \text{if} \varrho^{-1}(m) \neq \emptyset \text{ and} \\ \quad p^{\overline{\alpha}} \in \overline{\rho}^{-1}(n^{\overline{\beta}}) \cap \text{limsupp}\widetilde{\overline{X}} \neq \emptyset, & \\ \overline{0}, & \text{otherwise.} \end{cases}$$

where

$$\underline{w} = \bigvee_{p^{\underline{\alpha}} \in \underline{\rho}^{-1}(n^{\underline{\beta}}) \cap \text{limsupp}\underline{\widetilde{X}}} \underline{\gamma_X}(p^{\underline{\alpha}}),$$
$$w = \bigvee_{p \in \rho^{-1}(n) \cap \text{limsupp}\widetilde{X}} \gamma_X(p),$$
$$\overline{w} = \bigvee_{p^{\overline{\alpha}} \in \overline{\rho}^{-1}(n^{\overline{\beta}}) \cap \text{limsupp}\overline{\widetilde{X}}} \overline{\gamma_X}(p^{\overline{\alpha}})$$

and $\underline{\rho}(\underline{X}) = \underline{S}$, $\rho(X) = S$, $\overline{\rho}(\overline{X}) = \overline{S}$ are FSs in $\underline{N}$, N, $\overline{N}$; respectively.

(ii) for $\Lambda_S^V \in \text{VFPFSS}(M, N)$, the pre-image of Λ_S^V under the $\gamma_{\varrho,\rho}$ is the VFPFSS Γ_X^V over R defined by; $\forall p^{\underline{\alpha}} \in \underline{P}$, $\forall p \in P$, $\forall p^{\overline{\alpha}} \in \overline{P}$,

$$\mu_{\underline{\gamma_X}(p^{\underline{\alpha}})}(r) = \mu_{\underline{\lambda_S}\left(\underline{\rho}(p^{\underline{\alpha}})\right)}(\varrho(r)),$$
$$\mu_{\gamma_X(p)}(r) = \mu_{\lambda_S(\rho(p))}(\varrho(r)),$$
$$\mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r) = \mu_{\overline{\lambda_S}(\overline{\rho}(p^{\overline{\alpha}}))}(\varrho(r))$$

where $\underline{X} = \underline{\rho}^{-1}(\underline{S})$, $X = \rho^{-1}(S)$, $\overline{X} = \overline{\rho}^{-1}(\overline{S})$ are FSs in $\underline{P}$, P $\overline{\underline{P}}$, respectively.

Here, if $\underline{\rho}$, ρ, $\overline{\rho}$ and ϱ are injective (surjective, constant), then the VFP-fuzzy soft mapping $\gamma_{\varrho,\rho}$ is injective (surjective, constant).

Theorem 3.11 *Let* $\Gamma_X^V, \Gamma_{X_i}^V \in \text{VFPFSS}(R, P)$, $\Lambda_S^V, \Lambda_{S_i}^V \in \text{VFPFSS}(M, N)$*;* $\forall i \in I$ *where* I *is an index set. Let* $\gamma_{\varrho,\rho} : \text{VFPFSS}(R, P) \to \text{VFPFSS}(M, N)$ *be a VFP-fuzzy soft mapping. Then,*

(i) $\gamma_{\varrho,\rho}$ is injective$\Rightarrow \Gamma_X^V \widetilde{\subseteq} \gamma_{\varrho,\rho}^{-1}(\gamma_{\varrho,\rho}(\Gamma_X^V))$.
(ii) $\gamma_{\varrho,\rho}$ is surjective$\Rightarrow \gamma_{\varrho,\rho}^{-1}(\gamma_{\varrho,\rho}(\Lambda_S^V)) \widetilde{\subseteq} \Lambda_S^V$.
(iii) $\gamma_{\varrho,\rho}$ is injective$\Rightarrow \gamma_{\varrho,\rho}\left(\widetilde{\bigcap}_{i\in I}\Gamma_{X_i}^V\right) \widetilde{\subseteq} \widetilde{\bigcap}_{i\in I}\gamma_{\varrho,\rho}(\Gamma_{X_i}^V)$.
(iv) $\gamma_{\varrho,\rho}$ is surjective$\Rightarrow \gamma_{\varrho,\rho}\left(\Gamma_{\widetilde{P}}^V\right) \widetilde{\subseteq} \Lambda_{\widetilde{N}}^V$.
(v) $\Lambda_{S_1}^V \widetilde{\subseteq} \Lambda_{S_2}^V \Rightarrow \gamma_{\varrho,\rho}^{-1}(\Lambda_{S_1}^V) \widetilde{\subseteq} \gamma_{\varrho,\rho}^{-1}(\Lambda_{S_2}^V)$.
(vi) $\Gamma_{X_1}^V \widetilde{\subseteq} \Gamma_{X_2}^V \Rightarrow \gamma_{\varrho,\rho}(\Gamma_{X_1}^V) \widetilde{\subseteq} \gamma_{\varrho,\rho}(\Gamma_{X_2}^V)$.
(vii) $\widetilde{\bigcup}_{i\in I}\gamma_{\varrho,\rho}^{-1}(\Lambda_{S_i}^V) = \gamma_{\varrho,\rho}^{-1}\left(\widetilde{\bigcup}_{i\in I}\Lambda_{S_i}^V\right)$.
(viii) $\widetilde{\bigcap}_{i\in I}\gamma_{\varrho,\rho}^{-1}(\Lambda_{S_i}^V) = \gamma_{\varrho,\rho}^{-1}\left(\widetilde{\bigcap}_{i\in I}\Lambda_{S_i}^V\right)$.
(ix) $\widetilde{\bigcup}_{i\in I}\gamma_{\varrho,\rho}(\Gamma_{X_i}^V) = \gamma_{\varrho,\rho}\left(\widetilde{\bigcup}_{i\in I}\Gamma_{X_i}^V\right)$.
(x) $\gamma_{\varrho,\rho}^{-1}\left(\left[\Lambda_{S_i}^V\right]^c\right) = \left[\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V)\right]^c$.
(xi) $\Gamma_{\widetilde{P}}^V = \gamma_{\varrho,\rho}^{-1}\left(\Lambda_{\widetilde{N}}^V\right)$.
(xii) $\Gamma_{\emptyset}^V = \gamma_{\varrho,\rho}^{-1}\left(\Lambda_{\emptyset}^V\right)$.
(xiii) $\Lambda_{\emptyset}^V = \gamma_{\varrho,\rho}\left(\Gamma_{\emptyset}^V\right)$.

Proof We only prove (i), (vii), (x), (xi) and (xii). The remaining items can be proved similarly.

