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ON TOPOLOGIES OF FREE GROUPS

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All spaces are assumed to be Tychonoff.

Let X be a topological space, $F(X)$ the free algebraic group over a set X. Then $F_M(X)$, the free topological group over X in the sense of A. A. Markov, is the set $F(X)$ equipped with the topology having the following properties:

1) X is a subspace of $F_M(X)$;

2) each continuous mapping from X to an arbitrary topological group G extends to a continuous homomorphism from $F_M(X)$ to G.

Indeed, this is a very nice short characterization of the topology of $F_M(X)$ using category terms. Unfortunately, this characterization says nothing about the constructive form of open sets in $F_M(X)$. Consequently, one cannot answer many questions on the topological properties of the group $F_M(X)$. So we need an intrinsic description of the topology of free Markov group over X. This will be done in the first part of the paper.

In the second part, we define a new group topology ϱ on $F(X)$, resulting to a topological group $F_\varrho(X)$, and investigate its properties. The group $F_\varrho(X)$ is rather similar to $F_M(X)$, and may be characterized categorically replacing 2) above by

2') each continuous mapping f from X to an arbitrary topological group G such that the image $f[X]$ is thin in G, extends to a continuous homomorphism from $F_\varrho(X)$ to G.

We shall show among others that for a pseudocompact space X, $F_M(X) = F_\varrho(X)$, and use this result to estimate the Souslin number of $F_M(X)$. Further, we shall prove that the group $F_\varrho(X)$ is Weil-complete iff X is Dieudonné-complete.

1. THE TOPOLOGY OF THE GROUP $F_M(X)$

Let N^+ be the set of all positive integers. Let $^{-1}$ be some homeomorphism of the space X onto its copy X^{-1}, denote $\tilde{X}$ the topological sum $X \oplus X^{-1}$. For every $n \in N^+$ let $i_n: \tilde{X}^n \to F_n(X)$ be the natural map of $\tilde{X}^n$ onto the set $F_n(X)$ consisting of all words in the alphabet $\tilde{X}$ of length $\leqq n$. Next, let $j_n\ \tilde{X}^{2n} \to F_{2n}(X)$ be defined by the rule $j_n(x, y) = i_n(x) \cdot (i_n(y))^{-1}$ for every $x, y \in \tilde{X}^n$.

For each $n \in N^+$, denote by $\mathscr{U}_n$ the universal (i.e. the finest inducing the same topology) uniformity on the topological space $\tilde{X}^n$. Let $\mathscr{R}$ be the family of all sequences $E = \{U_n : n \in N^+\}$ such that $U_n \in \mathscr{U}_n$ whenever $n \in N^+$.

If $E \in \mathscr{R}$, $E = \{U_n : n \in N^+\}$ and if $n \in N^+$, let us define $V_n(E) = \bigcup \{j_{\pi(1)}[U_{\pi(1)}] \cdot \ldots$ $\ldots \cdot j_{\pi(n)}[U_{\pi(n)}] : \pi \in S_n\}$, where S_n is the permutation group of the set $\{1, \ldots, n\}$.

Finally, put $V(E) = \bigcup_{n \in N^+} V_n(E)$.

Theorem 1.1. *The family $\Sigma^* = \{V(E) : E \in \mathscr{R}\}$ is a neighborhood base of the unity in the group $F_M(X)$.*

Our proof of this theorem heavily depends on the following lemmas.

Lemma 1.1. *Let $m, n \in N^+$, $U \in \mathscr{U}_{n+m}$. Then there exists a $V_U \in \mathscr{U}_n$ such that $j_n(V_U) \subseteq j_{n+m}(U)$.*

Lemma 1.2. *Let $m, n \in N^+$, $g \in F_m(X)$ and $U \in \mathscr{U}_{n+m}$. Then there exists a $V_U \in \mathscr{U}_n$ such that $g \cdot j_n(V_U) \cdot g^{-1} \subseteq j_{n+m}(U)$.*

It is easily seen that Lemma 1.1 is a partial case of Lemma 1.2. As for the proof, since $g \in F_m(X)$, $g = i_m(x)$ for some $x \in \tilde{X}^m$. Define $V_U = \{(v, w) \in \tilde{X}^n \times \tilde{X}^n : ((v, x), (w, x)) \in U\}$. We shall omit the straightforward verification that this works.

Lemma 1.3. *Let H be a group with unity e and let $\{V_n : n \in N\}$ be a sequence of subsets of H such that $e \in V_n$ and $V_{n+1}^3 \subseteq V_n$ for each $n \in N$. Let $k \in N$, $p, k_1, \ldots, k_p \in N^+$ be such that $\sum_{i=1}^{p} 2^{-k_i} < 2^{-k}$. Then $V_{k_1} \cdot \ldots \cdot V_{k_p} \subseteq V_k$.*

P r o o f of Theorem 1.1. The proof will be divided into two sections. First we shall show that Σ^* generates some group topology $\mathscr{T}^*$ on $F(X)$ whose trace on X coincides with the original topology of X. Second we prove that $\mathscr{T}^*$ is finer than the topology of $F_M(X)$. This implies that $\mathscr{T}^*$ is indeed the the Markov topology, because the Markov topology is the finest group topology for $F(X)$ extending the topology of X.

I. The system Σ^* has the following properties:

(a) $\{e\} = \bigcap \Sigma^*$;
(b) for every pair $U, V \in \Sigma^*$ there is some $W \in \Sigma^*$ such that $W \subseteq U \cap V$;
(c) for every $U \in \Sigma^*$ there is some $V \in \Sigma^*$ with $V \cdot V^{-1} \subseteq U$;
(d) for every $U \in \Sigma^*$ and $g \in U$ there is some $V \in \Sigma^*$ with $V \cdot g \subseteq U$;
(e) for every $U \in \Sigma^*$ and $g \in F(X)$ there is some $V \in \Sigma^*$ with $g^{-1} \cdot V \cdot g \subseteq U$.

We shall show (c), (d) and (e), since (a) and (b) are trivial.