(i) Let $\Lambda_S^V = \gamma_{\varrho,\rho}(\Gamma_X^V)$ and $\Gamma_Y^V = \gamma_{\varrho,\rho}^{-1}(\Lambda_S^V)$. Since $\underline{X} \subseteq \underline{\rho}^{-1}\left(\underline{\rho}(\underline{X})\right) = \underline{\rho}^{-1}(\underline{S}) = \underline{Y}$, $X \subseteq \rho^{-1}(\rho(X)) = \rho^{-1}(S) = Y$ and $\overline{X} \subseteq \overline{\rho}^{-1}\left(\overline{\rho}(\overline{X})\right) = \overline{\rho}^{-1}(\overline{S}) = \overline{Y}$, then it is sufficient to show $\underline{\gamma_X}(p^{\underline{\alpha}}) \subseteq \underline{\gamma_Y}\left(p^{\underline{\beta}}\right)$, $\gamma_X(p) \subseteq \gamma_Y(p)$, $\overline{\gamma_X}\left(p^{\overline{\alpha}}\right) \subseteq \overline{\gamma_Y}\left(p^{\overline{\beta}}\right)$; $\forall p^{\underline{\alpha}}, p^{\underline{\beta}} \in \underline{P}$, $\forall p \in P$, $\forall p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$ $r \in R$,

$$\mu_{\underline{\gamma_Y}(p^{\underline{\alpha}})}(r) = \mu_{\underline{\lambda_S}(\underline{\rho}(p^{\underline{\alpha}}))}(\varrho(r)) = \bigvee_{r\in\varrho^{-1}(\varrho(r))} \mu_{\underline{q}}(r) \geq \mu_{\underline{\gamma_X}(p^{\underline{\alpha}})}(r),$$
$$\mu_{\gamma_Y(p)}(r) = \mu_{\lambda_S(\rho(p))}(\varrho(r)) = \bigvee_{r\in\varrho^{-1}(\varrho(r))} \mu_q(r) \geq \mu_{\gamma_X(p)}(r),$$

and

$$\mu_{\overline{\gamma_Y}(p^{\overline{\alpha}})}(r) = \mu_{\overline{\lambda_S}(\overline{\rho}(p^{\overline{\alpha}}))}(\varrho(r)) = \bigvee_{r\in\varrho^{-1}(\varrho(r))} \mu_{\overline{q}}(r) \geq \mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r),$$

where

$$\underline{q} = \bigvee_{p^{\underline{\alpha}} \in \underline{\rho}^{-1}\left(\underline{\rho}(p^{\underline{\alpha}})\right) \cap \text{limsupp}\underline{\widetilde{X}}},$$
$$q = \bigvee_{p \in \rho^{-1}(\rho(p)) \cap limsupp]\widetilde{X}},$$
$$\overline{q} = \bigvee_{p^{\overline{\alpha}} \in \overline{\rho}^{-1}(\overline{\rho}(p^{\overline{\alpha}})) \cap \text{limsupp}\overline{\widetilde{X}}}$$

Thus, the proof is complete.

(vii) If $\Gamma_{X_i}^V = \gamma_{\varrho,\rho}^{-1}(\Lambda_{S_i}^V)$ and $\Gamma_X^V = \gamma_{\varrho,\rho}^{-1}\left(\widetilde{\bigcup}_{i\in I}\Lambda_{S_i}^V\right)$, then $\underline{X} = \underline{\rho}^{-1}(\vee\underline{S}_i) = \vee\underline{\rho}^{-1}(\underline{S}_i) = \vee\underline{X}_i$, $X = \rho^{-1}(\vee S_i) = \vee\rho^{-1}(S_i) = \vee X_i$, $\overline{X} = \overline{\rho}^{-1}(\vee\overline{S}_i) = \vee\overline{\rho}^{-1}(\overline{S}_i) = \vee\overline{X}_i$; $\forall p^{\underline{\alpha}} \in \underline{P}$, $\forall p \in P$, $\forall p^{\overline{\alpha}} \in \overline{P}$ and $r \in R$,

$$\mu_{\underline{\gamma_X}(p^{\underline{\alpha}})}(r) = \bigvee_{i\in I} \mu_{\underline{\lambda_{S_i}}(\underline{\rho}(p^{\underline{\alpha}}))}(\varrho(r)) = \bigvee_{i\in I} \mu_{\underline{\gamma_{X_i}}(p^{\underline{\alpha}})}(r),$$
$$\mu_{\gamma_X(p)}(r) = \bigvee_{i\in I} \mu_{\lambda_{S_i}(\rho(p))}(\varrho(r)) = \bigvee_{i\in I} \mu_{\gamma_{X_i}(p)}(r),$$
$$\mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r) = \bigvee_{i\in I} \mu_{\overline{\lambda_{S_i}}(\overline{\rho}(p^{\overline{\alpha}}))}(\varrho(r)) = \bigvee_{i\in I} \mu_{\overline{\gamma_{X_i}}(p^{\overline{\alpha}})}(r).$$

Thus, the proof is complete.

(x) If $\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V) = \Gamma_X^V$ and $\gamma_{\varrho,\rho}^{-1}([\Lambda_S]^c) = \Gamma_Y^V$, then $\forall p^{\underline{\alpha}} \in \underline{P}$, $\forall p \in P$, $\forall p^{\overline{\alpha}} \in \overline{P}$ and $r \in R$,

$$\begin{aligned}\mu_{\underline{\gamma_Y}(p^{\underline{\alpha}})}(r) &= \mu_{\gamma_{\underline{\rho}^{-1}(\underline{S}^c)}(p^{\underline{\alpha}})}(r) = \mu_{\gamma_{(\underline{\rho}^{-1}(\underline{S}))}(p^{\underline{\alpha}})}(r) \\ &= \mu_{\underline{\gamma_{X^c}}(p^{\underline{\alpha}})}(r), \\ \mu_{\gamma_Y(p)}(r) &= \mu_{\gamma_{\rho^{-1}(S^c)}(p)}(r) \\ &= \mu_{\gamma_{(\rho^{-1}(S))}(p)}(r) = \mu_{\gamma_{X^c}(p)}(r), \\ \mu_{\overline{\gamma_Y}(p^{\overline{\alpha}})}(r) 7 & \mu_{\gamma_{\overline{\rho}^{-1}(\overline{S}^c)}(p^{\overline{\alpha}})}(r) \\ &= \mu_{\gamma_{(\overline{\rho}^{-1}(\overline{S}))}(p^{\overline{\alpha}})}(r) \\ &= \mu_{\overline{\gamma_{X^c}}(p^{\overline{\alpha}})}(r),\end{aligned}$$

where $\underline{\rho}^{-1}(\underline{S}), \rho^{-1}(S), \overline{\rho}^{-1}(\overline{S})$ and $\underline{\rho}^{-1}(\underline{S}^c), \rho^{-1}(S^c), \overline{\rho}^{-1}(\overline{S}^c)$ are FSs in $\underline{P}$, P, $\overline{P}$; respectively, i.e., $\left[\Gamma_X^V\right]^c$ and Γ_Y^V are equal. Thus, the proof is complete.

(xi) If $\Gamma_X^V = \gamma_{\varrho,\rho}^{-1}\left(\Lambda_{\widetilde{N}}^V\right)$, then $\forall p^{\overline{\alpha}} \in \overline{P}$ and $r \in R$, $\mu_{\overline{\gamma_X}(p^{\overline{\alpha}})}(r) = \mu_{\overline{\lambda_{\widetilde{N}}}(\overline{\rho}(p^{\overline{\alpha}}))}(\varrho(r)) = 1$, i.e., $\Gamma_{\widetilde{P}}^V = \Gamma_X^V$.

(xii) Here, since $\overline{\rho}^{-1}(\overline{N})$ is empty FS, then the proof is clear.

VFP-fuzzy soft topological spaces

In this section, an introduction to topological spaces has been made using the concepts given in the previous section. Some concepts of VFP-fuzzy soft topological spaces such as VFP-fuzzy soft open, VFP-fuzzy soft closed, VFP-fuzzy soft closure, VFP-fuzzy soft Q-neighborhood, VFP-fuzzy soft interior, base, VFP-fuzzy soft continuous, cover and VFP-fuzzy soft compact have been given and some related properties have been analyzed. In addition, some examples are given to better understand the defined concepts.

Definition 4.1 A VFP-fuzzy soft topological space is a pair (R, τ) where R is a nonempty set and τ is a family of VFPFSSs over R satisfying the following properties:

(i) $\Gamma_\emptyset^V, \Gamma_{\widetilde{P}}^V \in \tau$.
(ii) If $\Gamma_X^V, \Gamma_Y^V \in \tau$, then $\Gamma_X^V \widetilde{\cap} \Gamma_Y^V \in \tau$.
(iii) If $\Gamma_{X_i}^V \in \tau; \forall i \in I$, then $\widetilde{\bigcup}_{i\in I}\Gamma_{X_i}^V \in \tau$.

Then, τ is called a VFP-fuzzy soft topology on R. Every member of τ is called VFP-fuzzy soft open in (R, τ). Γ_Y^V is called VFP-fuzzy soft closed in (R, τ) if $\left[\Gamma_Y^V\right]^c \in \tau$.

Example 4.2 The families $\tau_{\text{indiscrete}} = \{\Gamma_\emptyset^V, \Gamma_{\widetilde{P}}^V\}$ and $\tau_{\text{discrete}} = \text{VFPFSS}(R, P)$ are VFP-fuzzy soft topology on R.

Example 4.3 Let $R = \{r_1, r_2, r_3, r_4\}$ and $P = \{p_1, p_2\}$. If

$$\Gamma_{X_1}^V = \left\{\begin{array}{l}(0.2/p_1, \{0.67/r_2, 0.6/r_3, 0.55/r_4, 0.5/r_5\}),\\(0.5/p_2, \{0.4/r_1, 0.6/r_3, 0.6/r_4, 0.75/r_7\}),\\(0.45/p_1, \{0.4/r_2, 0.4/r_3, 0.3/r_4, 0.3/r_5\}),\\(0.45/p_2, \{0.2/r_1, 0.2/r_3, 0.3/r_4, 0.4/r_7\}),\\(0.6/p_1, \{0.25/r_2, 0.2/r_3, 0.15/r_4\}),\\(0.9/p_2, \{0.1/r_1, 0.25/r_4, 0.2/r_7\})\end{array}\right\},$$

$$\Gamma_{X_2}^V = \left\{\begin{array}{l}(0.3/p_1, \{0.67/r_2, 0.7/r_3, 0.6/r_4, 0.5/r_5\}),\\(0.5/p_2, \{0.5/r_1, 0.6/r_3, 0.6/r_4, 0.75/r_7\}),\\(0.45/p_1, \{0.6/r_2, 0.4/r_3, 0.5/r_4, 0.3/r_5\}),\\(0.6/p_2, \{0.3/r_1, 0.2/r_3, 0.5/r_4, 0.4/r_7\}),\\(0.7/p_1, \{0.4/r_2, 0.2/r_3, 0.3/r_4\}),\\(0.9/p_2, \{0.1/r_1, 0.3/r_4, 0.2/r_7\})\end{array}\right\},$$

$$\Gamma_{X_3}^V = \left\{\begin{array}{l}(0.2/p_1, \{0.7/r_2, 0.6/r_3, 0.55/r_4, 0.58/r_5\}),\\(0.6/p_2, \{0.4/r_1, 0.7/r_3, 0.8/r_4, 0.95/r_7\}),\\(0.5/p_1, \{0.4/r_2, 0.5/r_3, 0.3/r_4, 0.5/r_5\}),\\(0.45/p_2, \{0.2/r_1, 0.5/r_3, 0.3/r_4, 0.5/r_7\}),\\(0.6/p_1, \{0.25/r_2, 0.4/r_3, 0.15/r_4\}),\\(0.95/p_2, \{0.1/r_1, 0.25/r_4, 0.3/r_7\})\end{array}\right\},$$

$$\Gamma_{X_4}^V = \left\{\begin{array}{l}(0.3/p_1, \{0.7/r_2, 0.7/r_3, 0.6/r_4, 0.58/r_5\}),\\(0.6/p_2, \{0.5/r_1, 0.7/r_3, 0.8/r_4, 0.95/r_7\}),\\(0.5/p_1, \{0.6/r_2, 0.5/r_3, 0.5/r_4, 0.5/r_5\}),\\(0.6/p_2, \{0.3/r_1, 0.5/r_3, 0.5/r_4, 0.5/r_7\}),\\(0.7/p_1, \{0.4/r_2, 0.4/r_3, 0.3/r_4\}),\\(0.95/p_2, \{0.1/r_1, 0.3/r_4, 0.3/r_7\})\end{array}\right\}$$

then $\tau = \left\{\Gamma_\emptyset^V, \Gamma_{X_1}^V, \Gamma_{X_2}^V, \Gamma_{X_3}^V, \Gamma_{X_4}^V, \Gamma_{\widetilde{P}}^V\right\}$ is a VFP-fuzzy soft topology on R.

Theorem 4.4 *Let (R, τ) be a* VFP-*fuzzy soft topological space and $\widetilde{\tau}$ be family of all* VFP-*fuzzy soft closed sets. Then,*

(i) $\Gamma_\emptyset^V, \Gamma_{\widetilde{P}}^V \in \widetilde{\tau}$.
(ii) If $\Gamma_X^V, \Gamma_Y^V \in \widetilde{\tau}$, *then* $\Gamma_X^V \widetilde{\cup} \Gamma_Y^V \in \widetilde{\tau}$.
(iii) If $\Gamma_{X_i}^V \in \widetilde{\tau}; \forall i \in I$, *then* $\widetilde{\bigcap}_{i\in I}\Gamma_{X_i}^V \in \widetilde{\tau}$.

Proof Straighforward.

Definition 4.5 Let (R, τ) be a VFP-fuzzy soft topological space and $\Gamma_X^V \in \text{VFPFSS}(R, P)$. The VFP-fuzzy soft closure of Γ_X^V in (R, τ), denoted by cls $\left[\Gamma_X^V\right]$, is the intersection of all VFP-fuzzy soft closed supersets of Γ_X^V, i.e., cls $\left[\Gamma_X^V\right] = \widetilde{\cap}\left\{\Gamma_Y^V : \Gamma_Y^V \in \widetilde{\tau}, \Gamma_X^V \widetilde{\subseteq} \Gamma_Y^V\right\}$. Clearly, cls $\left[\Gamma_X^V\right]$ is the smallest VFP-fuzzy soft closed set over R which contains Γ_X^V.

Theorem 4.6 *Let (R, τ) be a* VFP-*fuzzy soft topological space and $\Gamma_X^V, \Gamma_Y^V \in$* VFPFSS$(R, P)$. *Then,*

(i) Γ_X^V *is a VFP-fuzzy soft closed set* $\Leftrightarrow \Gamma_X^V = cls\left[\Gamma_X^V\right]$.
(ii) cls $\left[\Gamma_X^V \widetilde{\cup} \Gamma_Y^V\right] = cls\left[\Gamma_X^V\right] \widetilde{\cup}$ cls $\left[\Gamma_Y^V\right]$.
(iii) $cls\left[\Gamma_\emptyset^V\right] = \Gamma_\emptyset^V$ *and* cls $\left[\Gamma_{\widetilde{P}}^V\right] = \Gamma_{\widetilde{P}}^V$.

(iv) $\Gamma_X^V \widetilde{\subseteq} \text{cls}\left[\Gamma_X^V\right]$.
(v) $\text{cls}\left[cls\left[\Gamma_X^V\right]\right] = cls\left[\Gamma_X^V\right]$.
(vi) $\Gamma_X^V \widetilde{\subseteq} \Gamma_Y^V \Rightarrow \text{cls}\left[\Gamma_X^V\right] \widetilde{\subseteq} \text{cls}\left[\Gamma_Y^V\right]$.

Proof The items (iii), (iv), (v) and (vi) are obvious from the Definition 4.5.