Let $U \in \Sigma^*$, $U = V(E)$, $E = \{U_n : n \in N^+\}$. We may and shall assume that each U_n is symmetrical. Using Lemma 1.1, choose symmetrical $W_n \in \mathscr{U}_n$ satisfying $j_n(W_n) \subseteq$

$\subseteq j_{2n+1}(U_{2n+1}) \cap j_{2n}(U_{2n})$ for each $n \in N^+$; denote $\tilde{E} = \{W_n: n \in N^+\}$. Since each W_n is symmetrical, $V(\tilde{E})^{-1} = V(\tilde{E})$, moreover, $V(\tilde{E}) . V(\tilde{E}) \subseteq V(E)$. Indeed, let $\pi, \varrho \in S_n$, let $\sigma \in S_{2n}$ be defined by $\sigma(i) = 2\pi(i)$ for $i \leqq n$, $\sigma(i) = 2\varrho(i - n) - 1$ for $n + 1 \leqq \leqq i \leqq 2n$. Then $j_{\pi(1)}(W_{\pi(1)}) . \ldots . j_{\pi(n)}(W_{\pi(n)}) . j_{\varrho(1)}(W_{\varrho(1)}) . \ldots . j_{\varrho(n)}(W_{\varrho(n)}) \subseteq \subseteq j_{\sigma(1)}(U_{\sigma(1)}) . \ldots . j_{\sigma(2n)}(U_{\sigma(2n)}) \subseteq V_{2n}(E)$. Now since π and ϱ were arbitrary, we have $V_n(\tilde{E}) . V_n(\tilde{E}) \subseteq V_{2n}(E)$, therefore $V(\tilde{E}) . V(\tilde{E})^{-1} \subseteq V(E)$, which shows (c).

To show (d), let $E = \{U_n: n \in N^+\} \in \mathscr{R}$, $g \in V(E)$. Then for some $k \in N^+$ and $\pi \in S_k$, $g \in j_{\pi(1)}(U_{\pi(1)}) . \ldots . j_{\pi(k)}(U_{\pi(k)})$. With help of Lemma 1.1 find a sequence $\tilde{E} = \{V_n: n \in N^+\}$ with $j_n(V_n) \subseteq j_{n+k}(U_{n+k})$. Then $V(\tilde{E}) . g \subseteq V(E)$: Let $m \in N^+$, $\sigma \in S_m$. Define $\varrho \in S_{m+k}$ by $\varrho(i) = \sigma(i) + k$ for $i \leqq m$, $\varrho(i) = \pi(i - m)$ for $m < i \leqq \leqq m + k$. We have: $j_{\sigma(1)}(V_{\sigma(1)}) . \ldots . j_{\sigma(m)}(V_{\sigma(m)}) . g \subseteq j_{\sigma(1)}(V_{\sigma(1)}) . \ldots . j_{\sigma(m)}(V_{\sigma(m)}) . . j_{\pi(1)}(U_{\pi(1)}) . \ldots . j_{\pi(k)}(U_{\pi(k)}) \subseteq j_{\varrho(1)}(U_{\varrho(1)}) . \ldots . j_{\varrho(m+k)}(U_{\varrho(m+k)}) \subseteq V(E)$.

The proof of (e) is similar. Let $E = \{U_n: n \in N^+\} \in \mathscr{R}$, $g \in V(E)$. Then $g \in F_k(X)$ for some $k \in N^+$. For each $n \in N^+$, there is some symmetrical $V_n \in \mathscr{U}_n$ with $g^{-1} . . j_n(V_n) . g \subseteq j_{n+k}(U_{n+k})$ by Lemma 1.2. Let $\tilde{E} = \{V_n: n \in N^+\}$. Then $g^{-1} . V(\tilde{E}) . g \subseteq \subseteq V(E)$: If $n \in N^+$ and $\sigma \in S_n$, then $g^{-1} . j_{\sigma(1)}(V_{\sigma(1)}) . \ldots . j_{\sigma(n)}(V_{\sigma(n)}) . g \subseteq g^{-1} . . j_{\sigma(1)}(V_{\sigma(1)}) . g . g^{-1} . j_{\sigma(2)}(V_{\sigma(2)}) . g . \ldots . g^{-1} . j_{\sigma(n)}(V_{\sigma(n)}) . g \subseteq j_{\sigma(1)+k}(U_{\sigma(1)+k}) . \ldots . \ldots . j_{\sigma(n)+k}(U_{\sigma(n)+k}) \subseteq j_{\sigma(1)+k}(U_{\sigma(1)+k}) . \ldots . j_{\sigma(n)+k}(U_{\sigma(n)+k}) . j_1(U_1) . \ldots . j_k(U_k) \subseteq V(E)$.

It is well-known that the validity of (a)–(e) is equivalent to the existence of an admissible group topology $\mathscr{T}^*$ for $F(X)$.

Let $V \in \mathscr{T}^*$, $O = X \cap V$, $x \in O$. Then there is some $E \in \mathscr{R}$ and $g \in F(X)$ with $x \in V(E) . g \subseteq V$. Hence for some $\tilde{E} = \{U_n: n \in N^+\}$, we have $V(\tilde{E}) . x \subseteq V(E) . g$. Put $W(x) = \{y \in X: (y, x) \in U_1\}$. Then $x \in W(x) \subseteq X$ and $W(x)$ is a neighbourhood of x in X. Moreover, $W(x) \subseteq j_1(U_1) . x \subseteq V(\tilde{E}) . x \subseteq V(E) . g$. Thus $X \cap V$ is open in X, hence the restriction of $\mathscr{T}^*$ to X coincides with the original topology of X.

II. $\mathscr{T}^*$ is finer than the topology of $F_M(X)$. Let V be an open neighborhood of unity in $F_M(X)$, put $V_0 = V$ and let $\{V_n: n \in N^+\}$ be a sequence of neighborhoods of unity in $F_M(X)$ satisfying $V_n^3 \subseteq V_{n-1}$ for each $n \in N^+$.