(i) Let Γ_X^V be a VFP-fuzzy soft closed set. Since $\text{cls}\left[\Gamma_X^V\right]$ is the smallest VFP-fuzzy soft closed set which contains Γ_X^V, then $\text{cls}\left[\Gamma_X^V\right] \widetilde{\subseteq} \Gamma_X^V$. The opposite is clear. Thus, $\Gamma_X^V = cls\left[\Gamma_X^V\right]$.

(ii) Since $\Gamma_X^V \widetilde{\subseteq} \Gamma_X^V \widetilde{\cup} \Gamma_Y^V$ and $\Gamma_Y^V \widetilde{\subseteq} \Gamma_X^V \widetilde{\cup} \Gamma_Y^V$ by (vi), then $\text{cls}\left[\Gamma_X^V\right] \widetilde{\subseteq} \text{cls}\left[\Gamma_X^V \widetilde{\cup} \Gamma_Y^V\right]$, $\text{cls}\left[\Gamma_Y^V\right] \widetilde{\subseteq} \text{cls}\left[\Gamma_X^V \widetilde{\cup} \Gamma_Y^V\right]$, i.e., $\text{cls}\left[\Gamma_X^V\right] \widetilde{\cup} \text{cls}\left[\Gamma_Y^V\right] \widetilde{\subseteq} \text{cls}\left[\Gamma_X^V \widetilde{\cup} \Gamma_Y^V\right]$.
Conversely, since $\Gamma_X^V = cls\left[\Gamma_X^V\right]$ and $\Gamma_Y = cls\left[\Gamma_Y^V\right]$, then $cls\left[\Gamma_X^V\right] \widetilde{\cup} \text{cls}\left[\Gamma_Y^V\right]$ is a VFP-fuzzy soft closed set. Also, since $\Gamma_X^V \widetilde{\cup} \Gamma_Y^V \widetilde{\subseteq} \text{cls}\left[\Gamma_X^V\right] \widetilde{\cup} \text{cls}\left[\Gamma_Y^V\right]$ by (vi), then

$$\text{cls}\left[\Gamma_X^V \widetilde{\cup} \Gamma_Y^V\right] \widetilde{\subseteq} \text{cls}\left[\Gamma_X^V\right] \widetilde{\cup} \text{cls}\left[\Gamma_Y^V\right].$$

Definition 4.7 Let (R, τ) be a VFP-fuzzy soft topological space. A Γ_X^V in VFPFSS(R, P) is called VFP-fuzzy soft Q-neighborhood of a VFPFSS Γ_Y^V if there exists a VFP-fuzzy soft open set Γ_Z^V in τ such that $\Gamma_Y^V q \Gamma_Z^V$ and $\Gamma_Z^V \widetilde{\subseteq} \Gamma_X^V$.

Theorem 4.8 *Let* $p_{\theta}^{r_{\ddot{\theta}}}, \Gamma_X^V \in$ VFPFSS(R, P). *Then,* $p_{\theta}^{r_{\ddot{\theta}}} \widetilde{\in} cls\left[\Gamma_X^V\right]$ *if and only if each VFP-fuzzy soft Q-neighborhood of* $p_{\theta}^{r_{\ddot{\theta}}}$ *is VFP-fuzzy soft quasi-coincident with* Γ_X^V.

Proof $(\Rightarrow)$ Assume that Γ_Z^c is VFP-fuzzy soft Q-neighborhood of $p_{\theta}^{r_{\ddot{\theta}}}$ and $\Gamma_Z \overline{q} \Gamma_X^V$. Then, there exists a VFP-fuzzy soft open set Γ_Y^V such that $p_{\theta}^{r_{\ddot{\theta}}}\ q\ \Gamma_Y^V \widetilde{\subseteq} \Gamma_Z^c$. Since $\Gamma_Z^V \overline{q} \Gamma_X^V$ by Proposition 3.8 (i), then $\Gamma_X^V \widetilde{\subseteq} \left[\Gamma_Z^V\right]^c \widetilde{\subseteq} \left[\Gamma_Y^V\right]^c$. Also, since $p_{\theta}^{r_{\ddot{\theta}}}\ q\ \Gamma_Y^V$, then $p_{\theta}^{r_{\ddot{\theta}}} \widetilde{\notin} \left[\Gamma_Y^V\right]^c$, that is, contradiction is obtained.

$(\Leftarrow)$ Assume that $p_{\theta}^{r_{\ddot{\theta}}} \widetilde{\notin} cls\left[\Gamma_X^V\right]$. Then there exists a VFP-fuzzy soft closed set Γ_Y^V which is containing Γ_X^V such that $p_{\theta}^{r_{\ddot{\theta}}} \widetilde{\notin} \Gamma_Y^V$. By Proposition 3.8 (i) and (vi), we have $p_{\theta}^{r_{\ddot{\theta}}}\ q\ \left[\Gamma_Y^V\right]^c$ and $\left[\Gamma_Y^V\right]^c$ is a VFP-fuzzy soft Q-neighborhood of $p_{\theta}^{r_{\ddot{\theta}}}$, thus $\Gamma_X^V\ \overline{q}\ \left[\Gamma_Y^V\right]^c$. That is, contradiction is obtained.

Definition 4.9 Let (R, τ) be a VFP-fuzzy soft topological space and $\Gamma_X^V \in VFPFSS(R, P)$. The VFP-fuzzy soft interior of Γ_X^V, denoted by $int\left[\Gamma_X^V\right]$, is the union of all VFP-fuzzy soft open subsets of Γ_X^V, i.e., $\text{int}\left[\Gamma_X^V\right] = \widetilde{\cup}\left\{\Gamma_Y^V : \Gamma_Y^V \in \tau, \Gamma_Y^V \widetilde{\subseteq} \Gamma_X^V\right\}$. Clearly, $\text{int}\left[\Gamma_X^V\right]$ is the largest VFP-fuzzy soft open set contained in Γ_X^V.

Theorem 4.10 *Let* (R, τ) *be a* VFP*-fuzzy soft topological space and* $\Gamma_X^V, \Gamma_Y^V \in$ VFPFSS(R, P). *Then,*

(i) Γ_X^V *is a* VFP*-fuzzy soft open set* $\Leftrightarrow \Gamma_X^V = int\left[\Gamma_X^V\right]$.
(ii) $int\left[\Gamma_X^V \widetilde{\cap} \Gamma_Y^V\right] = int\left[\Gamma_X^V\right] \widetilde{\cap}\ int\left[\Gamma_Y^V\right]$.
(iii) $int\left[\Gamma_{\emptyset}^V\right] = \Gamma_{\emptyset}^V$ *and* $int\left[\Gamma_{\widetilde{P}}^V\right] = \Gamma_{\widetilde{P}}^V$.
(iv) $int\left[\Gamma_X^V\right] \widetilde{\subseteq} \Gamma_X^V$.
(v) $int\left[int\left[\Gamma_X^V\right]\right] = int\left[\Gamma_X^V\right]$.
(vi) $\Gamma_X^V \widetilde{\subseteq} \Gamma_Y^V \Rightarrow int\left[\Gamma_X^V\right] \widetilde{\subseteq} int\left[\Gamma_Y^V\right]$.

Proof It can be proved similar to Theorem 4.6.

Theorem 4.11 *Let* (R, τ) *be a VFP-fuzzy soft topological space and* $\Gamma_X^V \in VFPFSS(R, P)$. *Then,*

(i) $\left[\text{int}\left[\Gamma_X^V\right]\right]^c = \text{cls}\left[\left[\Gamma_X^V\right]^c\right]$.
(ii) $\left[\text{cls}\left[\Gamma_X^V\right]\right]^c = \text{int}\left[\left[\Gamma_X^V\right]^c\right]$.