Put $U_n = \{(x, y): x, y \in \tilde{X}^n$ and $i_n(x) . (i_n(y))^{-1} \subseteq V_n\}$. Since every i_n is continuous, we have $U_n \in \mathscr{U}_n$. Now clearly the sequence $E = \{U_n: n \in N^+\}$ belongs to $\mathscr{R}$ and $V_n(E) \subseteq V_0 = V$ for each $n \in N^+$ by Lemma 1.3. Thus $V(E) \subseteq V$, too.

Theorem 1.2. *Let X be a closed subspace of a space Y and suppose that Y^n is paracompact for each $n \in N^+$. Then $F_M(X)$ embeds into $F_M(Y)$ as a closed topological subgroup.*

Sketch of the proof. We have to show that the natural isomorphism φ between $F_M(X)$ and the subgroup $A(X) \subseteq F_M(Y)$ algebraically generated by X, is a homeomorphism. The continuity of φ is obvious. In order to show the continuity of φ^{-1} we need the following: For each $E \in \mathscr{R}_X$ there is an $\tilde{E} \in \mathscr{R}_Y$ with $V(\tilde{E}) \cap A(X) \subseteq \subseteq \varphi(V(E))$, the meaning of $\mathscr{R}_X$ and $\mathscr{R}_Y$ is clear. The last statement can be proved using the forthcoming two well-known assertions.

Assertion 1.1. *Let Z be paracompact. Then the family of all open neighborhoods of the diagonal in Z^2 is a base for the universal uniformity $\mathscr{U}_Z$ of Z.*

Assertion 1.2. *Let T be a closed subspace of a collectionwise normal (in particular, paracompact) space Z. Then $(T, \mathscr{U}_T)$ is a uniform subspace of $(Z, \mathscr{U}_Z)$.*

2. THE NEW GROUP TOPOLOGY ϱ ON $F(X)$ AND ITS RELATION TO MARKOV FREE TOPOLOGY

In this part, we equip the free algebraic group $F(X)$ over a topological space X by a group topology ϱ, which is coarser than the free Markov topology, but still similar to. The resulting space will be denoted by $F_\varrho(X)$.

Let d be an arbitrary continuous pseudometric on X, let G be the set of all elements of $F(X)$ which have even length. For $g \in G$, we shall define a real number $\|g\|_d$ as follows. $\|e\|_d = 0$ for the unity e of $F(X)$. $\|x \cdot y^{-1}\|_d = \|x^{-1} \cdot y\|_d = d(x, y)$ and $\|x \cdot y\|_d = \|x^{-1} \cdot y^{-1}\|_d = 1$ for each $x, y \in X$. Thus we have defined $\|g\|_d$ for every $g \in G$ of length 2. Let $n \in N^+$ and suppose $\|g\|_d$ has been defined for each $g \in G$ of length $\leqq 2n$. For every $g \in G$, $g = x_1 \cdot \ldots \cdot x_{2n+2}$ with $x_1, \ldots, x_{2n+2} \in \tilde{X} = X \oplus \oplus X^{-1}$, let $\|g\|_d$ be the minimum of the numbers $\|x_1 \cdot \ldots \cdot x_{2i}\|_d + \|x_{2i+1} \cdot \ldots \cdot x_{2n+2}\|_d$, $1 \leqq i \leqq n$, and $\|x_1 \cdot x_{2n+2}\|_d + 2\|x_2 \cdot \ldots \cdot x_{2n+1}\|_d$.

Let $\mathscr{D}$ be the set of all continuous pseudometrics on X. For every $d \in \mathscr{D}$ let $V_d = \{g \in G: \|g\|_d < 1\}$. Clearly, $\|\cdot\|_d$ is a pseudonorm on G for each $d \in \mathscr{D}$, i.e. the following is valid:

1) $\|e\|_d = 0$,
2) $\|g^{-1}\|_d = \|g\|_d$ for each $g \in G$,
3) $\|g \cdot h\|_d \leqq \|g\|_d + \|h\|_d$ for each $g, h \in G$.

Theorem 2.1. *The family $\{V_d: d \in \mathscr{D}\}$ is a neighborhood base of unity in some group topology ϱ on $F(X)$. The topology ϱ induces the original topology on X. Moreover, the set $F_n(X)$ is closed in $F_\varrho(X)$ for every $n \in N^+$ and*

(*) for each V with $e \in V \in \varrho$ there is some W with $e \in W \in \varrho$ such that $x \cdot W \cdot x^{-1} \subseteq V$ for each $x \in X$.

The proof of this theorem is very similar to the part I of the proof of Theorem 1.1. Therefore we shall verify only that (*) holds. Fix a continuous pseudometric d on X such that $e \in V_d \subseteq V$. Then by the definition of $\|\cdot\|_d$, $\|x \cdot g \cdot x^{-1}\|_d \leqq 2\|g\|_d$ for each $g \in G$ and $x \in X$. Hence it suffices to put $\tilde{d} = 2d$ in order to obtain $x \cdot V_{\tilde{d}} \cdot x^{-1} \subseteq V_d$.

Recall that a topological group H has a quasi-invariant basis iff for every open neighborhoood V of the unity there is a countable family γ consisting of open neighborhoods of unity such that for each $g \in H$ there is some $W \in \gamma$ with $g \cdot W \cdot g^{-1} \subseteq V$.

By (*), the group $F_\varrho(X)$ has a quasi-invariant basis. Thus $F_\varrho(X)$ is embeddable as a subgroup into some product of metrizable groups [2]. So we have the following

Proposition 2.1. *For each space X the group $F_\varrho(X)$ is topologically isomorphic to a subgroup of a product of metrizable groups.*

Now let Y be a subspace of X and let $\mathscr{U}_Y$, $\mathscr{U}_X$ be the universal uniformities on Y and X respectively.

Proposition 2.2. *Let Y be a subspace of X. Then $(Y, \mathscr{U}_Y)$ is a uniform subspace of $(X, \mathscr{U}_X)$ if and only if $F_\varrho(Y)$ is naturally embeddable into $F_\varrho(X)$.*

Proof. The natural monomorphism $\varphi: F_\varrho(Y) \to F_\varrho(X)$ is continuous. To prove the continuity of φ^{-1}, fix any continuous pseudometric d on Y and let V_d be the corresponding open neighborhood of unity, $V_d \subseteq F_\varrho(Y)$. We may suppose that $d(x, y) \leqq 1$ for each $x, y \in Y$. Since $(Y, \mathscr{U}_Y)$ is a uniform subspace of $(X, \mathscr{U}_X)$, there is a continuous pseudometric $\tilde{d}$ on X extending d. One can easily check that $\varphi^{-1}[V_{\tilde{d}} \cap \varphi[F_\varrho(Y)]] = V_d$, which shows that φ^{-1} is continuous.

Now suppose $F_\varrho(Y)$ naturally embeds into $F_\varrho(X)$ Let d be an arbitrary continuous pseudometric on Y, $U_d \in \mathscr{U}_Y$, $U_d = \{(x, y) \in Y^2: d(x, y) < 1\}$. Then $V_d = \{g: \|g\|_d < 1\}$ is a neighborhood of unity in $F_\varrho(Y)$. Since $F_\varrho(Y) \subseteq F_\varrho(X)$, there is some continuous pseudometric $\tilde{d}$ on X with $V_{\tilde{d}} \cap F_\varrho(Y) \subseteq V_d$. Now the set $U_{\tilde{d}} = \{(x, y) \in X^2: \tilde{d}(x, y) < 1\}$ belongs to $\mathscr{U}_X$ and $U_{\tilde{d}} \cap Y^2 \subseteq U_d$. Thus $(Y, \mathscr{U}_Y)$ is a uniform subspace of $(X, \mathscr{U}_X)$.

Corollary 2.1. *Let Y be a closed subspace of a paracompact space X. Then $F_\varrho(Y)$ is naturally embeddable into $F_\varrho(X)$ as a closed topological subgroup.*

The importance of the next definition will be exhibited in Theorems 2.2, 2.3.

Definition 2.1. A subset X of a topological group H with the unity e is called *thin in H* provided that for each V open in H with $e \in V$ there is an open $W \subseteq H$ such that $e \in W$ and $x \,.\, W \,.\, x^{-1} \subseteq V$ for every $x \in X$.

Let us note that by $(*)$ in Theorem 2.1, X is thin in $F_\varrho(X)$.

Theorem 2.2. *Let X be a topological space and let $\mathscr{T}$ be any group topology on $F(X)$ which extends the topology of X. If X is thin in $F_{\mathscr{T}}(X)$, then the topology ϱ is finer than $\mathscr{T}$.*

Before giving the proof of this main result, we need the following

Definition 2.2. An element $g \in G$ is *decomposable with respect to a pseudometric d on X* provided that there exist elements $g_1, \ldots, g_n \in G$ such that

1) $g = g_1 \cdot \ldots \cdot g_n$,

2) $\|g\|_d = \sum_{i=1}^{n} \|g_i\|_d$,

3) $l(g_i) < l(g)$ for each $i = 1, \ldots, n$,

where $l(g)$ denote the length of an element $g \in F(X)$.

Proof of Theorem 2.2. Let $V \in \mathcal{T}$ be an arbitrary neighborhood of e. We need to find a continuous pseudometric $\tilde{d}$ on X such that $V_{\tilde{d}} \subseteq V$.

Let $V_0 = V$. There exists a sequence $\xi = \{V_n: n \in N^+\}$ such that for each $n \in N^+$, $e \in V_n$, $V_n^{-1} = V_n$, $V_n^3 \subseteq V_{n-1}$ (for $\mathcal{T}$ is a group topology on $F(X)$) and for each $x \in \tilde{X}$, $x . V_n . x^{-1} \subseteq V_{n-1}$ (for $\tilde{X}$ as well as X is thin in $F_{\mathcal{T}}(X)$). For each $n \in N^+$, put $U_n = \{(x, y) \in X^2: x^{-1} . y \in V_{2n}\}$. Then U_n is an open entourage of the diagonal in $X \times X$ and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$. Thus there is a continuous pseudometric d on X such that $\{(x, y) \in X^2: d(x, y) < 2^{-n}\} \subseteq U_n$. Put $\tilde{d} = 4d$. We claim that $V_{\tilde{d}} \subseteq V$.

It suffices to show the following: If $g \in G$, $n \in N^+$ and $\|g\|_d < 2^{-n}$, then $g \in V_{2n-2}$. Induction on the length of g:

If $g = x . y^{-1}$, then $d(x, y) < 2^{-n}$, hence $x . y^{-1} \in V_{2n}$.

Suppose $g = x_1 . \dots . x_{2p}$ with $p > 1$.

If g is not decomposable, then for some $g_1 \in G$ and $x, y \in X$, $g = x . g_1 . y$ and $\|g\|_d = \|x . y\|_d + 2\|g_1\|_d$. Since $\|g\|_d < 2^{-n}$ and $n \in N^+$, we have $\|x . y\|_d < 1$, thus either $g = x . g_1 . z^{-1}$ where $x, z \in X$ or $g = t^{-1} . g_1 . y$ where $t, y \in X$. Assume the first possibility. Clearly $\|g_1\|_d < 2^{-n-1}$ (otherwise $\|g\|_d \geqq 2^{-n}$), thus by the inductive assumption, $g_1 \in V_{2n}$. Further, $x . z^{-1} \in V_{2n}$ for $\|x . z^{-1}\|_d < 2^{-n}$. Thus $x . g_1 . x^{-1} \in V_{2n-1}$ and $x . g_1 . z^{-1} = x . g_1 . x^{-1} . x . z^{-1} \in V_{2n-1} . V_{2n} \subseteq V_{2n-2}$.