Proof (i)

$$\begin{aligned}\left[int\left[\Gamma_X^V\right]\right]^c &= \left[\widetilde{\cup}\left\{\Gamma_Y^V : \Gamma_Y^V \in \tau, \Gamma_Y^V \widetilde{\subseteq} \Gamma_X^V\right\}\right]^c \\ &= \widetilde{\cap}\left\{\left[\Gamma_Y^V\right]^c : \Gamma_Y^V \in \tau, \Gamma_Y^V \widetilde{\subseteq} \Gamma_X^V\right\} \\ &= \widetilde{\cap}\left\{\left[\Gamma_Y^V\right]^c : \left[\Gamma_Y^V\right]^c \in \widetilde{\tau}, \left[\Gamma_X^V\right]^c \widetilde{\subseteq} \left[\Gamma_Y^V\right]^c\right\} \\ &= cls\left[\left[\Gamma_X^V\right]^c\right]\end{aligned}$$

(ii) It is obvious from (i).

Definition 4.12 Let (R, τ) be a VFP-fuzzy soft topological space. A subcollection $\mathscr{B}$ of τ is called a base for τ if every member of τ can be expressed as a union of members of $\mathscr{B}$.

Example 4.13 Consider Example 4.3. Then, the family $\mathscr{B} = \left\{\Gamma_{\emptyset}^V, \Gamma_{X_1}^V, \Gamma_{X_2}^V, \Gamma_{X_3}^V, \Gamma_{\widetilde{P}}^V\right\}$ is a basis for τ.

Proposition 4.14 *Let* (R, τ) *be a VFP-fuzzy soft topological space and* $\mathscr{B}$ *is subfamily of* τ. $\mathscr{B}$ *is a base for* τ *if and only if for each* VFP*-fuzzy soft open Q-neighborhood* Γ_X^V *of* $p_{\theta}^{r_{\ddot{\theta}}}$, *there exists a* $\Gamma_Y^V \in \mathscr{B}$ *such that* $p_{\theta}^{r_{\ddot{\theta}}}\ q\ \Gamma_Y^V \widetilde{\subseteq} \Gamma_X^V$*;* $\forall p_{\theta}^{r_{\ddot{\theta}}} \in$ VFPFSS(R, P).

Proof $(\Rightarrow)$ There exists a subfamily $\widetilde{\mathscr{B}}$ of $\mathscr{B}$ such that $\Gamma_X^V = \widetilde{\bigcup}\left\{\Gamma_Y^V : \Gamma_Y^V \in \widetilde{\mathscr{B}}\right\}$. If $p_{\theta}^{r_{\ddot{\theta}}} \overline{q} \Gamma_Y^V$; $\forall \Gamma_Y^V \in \widetilde{\mathscr{B}}$, then $\overline{\theta}^1 + \mu_{\overline{Y}}(p^{\overline{\beta}}) \leq 1$ and $\overline{\theta}^2 + \mu_{\overline{\gamma Y}(p^{\overline{\beta}})}(r) \leq 1$; $\forall \Gamma_Y^V \in \widetilde{\mathscr{B}}$. That is, contradiction with

$$\mu_{\overline{X}}(p^{\overline{\alpha}}) = \sup\{\mu_{\overline{Y}}(p^{\overline{\beta}}) : \Gamma_Y^V \in \widetilde{\mathscr{B}}\}$$

and

$$\mu_{\overline{\gamma X}(p^{\overline{\alpha}})}(r) = \sup\left\{\mu_{\overline{\gamma Y}(p^{\overline{\beta}})}(r) : \Gamma_Y^V \in \widetilde{\mathscr{B}}\right\}$$

is obtained.

$(\Leftarrow)$ If $\mathscr{B}$ is not a base for τ, then $\Gamma_Z = \widetilde{\bigcup}\left\{\Gamma_Y^V \in \mathscr{B} : \Gamma_Y^V \widetilde{\subseteq} \Gamma_X^V\right\} \neq \Gamma_X^V$; $\exists \Gamma_X^V \in \tau$. Here, since $\Gamma_Z^V \neq$

Γ_X^V, then "$\mu_{\underline{Z}}(p^{\underline{\delta}}) < \mu_{\underline{X}}(p^{\underline{\alpha}}), \mu_Z(p) < \mu_X(p), \mu_{\overline{Z}}(p^{\overline{\delta}}) < \mu_{\overline{X}}(p^{\overline{\alpha}})$" or "$\mu_{\underline{\gamma Z}(p^{\underline{\delta}})}(r) < \mu_{\underline{\gamma X}(p^{\underline{\alpha}})}(r), \mu_{\gamma Z(p)}(r) < \mu_{\gamma X(p)}(r), \mu_{\overline{\gamma Z}(p^{\overline{\delta}})}(r) < \mu_{\overline{\gamma X}(p^{\overline{\alpha}})}(r)$"; $\exists p^{\underline{\alpha}}, p^{\underline{\delta}} \in \underline{P}$, $\exists p \in P$, $\exists p^{\overline{\alpha}}, p^{\overline{\delta}} \in \overline{P}$ and $\exists r \in R$. In this case, since $\overline{\theta}^1 = 1 - \mu_{\overline{Z}}(p^{\overline{\delta}})$ or $\overline{\theta}^2 = 1 - \mu_{\overline{\gamma Z}(p^{\overline{\delta}})}(r)$, then $p_{\dot{\theta}}^{r_{\ddot{\theta}}}\ q\ \Gamma_X^V$ and $p_{\dot{\theta}}^{r_{\ddot{\theta}}}\ \overline{q}\ \Gamma_Z^V$. Thus, $p_{\dot{\theta}}^{r_{\ddot{\theta}}}\ \overline{q}\ \Gamma_Y^V$; $\forall \Gamma_Y^V \in \mathscr{B}$ which contained in Γ_X^V, that is, contradiction is obtained.

Definition 4.15 Let (R, τ_1) and (M, τ_2) be two VFP-fuzzy soft topological spaces. A VFP-fuzzy soft mapping $\gamma_{\varrho,\rho} : (R, \tau_1) \to (M, \tau_2)$ is called VFP-fuzzy soft continuous if $\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V) \in \tau_1; \forall \Lambda_S^V \in \tau_2$.

Example 4.16 Let $R = \{r_1, r_2, r_3, r_4\}$, $M = \{m_1, m_2, m_3, m_4\}$, $\underline{P} = \{p_1^{\underline{\alpha_1}}, p_2^{\underline{\alpha_2}}\}$, $P = \{p_1, p_2\}$, $\overline{P} = \{p_1^{\overline{\alpha_1}}, p_2^{\overline{\alpha_2}}\}$, $\underline{N} = \{n_1^{\underline{\beta_1}}, n_2^{\underline{\beta_2}}\}$, $N = \{n_1, n_2\}$, $\overline{N} = \{n_1^{\overline{\beta_1}}, n_2^{\overline{\beta_2}}\}$ and $\tau_1 = \left\{\Gamma_\emptyset^V, \Gamma_X^V, \Gamma_{\widetilde{P}}^V\right\}$, $\tau_2 = \left\{\Lambda_\emptyset^V, \Lambda_S^V, \Lambda_{\widetilde{N}}^V\right\}$ be VFP-fuzzy soft topologies on R and M respectively, where

$$\Gamma_X^V = \left\{\begin{array}{l}\big(0.18/p_1, \{0.72/r_1, 0.65/r_3, 0.75/r_4\}\big),\\ \big(0.34/p_2, \{0.42/r_2, 0.5/r_3, 0.6/r_4\}\big),\\ \big(0.5/p_1, \{0.4/r_1, 0.5/r_3, 0.43/r_4\}\big),\\ \big(0.57/p_2, \{0.2/r_2, 0.4/r_3, 0.3/r_4\}\big),\\ \big(0.7/p_1, \{0.25/r_1, 0.3/r_3, 0.2/r_4\}\big),\\ \big(0.85/p_2, \{0.1/r_2, 0.25/r_3, 0.15/r_4\}\big)\end{array}\right\}$$

and

$$\Lambda_S^V = \left\{\begin{array}{l}\big(0.25/n_1, \{0.8/m_1, 0.6/m_3, 0.7/m_4\}\big),\\ \big(0.4/n_2, \{0.67/m_2, 0.55/m_3, 0.45/m_4\}\big),\\ \big(0.55/n_1, \{0.65/m_1, 0.5/m_3, 0.6/m_4\}\big),\\ \big(0.7/n_2, \{0.5/m_2, 0.35/m_3, 0.2/m_4\}\big),\\ \big(0.8/n_1, \{0.4/m_1, 0.45/m_3, 0.2/m_4\}\big),\\ \big(0.9/n_2, \{0.3/m_2, 0.15/m_3, 0.1/m_4\}\big)\end{array}\right\}.$$