If g is decomposable, then $g = g_1 . \dots . g_k$ for some $g_1, \dots, g_k \in G$ with $l(g_j) < l(g)$ and $\|g\|_d = \|g_1\|_d + \dots + \|g_k\|_d < 2^{-n}$. We may and shall assume that no g_j is decomposable. Pick integers i_j ($j = 1, \dots, k$) in such a way that $2^{-i_j-1} \leqq \|g_j\|_d < 2^{-i_j}$. Then by the inductive assumption, each g_j belongs to V_{2i_j-2} and $\sum_{j=1}^{k} 2^{-i_j} < 2^{-n+1}$.

Assume first $i_j \geqq n + 1$ for all $j = 1, \dots, k$. Then $\sum_{j=1}^{k} 2^{-2i_j+2} \leqq \sum_{j=1}^{k} 2^{-i_j-n+1} < 2^{-2n+2}$. According to Lemma 1.3, $g \in V_{2n-2}$.

If $i_m = n$ for some $m \in N^+$, $1 \leqq m \leqq k$, then $\|g_1 . \dots . g_{m-1}\|_d < 2^{-n-1}$, $\|g_{m+1} . \dots . g_k\|_d < 2^{-n-1}$. By the inductive assumption, $g_1 . \dots . g_{m-1} \in V_{2n}$, $g_{m+1} . \dots . g_k \in V_{2n}$ and $g_m \in V_{2n-1} . V_{2n}$ since g_m is not decomposable. Thus $g \in V_{2n} . V_{2n-1} . V_{2n} . V_{2n} \subseteq V_{2n} . V_{2n-1} . V_{2n-1} \subseteq V_{2n-2}$.

The next result is closely related to Theorem 2.2.

Theorem 2.3. *Let $f: X \to H$ be a continuous mapping from a topological space X to a topological group H. Suppose $f[X]$ to be thin in H. Then there exists a continuous homomorphism $\bar{f}: F_\varrho(X) \to H$ which extends f.*

Proof. Let $\bar{f}$ be the algebraical extension of f with the domain $F(X)$. Set $\mathcal{B} = \{U \cap \bar{f}^{-1}[V]: U$ is open in $F_\varrho(X)$, V open in $H\}$. Then $\mathcal{B}$ is a base for some group topology $\mathcal{T}$ on $F(X)$, which is obviously finer than ϱ and which coincides with ϱ on X. Since X is thin in $F_\varrho(X)$ and $f[X]$ is thin in H, X is thin in $F_{\mathscr{T}}(X)$, too. By Theorem 2.2, $\mathscr{T} = \varrho$, which was to be proved.

Corollary 2.1. *Let X, Y be topological spaces, $f: X \to Y$ continuous. Then there is a continuous homomorphism ϱf: $F_\varrho(X) \to F_\varrho(Y)$ extending f.*

This corollary may be proved independently of Theorem 2.3. Indeed, let ϱf be an algebraic extension of f. Let V_d be an open neighborhood of unity in $F_\varrho(Y)$ corresponding to some continuous pseudometric d on Y. If we define for $x, y \in X$, $\tilde{d}(x, y) = d(f(x), f(y))$, then $\varrho f(V_{\tilde{d}}) \subseteq V_d$, as can be easily checked. Thus ϱf is continuous.

Theorem 2.4. *The group $F_\varrho(X)$ is Weil-complete if and only if the space X is Dieudonné-complete.*

Sketch of the proof. The "only if" part follows by the fact that the left uniformity $\mathscr{U}_l$ of the group $F_\varrho(X)$ induces the universal uniformity $\mathscr{U}_X$ of X and X is closed in $F_\varrho(X)$.

The "if" part is more difficult. First we note that the closed subspace $F_1(X) = X \cup X^{-1} \subseteq F_\varrho(X)$ is homeomorphic to the topological sum $X \oplus X^{-1}$, hence the space $F_1(X)$ is Dieudonné-complete. Since the universal uniformity $\mathscr{U}_{F_1(X)}$ of the space $F_1(X)$ is induced by the left uniformity $\mathscr{U}_l$ of $F_\varrho(X)$, the space $(F_1(X), \mathscr{U}_{F_1(X)})$ is a complete subspace of the uniform space $(F_\varrho(X), \mathscr{U}_l)$. Further, proceeding by induction, we obtain that $(F_n(X), \mathscr{V}_n)$ is complete for each $n \in N^+$; here $\mathscr{V}_n$ denotes the uniformity on $F_n(X)$ induced by $\mathscr{U}_l$.

This fact together with the property $(*)$ implies the completeness of the uniform space $(F_\varrho(X), \mathscr{U}_l)$, i.e. Weil-completeness of $F_\varrho(X)$. For showing this we need a modified Graev's construction ([5], Theorem 6).

The next result follows from Theorem 2.4 using the continuity of the maps $i_n: \tilde{X}^n \to F_n(X)$.

Corollary 2.2. *Let X be a Dieudonné-complete space. Then the uniform space $(F_n(X), \mathscr{W}_n)$ is complete for each $n \in N^+$.* (Here $\mathscr{W}_n$ is the uniformity on $F_n(X)$ induced by the left uniformity of the group $F_M(X)$.)

Problem 2.1. Suppose the space X to be Dieudonné-complete. Is then the group $F_M(X)$ Weil-complete? Raikov-complete?

Let us denote $\text{Top}_{3\frac{1}{2}}$ the category of all Tychonoff spaces and their continuous mappings. Let Hom be the category of all topological groups and their continuous homomorphisms. We define the functor ϱ: $\text{Top}_{3\frac{1}{2}} \to$ Hom by $\varrho(X) = F_\varrho(X)$ for objects, $\varrho(f) = \varrho f$ for morphisms. It is easily seen that ϱ is already a functor.

Let μ: $\text{Top}_{3\frac{1}{2}} \to \text{Top}_{3\frac{1}{2}}$ be the functor of Dieudonné-completion, and R: Hom $\to$ Hom the functor of Raikov-completion (as for the definition, see [4], 8.5.8, 8.5.13 and [6]).

Theorem 2.5. *The functors $\varrho \circ \mu$ and $R \circ \varrho$ are naturally equivalent.*

Proof. First we show the equality $\varrho \circ \mu = R \circ \varrho$ for objects.

Let X be a topological space. The group $RF_\varrho(X)$ is complete in its two-sided uniformity $\mathscr{V}$. The group $F_\varrho(\mu X)$ is Weil-complete because the space μX is Dieudonné-complete; hence the group $F_\varrho(\mu X)$ is complete in its two-sided uniformity $\mathscr{U}$. Let i be the embedding of X into μX and let φ be its extension to a topological monomorphism, $\varphi: F_\varrho(X) \to F_\varrho(\mu X)$ (see Prop. 2.2). Let $\mathscr{W}_1$ be the two-sided uniformity of the group $F_\varrho(X)$. Since $(F_\varrho(X), \mathscr{W}_1)$ is a uniform subspace of $(RF_\varrho(X), \mathscr{V})$ and since $F_\varrho(X)$ is dense in $RF_\varrho(X)$, the uniform continuity of φ and the completeness of $(F_\varrho(\mu X), \mathscr{U})$ imply that φ can be extended to a continuous mapping $\psi_1: RF_\varrho(X) \to \to F_\varrho(\mu X)$.

Let $\mathscr{W}_2$ be the uniformity of the group $\varphi(F_\varrho(X)) \subseteq F_\varrho(\mu X)$ induced by $\mathscr{U}$. Then the map $\varphi^{-1}: (\varphi(F_\varrho(X)), \mathscr{W}_2) \to (R\, F_\varrho(X), \mathscr{V})$ is uniformly continuous. The completeness of the space $(RF_\varrho(X), \mathscr{V})$ implies that there exists a continuous extension $\psi_2: F_\varrho(\mu X) \to RF_\varrho(X)$ of φ^{-1}. Obviously $\psi_1 \circ \psi_2$ maps $\varphi(F_\varrho(X))$ identically onto itself and $\psi_2 \circ \psi_1$ is an identity on $F_\varrho(X)$. But $F_\varrho(X)$ is dense in $RF_\varrho(X)$ and $\varphi(F_\varrho(X))$ is dense in $F_\varrho(\mu X)$; hence $\psi_1 \circ \psi_2$ is an identity mapping from $F_\varrho(\mu X)$ onto itself, $\psi_2 \circ \psi_1$ is an identity mapping from $RF_\varrho(X)$ onto itself. Therefore ψ_1 is a topological isomorphism between $RF_\varrho(X)$ and $F_\varrho(\mu X)$.

Now let $f: X \to Y$ be a continuous mapping. Then the equality $R(\varrho f) = \varrho(\mu f)$ follows by the equality $RF_\varrho(X) = F_\varrho(\mu X)$ which has just been proved and by the fact that $\varrho(\mu f)$ and $R(\varrho f)$ agree on a dense subset $F_\varrho(X)$.

Let X be a subspace of a topological group H and let $\mathscr{V}$ be the uniformity on X induced by the right uniformity $\mathscr{U}_r$ of the group H.

Lemma 2.1. *If the uniform space $(X, \mathscr{V})$ is totally bounded then X is thin in H.*

The routine proof may be left to the reader.

It is well-known that every pseudocompact space X is totally bounded in its universal uniformity $\mathscr{U}_X$. Thus Theorem 2.2 together with Lemma 2.1 give us immediately

Theorem 2.6. $F_\varrho(X) = F_M(X)$ *for every pseudocompact space X.*

The remaining part of the paper gives bounds for the Souslin number of topological groups. We shall start with $F_M(X)$ for a pseudocompact space X.

Theorem 2.7. *Let X be a pseudocompact space. Then the Souslin number of the group $F_M(X)$ is countable.*

For the proof, we shall need one combinatorial fact.

Lemma 2.2. *Let X be a set, $m, n \in N^+$ and let $\{(x_{i,1}, \ldots, x_{i,m}, \gamma_i): i \in I\}$ be an infinite family of ordered $(m + 1)$-tuples such that $x_{i,1}, \ldots, x_{i,m} \in X$ and γ_i is a cover of X with $|\gamma_i| \leqq n$ for each $i \in I$.*

Then there exists an infinite $J \subseteq I$ such that for each $i, j \in J$, if $i \neq j$, then $\mathrm{St}(x_{i,k}, \gamma_j) \cap \mathrm{St}(x_{j,k}, \gamma_i) \neq \emptyset$ *for* $k = 1, \ldots, m$.

The lemma follows easily by m-tuple application of Ramsey theorem $\omega \to (\omega)^2_n$.

Proof of Theorem 2.7. Let $\{O_\alpha: \alpha < \omega_1\}$ be a family of open non-empty subsets of $F_M(X)$. We have to find distinct $\alpha, \beta < \omega_1$ with $O_\alpha \cap O_\beta \neq \emptyset$.

For each $\alpha < \omega_1$ choose a point $g_\alpha \in O_\alpha$. By Theorem 2.6, there is a continuous pseudometric d_α on X such that $g_\alpha \,.\, V_{d_\alpha} \subseteq O_\alpha$ and $V_{d_\alpha} \,.\, g_\alpha \subseteq O_\alpha$. Since each g_α is of finite length, we may assume $l(g_\alpha) = m$ for all $\alpha < \omega_1$. Each element g_α can be written in the form $g_\alpha = x_{\alpha,1}^{\varepsilon_{\alpha,1}} \,.\, x_{\alpha,2}^{\varepsilon_{\alpha,2}} \,.\, \ldots \,.\, x_{\alpha,m}^{\varepsilon_{\alpha,m}}$ with $x_{\alpha,1}, \ldots, x_{\alpha,m} \in X$ and $\varepsilon_{\alpha,1}, \ldots$ $\ldots, \varepsilon_{\alpha,m} \in \{-1, +1\}$. Again we may and shall assume that for one particular m-tuple $(\varepsilon_1, \ldots, \varepsilon_m)$ and for all $\alpha < \omega_1$, $g_\alpha = x_{\alpha,1}^{\varepsilon_1} \ldots x_{\alpha,m}^{\varepsilon_m}$.