Define $\underline{\rho} : \underline{P} \to \underline{N}$, $\rho : P \to N$, $\overline{\rho} : \overline{P} \to \overline{N}$ and $\varrho : R \to M$ as $\underline{\rho}\left(p_1^{\underline{\alpha_1}}\right) = n_2^{\underline{\beta_2}}$, $\underline{\rho}\left(p_2^{\underline{\alpha_2}}\right) = n_1^{\underline{\beta_1}}$, $\rho(p_1) = n_2$, $\rho(p_2) = n_1$, $\overline{\rho}\left(p_1^{\overline{\alpha_1}}\right) = n_2^{\overline{\beta_2}}$, $\overline{\rho}\left(p_2^{\overline{\alpha_2}}\right) = n_1^{\overline{\beta_1}}$ and $\varrho(r_1) = m_2$, $\varrho(r_2) = m_3$, $\varrho(r_3) = m_4$, $\varrho(r_4) = m_1$. Then the VFP-fuzzy soft mapping $\gamma_{\varrho,\rho} : (R, \tau_1) \to (M, \tau_2)$ is VFP-fuzzy soft continuous.

Note that the VFP-fuzzy soft constant mapping $\gamma_{\varrho,\rho} : (R, \tau_1) \to (M, \tau_2)$ is not VFP-fuzzy soft continuous in general.

Example 4.17 Let $R = \{r_1, r_2, r_3, r_4\}$, $M = \{m_1, m_2, m_3, m_4\}$, $\underline{P} = \{p_1^{\underline{\alpha_1}}, p_2^{\underline{\alpha_2}}\}$, $P = \{p_1, p_2\}$, $\overline{P} = \{p_1^{\overline{\alpha_1}}, p_2^{\overline{\alpha_2}}\}$, $\underline{N} = \{n_1^{\underline{\beta_1}}, n_2^{\underline{\beta_2}}\}$, $N = \{n_1, n_2\}$, $\overline{N} = \{n_1^{\overline{\beta_1}}, n_2^{\overline{\beta_2}}\}$ and $\tau_1 = \left\{\Gamma_\emptyset^V, \Gamma_{\widetilde{P}}^V\right\}$, $\tau_2 = \left\{\Lambda_\emptyset^V, \Lambda_S^V, \Lambda_{\widetilde{N}}^V\right\}$ be VFP-fuzzy soft topologies on R and M respectively, where

$$\Lambda_S^V = \left\{\begin{array}{l}\big(0.25/n_1, \{0.8/m_1, 0.6/m_3, 0.7/m_4\}\big),\\ \big(0.4/n_2, \{0.67/m_2, 0.55/m_3, 0.45/m_4\}\big),\\ \big(0.55/n_1, \{0.65/m_1, 0.5/m_3, 0.6/m_4\}\big),\\ \big(0.7/n_2, \{0.5/m_2, 0.35/m_3, 0.2/m_4\}\big),\\ \big(0.8/n_1, \{0.4/m_1, 0.45/m_3, 0.2/m_4\}\big),\\ \big(0.9/n_2, \{0.3/m_2, 0.15/m_3, 0.1/m_4\}\big)\end{array}\right\}.$$

Define $\underline{\rho} : \underline{P} \to \underline{N}$, $\rho : P \to N$, $\overline{\rho} : \overline{P} \to \overline{N}$ and $\varrho : R \to M$ as $\underline{\rho}\left(p_1^{\underline{\alpha_1}}\right) = \underline{\rho}\left(p_2^{\underline{\alpha_2}}\right) = n_1^{\underline{\beta_1}}$, $\rho(p_1) = \rho(p_2) = n_1$, $\overline{\rho}\left(p_1^{\overline{\alpha_1}}\right) = \overline{\rho}\left(p_2^{\overline{\alpha_2}}\right) = n_1^{\overline{\beta_1}}$ and $\varrho(r_1) = \varrho(r_2) = \varrho(r_3) = \varrho(r_4) = m_1$. Then, the VFP-fuzzy soft mapping $\gamma_{\varrho,\rho} : (R, \tau_1) \to (M, \tau_2)$ is a VFP-fuzzy soft constant mapping and is not VFP-fuzzy soft continuous.

Note that, a constant FS on P taking value $\theta \in [0, 1]$ will be denoted by θ_P.

Definition 4.18 Let $\Gamma_X^V \in VFPFSS(R, P)$. Γ_X^V is called $\dot{\theta}, \ddot{\theta} - X-$universal VFPFSS if $\mu_{\underline{X}}(p^{\underline{\alpha}}) = \underline{\theta}^1$, $\mu_X(p) = \theta^1$, $\mu_{\overline{X}}(p^{\overline{\alpha}}) = \overline{\theta}^1$ and $\mu_{\underline{\gamma X}(p^{\underline{\alpha}})}(r) = \underline{\theta}^2$, $\mu_{\gamma X(p)}(r) = \theta^2$, $\mu_{\overline{\gamma X}(p^{\overline{\alpha}})}(r) = \overline{\theta}^2$; $\forall p^{\underline{\alpha}} \in \underline{P}$, $\forall p \in P$, $\forall p^{\overline{\alpha}} \in \overline{P}$ and $\dot{\theta} = \left(\underline{\theta}^1, \theta^1, \overline{\theta}^1\right)$, $\ddot{\theta} = \left(\underline{\theta}^2, \theta^2, \overline{\theta}^2\right)$; denoted by $\left[\Gamma_X^V\right]_{\widetilde{\dot{\theta}, \ddot{\theta}}}$.

Definition 4.19 A VFP-fuzzy soft topology is called enriched if $\left[\Gamma_X^V\right]_{\widetilde{\dot{\theta}, \ddot{\theta}}} \in \tau$; $\underline{\theta}^1, \theta^1, \overline{\theta}^1, \underline{\theta}^2, \theta^2, \overline{\theta}^2 \in (0, 1]$ and $\dot{\theta} = \left(\underline{\theta}^1, \theta^1, \overline{\theta}^1\right)$, $\ddot{\theta} = \left(\underline{\theta}^2, \theta^2, \overline{\theta}^2\right)$.