Put $\varrho_\alpha = 2^{m+1} \,.\, d_\alpha$ for $\alpha < \omega_1$. Since X is pseudocompact, for every $\alpha < \omega_1$ there is a finite subset $K_\alpha \subseteq X$ such that $X \subseteq \bigcup\{x \,.\, V_{\varrho_\alpha}: x \in K_\alpha\}$. Denote $\gamma_\alpha = \{X \cap x \,.\, V_{\varrho_\alpha}: x \in K_\alpha\}$. Then γ_α is a finite cover of X and we may and shall for the last time assume that for some $n \in N^+$, $|\gamma_\alpha| = n$ for all $\alpha < \omega_1$. Consider the family $\{(x_{\alpha,1}, \ldots, x_{\alpha,m}, \gamma_\alpha): \alpha < \omega_1\}$. By Lemma 2.2 there are distinct α, β such that $\mathrm{St}(x_{\alpha,k}, \gamma_\beta) \cap \mathrm{St}(x_{\beta,k}, \gamma_\alpha) \neq \emptyset$ for $k = 1, 2, \ldots, m$. We claim that $O_\alpha \cap O_\beta$ is non-void for these particular α, β.

It suffices to show that $g_\alpha \,.\, V_{d_\alpha} \cap V_{d_\beta} \,.\, g_\beta \neq \emptyset$. To this end, pick $a_k \in \mathrm{St}(x_{\alpha,k}, \gamma_\beta) \cap \mathrm{St}(x_{\beta,k}, \gamma_\alpha)$ and let $a = a_m^{-\varepsilon_m} \,.\, \ldots \,.\, a_1^{-\varepsilon_1}$. Then $g_\alpha \,.\, a \,.\, g_\beta \in g_\alpha \,.\, V_{d_\alpha} \cap V_{d_\beta} \,.\, g_\beta$, i.e. $\|a \,.\, g_\beta\|_{d_\alpha} < 1$ and $\|g_\alpha \,.\, a\|_{d_\beta} < 1$. We shall show the first inequality only. By the definition of $\|\cdot\|_d$, $\|a \,.\, g_\beta\|_{d_\alpha} \leqq \sum_{k=1}^{m} 2^{m-k} \,.\, d_\alpha(a_k, x_{\beta,k})$. Since $a_k \in \mathrm{St}(x_{\beta,k}, \gamma_\alpha)$ for every $k \leqq m$, there is a point $x_k \in K_\alpha$ such that $\{a_k, x_{\beta,k}\} \subseteq x_k \,.\, V_{\varrho_\alpha}$. Therefore $\varrho_\alpha(x_k, a_k) < 1$ as well as $\varrho_\alpha(x_k, x_{\beta k}) < 1$, consequently $\varrho_\alpha(a_k, x_{\beta,k}) < 2$. Thus $d_\alpha(a_k, x_{\beta,k}) = 2^{-m-1} \,.\, \varrho_\alpha(a_k, x_{\beta,k}) < 2^{-m}$. Since the last inequality holds for all $k \leqq m$, we have $\|a \,.\, g_\beta\|_{d_\alpha} < 2^{-m} \,.\, \sum_{k=1}^{m} 2^{m-k} < 1$.

Remark. A stronger version of Lemma 2.2 can be used to improve Theorem 2.7 as follows:

If τ is a regular uncountable cardinal and γ is a family of open non-void subsets of $F_M(X)$, $|\gamma| \geqq \tau$, then there exists a subfamily $\mu \subseteq \gamma$ of cardinality τ such that $U \cap V \neq \emptyset$ for each $U, V \in \mu$ (X is assumed to be pseudocompact).

Problem 2.2. Can one choose a subfamily $\mu \subseteq \gamma$ of cardinality τ to be centered?

Corollary 2.3. *If a topological group H is algebraically generated by its pseudocompact subspace, then $c(H) \leqq \aleph_0$.*

Proof. Denote by X the pseudocompact subspace, let $\varphi: F_M(X) \to H$ continuously extend the identity $i: X \to X$. Apply Theorem 2.7.

The following definition is due to I. Guran.

Definition 2.3. Let τ be a cardinal number. A topological group H is *τ-bounded* if for every open neighborhood V of the unity there exists a subset $K \subseteq H$ such that $|K| \leqq \tau$ and $H = K \,.\, V$.

It is known [7] that any group with a dense Lindelöf subspace must be $\aleph_0$-bounded as well as any group H with $c(H) \leqq \aleph_0$.

Theorem 2.8. *Let τ be an infinite cardinal, let H be a τ-bounded group. Then $c(H) \leqq 2^\tau$.*

We shall need the following lemma, which may be compared with Lemma 2.2.

Lemma 2.3. *Let H be a set, τ an infinite cardinal, $\lambda = (2^\tau)^+$ and let $\{(x_\alpha, \gamma_\alpha): \alpha<\lambda\}$ be a family of pairs such that $x_\alpha \in H$, γ_α is a cover of H, $|\gamma_\alpha| \leqq \tau$ for each $\alpha < \lambda$. Then there exist distinct $\alpha, \beta < \lambda$ such that* $\mathrm{St}(x_\alpha, \gamma_\beta) \cap \mathrm{St}(x_\beta, \gamma_\alpha) \neq \emptyset$.

Proof. Enumerate $\gamma_\alpha = \{A_{\alpha,k}: k < \tau\}$ for $\alpha < \lambda$. For $\{\alpha, \beta\} \in [\lambda]^2$ with $\alpha < \beta$ choose a pair $(k, m) \in \tau \times \tau$ with $x_\alpha \in A_{\beta,k}$ and $x_\beta \in A_{\alpha,m}$; this defines a mapping $\varphi: [\lambda]^2 \to \tau \times \tau$. By Erdös-Radó theorem $(2^\tau)^+ \to (\tau^+)^2_\tau$ there is a pair $(k, m) \in \in \tau \times \tau$ and $I \subseteq \lambda$ with $|I| = \tau^+$ such that $\varphi(\{\alpha, \beta\}) = (k, m)$ for each $\alpha < \beta$, $\alpha, \beta \in I$.

Let $\alpha < \delta < \beta$, $\alpha, \delta, \beta \in I$. Then $x_\alpha \in A_{\beta,k}$ and $x_\beta \in A_{\alpha,m}$ for $\varphi(\{\alpha, \beta\}) = (k, m)$, similarly $x_\alpha \in A_{\delta,k}$ and $x_\delta \in A_{\alpha,m}$, $x_\delta \in A_{\beta,k}$ and $x_\beta \in A_{\delta,m}$.

Thus $\mathrm{St}(x_\alpha, \gamma_\beta) \cap \mathrm{St}(x_\beta, \gamma_\alpha)$ is non-void, because $x_\delta \in A_{\beta,k} \cap A_{\alpha,m} \subseteq \mathrm{St}(x_\alpha, \gamma_\beta) \cap \cap \mathrm{St}(x_\beta, \gamma_\alpha)$.

Proof of Theorem 2.8. Let $\{O_\alpha: \alpha < \lambda\}$ be an arbitrary family of open non-empty subsets of H, $\lambda = (2^\tau)^+$. For each $\alpha < \lambda$ choose a point $x_\alpha \in O_\alpha$ and an open neighborhood U_α of the unity such that $x_\alpha \,.\, U_\alpha \subseteq O_\alpha$ and $U_\alpha \,.\, x_\alpha \subseteq O_\alpha$. Let V_α be an open neighborhood of unity satisfying $V_\alpha^2 \subseteq U_\alpha$, $V_\alpha^{-1} = V_\alpha$. By τ-boundedness of H, there is a subset $K_\alpha \subseteq H$ such that $|K_\alpha| \leqq \tau$ and $H = K_\alpha \,.\, V_\alpha$. Set $L_\alpha = K_\alpha \cup K_\alpha^{-1}$. Then $L_\alpha \,.\, V_\alpha = H = V_\alpha \,.\, L_\alpha$. Let $\gamma_\alpha = \{x \,.\, V_\alpha: x \in L_\alpha\} \cup \{V_\alpha \,.\, x: x \in L_\alpha\}$. Clearly $|\gamma_\alpha| \leqq \tau$, hence Lemma 2.3 applies: There are $\alpha < \beta < \lambda$ with $P = \mathrm{St}(x_\alpha, \gamma_\beta) \cap \cap \mathrm{St}(x_\beta, \gamma_\alpha) \neq \emptyset$. Pick a point $x \in P$. Then for some $a, b \in H$, $\{x_\alpha, x\} \subseteq a \,.\, V_\beta$ and $\{x_\beta, x\} \subseteq V_\alpha \,.\, b$; hence $a = x_\alpha \,.\, u_\beta$ and $b = u_\alpha \,.\, x_\beta$ for some $u_\beta \in V_\beta$ and $u_\alpha \in V_\alpha$. Consequently $x \in x_\alpha \,.\, u_\beta \,.\, V_\beta \cap V_\alpha \,.\, u_\alpha \,.\, x_\beta$, thence $x_\alpha \,.\, V_\beta^2 \cap V_\alpha^2 \,.\, x_\beta \neq \emptyset$. Since the last is equivalent to $V_\alpha^2 \,.\, x_\alpha \cap x_\beta \,.\, V_\beta^2 \neq \emptyset$ and $V_\alpha^2 \,.\, x_\alpha \subseteq O_\alpha$, $x_\beta \,.\, V_\beta^2 \subseteq O_\beta$, we have $O_\alpha \cap O_\beta \neq \emptyset$, which was to be proved.

Let τ be an infinite cardinal. Recall that a space X is said to be *pseudo-τ-compact*, if each open family of cardinality $\geqq \tau$ has a cluster point.

The following notion was introduced by I. I. Guran.

Definition 2.4. A uniform space $(X, \mathscr{U})$ is *τ-bounded* provided that for each member $U \in \mathscr{U}$ there is a subset $K \subseteq X$ such that $|K| \leqq \tau$ and $X = \bigcup_{x \in K} B(x, U)$, where $B(x, U) = \{y \in X: (x, y) \in U\}$.

A. V. Archangelśkij noted that a space X with the universal uniformity $\mathscr{U}_X$ is τ-bounded iff X is pseudo-τ^+-compact. It is known [8] that the group $F_M(X)$ is τ-bounded iff the uniform space $(X, \mathscr{U}_X)$ is τ-bounded. Combining Theorem 2.8 with the facts just mentioned we obtain the following result.

Corollary 2.4. *If X is pseudo-τ^+-compact, then the Souslin number of the group $F_M(X)$ does not exceed 2^τ.*

Example. There is a Lindelöf group H with $c(H) > \aleph_0$.

Let T be an uncountable set, let $X = T \cup \{*\}$ be a one-point Lindelöfication of T, i.e. each point $t \in T$ is isolated and the family $\{\{*\} \cup (T - K): |K| \leqq \aleph_0\}$ is an open base of $*$. Then X is a Lindelöf P-space, i.e. the intersection of any countable family of open sets is open. Moreover, X^n is Lindelöf for each $n \in N^+$, hence $F_M(X)$ is Lindelöf, too. Further, $F_M(X)$ is a P-space. Obviously the pseudocharacter of X is uncountable, and the same holds for $F_M(X)$. Summarizing, $F_M(X)$ is a regular P-space of an uncountable pseudocharacter, which in turn implies $c(F_M(X)) > \aleph_0$.

Problem 2.3. Is it true that $c(H) \leqq \aleph_1$ for every Lindelöf ($\aleph_0$-bounded, resp.) group H?

Added in proof. Problem 2.1 was recently partially solved by V. Uspenskij. He proved the following.

Suppose X to be an $\aleph_0$-bounded (or, equivalently, pseudo-$\aleph_1$-compact) Dieudonné complete space. Then the free topological group $F_M(X)$ is complete.

We have proved in [11] that there exists an $\aleph_0$-bounded group H with $c(H) = 2^{\aleph_0}$. This is a partial answer to Problem 2.3. The group H in question is not Lindelöf.

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