Theorem 4.20 *If (M, τ_2) be a VFP-fuzzy soft topological space, (R, τ_1) be a enriched* VFP-*fuzzy soft topological space and $\gamma_{\varrho,\rho}$:* VFPFSS$(R, P) \to$ VFPFSS(M, N) *be a constant* VFP-*fuzzy soft mapping, then $\gamma_{\varrho,\rho}$ is VFP-fuzzy soft continuous.*

Proof Let $\underline{\rho} : \underline{P} \to \underline{N}$, $\rho : P \to N$, $\overline{\rho} : \overline{P} \to \overline{N}$ and $\varrho : R \to M$ be constant mapping defined as $\underline{\rho}(p^{\underline{\alpha}}) = n_0^{\underline{\beta}}$, $\rho(p) = n_0$, $\overline{\rho}(p^{\overline{\alpha}}) = n_0^{\overline{\beta}}$ and $\Lambda_S^V \in \tau_2$, $\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V) = \Gamma_X^V$. Then, $\underline{X} = \underline{\rho}^{-1}(\underline{S}) = \underline{\theta}_P^1$, $X = \rho^{-1}(S) = \theta_P^1$, $\overline{X} = \overline{\rho}^{-1}(\overline{S}) = \overline{\theta}_P^1$ where $\mu_{\underline{S}}(n^{\underline{\beta}}) = \underline{\theta}^1$, $\mu_S(n) = \theta^1$, $\mu_{\overline{S}}(n^{\overline{\beta}}) = \overline{\theta}^1$ and $\mu_{\underline{\gamma X}(p^{\underline{\alpha}})}(r) = \mu_{\underline{\lambda S}(\underline{\rho}(p^{\underline{\beta}}))}(\varrho(r)) = \mu_{\underline{\lambda S}(n_0^{\underline{\beta}})}(m_0) = \underline{\theta}^2$, $\mu_{\gamma X(p)}(r) = \mu_{\lambda S(\rho(p))}(\varrho(r)) = \mu_{\lambda S(n_0)}(m_0) = \theta^2$, $\mu_{\overline{\gamma X}(p^{\overline{\alpha}})}(r) = \mu_{\overline{\lambda S}(\overline{\rho}(p^{\overline{\beta}}))}(\varrho(r)) = \mu_{\overline{\lambda S}(n_0^{\overline{\beta}})}(m_0) = \overline{\theta}^2$; $\forall n^{\underline{\beta}}, p^{\underline{\alpha}} \in \underline{P}$, $\forall p \in P$, $\forall n^{\overline{\beta}}, p^{\overline{\alpha}} \in \overline{P}$. Thus, since $\Gamma_X^V = \left[\Gamma_P^V\right]_{\widetilde{\dot{\theta}, \ddot{\theta}}} \in \tau_1$, then $\gamma_{\varrho,\rho} : (R, P) \to (M, N)$ is VFP-fuzzy soft continuous.

Theorem 4.21 *Let (R, τ_1) and (M, τ_2) be two* VFP-*fuzzy soft topological spaces and $\gamma_{\varrho,\rho} : (R, P) \to (M, N)$ be a* VFP-*fuzzy soft mapping. Then, the following are equivalent:*

(i) $\gamma_{\varrho,\rho}\left(\text{cls}\left[\Gamma_X^V\right]\right)\tilde{\subseteq}\ \text{cls}\left[\gamma_{\varrho,\rho}\left(\Gamma_X^V\right)\right];\ \forall\Gamma_X^V \in VFPFSS(R,P)$,

Example 4.25 Let $\underline{P}=\{p_1^{\underline{\alpha_1}},p_2^{\underline{\alpha_2}},...\}$, $P=\{p_1,p_2,...\}$, $\overline{P}=\{p_1^{\overline{\alpha_1}},p_2^{\overline{\alpha_2}},...\}$ and $R=\{r_1,r_2,...\}$. If

$$\Gamma_{X_k}^V=\left\{\begin{array}{l}\left((1/4)/p_1^{\underline{\alpha_1}},\{1/r_1\}\right),\left((1/8)/p_2^{\underline{\alpha_2}},\{1/r_1,(1/2)/r_2\}\right),\\ ...,\left((1/4k)/p_k^{\underline{\alpha_k}},\{1/r_1,(1/2)/r_2,...,(1/k)/r_k\}\right),\\ \left((1/2)/p_1,\{(1/2)/r_1\}\right),\left((1/4)/p_2,\{(1/2)/r_1,(1/4)/r_2\}\right),\\ ...,\left((1/2k)/p_k,\{(1/2)/r_1,(1/4)/r_2,...,(1/2k)/r_k\}\right),\\ \left(1/p_1^{\overline{\alpha_1}},\{(1/4)/r_1\}\right),\left((1/2)/p_2^{\overline{\alpha_2}},\{(1/4)/r_1,(1/8)/r_2\}\right),\\ ...,\left((1/k)/p_k^{\overline{\alpha_k}},\{(1/4)/r_1,(1/8)/r_2,...,(1/4k)/r_k\}\right)\end{array}:k=1,2,...\right\},$$

(ii) $\text{cls}\left[\gamma_{\varrho,\rho}^{-1}\left(\lambda_S^V\right)\right]\tilde{\subseteq}\gamma_{\varrho,\rho}^{-1}\left(\text{cls}\left[\Lambda_S^V\right]\right);\ \forall\Lambda_S^V \in VFPFSS(M,N)$,

(iii) $\gamma_{\varrho,\rho}^{-1}\left(int\left[\Lambda_S^V\right]\right)\tilde{\subseteq}\ int\left[\gamma_{\varrho,\rho}^{-1}\left(\Lambda_S^V\right)\right];\ \forall\Lambda_S^V \in VFPFSS(M,N)$,

(iv) $\gamma_{\varrho,\rho}$ *is VFP-fuzzy soft continuous,*

(v) $\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V)$ *is VFP-fuzzy soft closed;* $\forall\Lambda_S^V\in\tilde{\tau_2}$.

Proof (i) $\Rightarrow$ (ii) If $\Gamma_X^V=\gamma_{\varrho,\rho}^{-1}\left(\Lambda_S^V\right)$, then $\gamma_{\varrho,\rho}\left(cls\left[\gamma_{\varrho,\rho}^{-1}\left(\Lambda_S^V\right)\right]\right)\tilde{\subseteq}\ \text{cls}\left[\gamma_{\varrho,\rho}\left(\gamma_{\varrho,\rho}^{-1}\left(\Lambda_S^V\right)\right)\right]\tilde{\subseteq}\ \text{cls}\left[\Lambda_S^V\right]$. Thus by Theorem 3.11 (i), $\text{cls}\left[\gamma_{\varrho,\rho}^{-1}\left(\Lambda_S^V\right)\right]\tilde{\subseteq}\ \gamma_{\varrho,\rho}^{-1}\left(\gamma_{\varrho,\rho}\left(cls\left[\gamma_{\varrho,\rho}^{-1}\left(\Lambda_S^V\right)\right]\right)\right)\tilde{\subseteq}\ \gamma_{\varrho,\rho}^{-1}\left(\text{cls}\left[\Lambda_S^V\right]\right)$.

(ii) $\Leftrightarrow$ (iii) It is obvious from Theorem 3.11 (x) and Theorem 4.11.

(iii) $\Rightarrow$ (iv) Since $\Lambda_S^V\in\tau_2$, then $\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V)\gamma_{\varrho,\rho}^{-1}\left(int\left[\Lambda_S^V\right]\right)\tilde{\subseteq}\gamma_{\varrho,\rho}^{-1}(\lambda_S^V)$, i.e., $\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V)$ is a VFP-fuzzy soft open and so $\gamma_{\varrho,\rho}$ is VFP-fuzzy soft continuous.

(iv) $\Rightarrow$ (v) It is obvious from Theorem 3.11 (x).

(v) $\Rightarrow$ (i) Since $\Gamma_X^V\tilde{\subseteq}\gamma_{\varrho,\rho}^{-1}\ (\gamma_{\varrho,\rho}(\Gamma_X^V))$, i.e, $\Gamma_X^V\tilde{\subseteq}\gamma_{\varrho,\rho}^{-1}\left(cls\left[\gamma_{\varrho,\rho}(\Gamma_X^V)\right]\right)\in\tilde{\tau_1}$, then

$$cls\left[\Gamma_X^V\right]\tilde{\subseteq}\gamma_{\varrho,\rho}^{-1}\ \left(cls\left[\gamma_{\varrho,\rho}(\Gamma_X^V)\right]\right).$$

By Theorem 3.11 (ii), we have $\gamma_{\varrho,\rho}\left(cls\left[\Gamma_X^V\right]\right)\tilde{\subseteq}\gamma_{\varrho,\rho}\left(\gamma_{\varrho,\rho}^{-1}\ \left(cls\left[\gamma_{\varrho,\rho}(\Gamma_X^V)\right]\right)\right)\tilde{\subseteq}\ cls\left[\gamma_{\varrho,\rho}(\Gamma_X^V)\right]$.

Theorem 4.22 *Let* $\gamma_{\varrho,\rho}:(R,P)\to(M,N)$ *be a VFP-fuzzy soft mapping and* $\mathscr{B}$ *be a base for* τ_2. *Then* $\gamma_{\varrho,\rho}$ *is VFP-fuzzy soft continuous* $\Leftrightarrow\gamma_{\varrho,\rho}^{-1}(\Lambda_S^V)\in\tau_1;\ \forall\Lambda_S^V\in\mathscr{B}$.

Proof Straightforward.

Definition 4.23 A family $\mathscr{S}$ of VFPFSSs is a cover of a VFPFSS Γ_X^V if $\Gamma_X^V\tilde{\subseteq}\tilde{\bigcup}\left\{\Gamma_{X_i}^V:\Gamma_{X_i}^V\in\mathscr{S},i\in I\right\}$. It is a VFP-fuzzy soft open cover if each member of $\mathscr{S}$ is a VFP-fuzzy soft open set. A subcover of $\mathscr{S}$ is a subfamily of $\mathscr{S}$ which is also a cover.

Definition 4.24 A VFP-fuzzy soft topological space (R,τ) is VFP-fuzzy soft compact if each VFP-fuzzy soft open cover of $\Gamma_{\widetilde{P}}^V$ has a finite subcover.

then $\tau=\left\{\Gamma_{X_k}^V:k=1,2,...\right\}\cup\left\{\Gamma_\emptyset^V,\Gamma_{\widetilde{P}}^V\right\}$ a VFP-fuzzy soft topology on R and (R,τ) is VFP-fuzzy soft compact.

Definition 4.26 A family $\mathscr{S}$ of VFPFSSs has the finite intersection property if the intersection of the members of each finite subfamily of $\mathscr{S}$ is not empty VFPFSS.

Theorem 4.27 *A VFP-fuzzy soft topological space is VFP-fuzzy soft compact if and only if each family of VFP-fuzzy soft closed sets with the finite intersection property has a non-empty VFP-fuzzy soft intersection.*

Proof If $\mathscr{S}$ is a family of VFPFSSs in a VFP-fuzzy soft topological space (R,τ), then $\mathscr{S}$ is a cover of $\Gamma_{\widetilde{P}}^V$ if and only if one of the following conditions holds:

(i) $\tilde{\bigcup}\left\{\Gamma_{X_i}^V:\Gamma_{X_i}^V\in\mathscr{S},i\in I\right\}=\Gamma_{\widetilde{P}}^V$.

(ii) $\left[\tilde{\bigcup}\left\{\Gamma_{X_i}^V:\Gamma_{X_i}^V\in\mathscr{S},i\in I\right\}\right]^c=\left[\Gamma_{\widetilde{P}}^V\right]^c=\Gamma_\emptyset^c$.

(iii) $\tilde{\bigcap}\left\{\left[\Gamma_{X_i}^V\right]^c:\Gamma_{X_i}^V\in\mathscr{S},i\in I\right\}=\Gamma_\emptyset^V$.

Hence, this shows that VFP-fuzzy soft topological space is VFP-fuzzy soft compact.

Theorem 4.28 *Let* (R,τ_1) *and* (M,τ_2) *be VFP-fuzzy soft topological spaces and* $\gamma_{\varrho,\rho}:VFPFSS(R,P)\to VFPFSS(M,N)$ *be a VFP-fuzzy soft mapping. If* (R,τ_1) *is VFP-fuzzy soft compact and* $\gamma_{\varrho,\rho}$ *is VFP-fuzzy soft continuous surjection, then* (M,τ_2) *is VFP-fuzzy soft compact.*

Proof If $\mathscr{S}=\left\{\Lambda_{S_i}^V:\Lambda_{S_i}^V\in\tau_2,i\in I\right\}$ is a cover of $\Lambda_{\widetilde{N}}^V$, then $\left\{\gamma_{\varrho,\rho}^{-1}\left(\Lambda_{S_i}^V\right):\Lambda_{S_i}^V\in\mathscr{S}\right\}$ is a cover of $\Gamma_{\widetilde{P}}^V$ by VFP-fuzzy soft continuous $\gamma_{\varrho,\rho}$. Since (R,τ_1) is VFP-fuzzy soft compact, then $\left\{\gamma_{\varrho,\rho}^{-1}\left(\Lambda_{S_i}^V\right):i\in I_0\right\}$ covers $\Gamma_{\widetilde{P}}^V$; $\exists I_0\in I$. Moreover, we have $\gamma_{\varrho,\rho}\left(\tilde{\bigcup}\left\{\gamma_{\varrho,\rho}^{-1}\left(\Lambda_{S_i}^V\right):i\in I_0\right\}\right)=\gamma_{\varrho,\rho}\left(\Gamma_{\widetilde{P}}^V\right)$ and so $\tilde{\bigcup}\left\{\Lambda_{S_i}^V:i\in I_0\right\}=\Lambda_{\widetilde{N}}^V$. Hence, (M,τ_2) is VFP-fuzzy soft compact.

Conclusion

To express many relationships on mathematical models, various topological structures have been built on many different set types that have been introduced to the literature. In general, more general topological structures can be obtained thanks to the generalization of set types, and thus the relationships on mathematical models are expressed better. The virtual fuzzy parametrized fuzzy soft set theory, which is one of the most important hybrid set types put forward in recent years, enables a decision maker to express membership degrees more accurately, and thanks to this feature, it has managed to attract the attention of many researchers. Therefore, the purpose of this paper is to generalize the concept of topology in expressing the relationships in a better way by establishing a topological structure on virtual fuzzy parametrized fuzzy soft sets. For this, first, some concepts such as point, quasi-coincident and mapping were defined for the virtual fuzzy parametrized fuzzy soft set. Then, with the help of these auxiliary concepts, virtual fuzzy parametrized fuzzy soft topological spaces are defined and analyzed in detail. Moreover, for virtual fuzzy parametrized fuzzy soft topological spaces, concepts such as open, closed, closure, Q-neighborhood, interior, base, continuous, cover and compact are defined and some related properties are given. In addition, many examples have been added to make the concepts given throughout the paper easier to understand.

In an environment of uncertainty, it is very important for the decision maker to be able to express the membership degrees in the most accurate way. The topological structure built on virtual fuzzy parametrized fuzzy soft sets, which is one of the mathematical approaches put forward for this, can also be re-evaluated for mathematical models such as virtual fuzzy parametrized soft set [14], virtual neutrosophic parametrized soft set [15]. We hope that the many concepts and many characteristic properties given in this paper will be useful for researchers to further advance and promote in virtual fuzzy parametrized fuzzy soft set theory.

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