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ON TORSION IN GROUPS WHOSE AUTOMORPHISM GROUPS HAVE FINITE RANK

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1. Introduction. Our object is to study the effect on the elements of finite order in a group of imposing finiteness conditions on the automorphism group. That some effect is to be expected is suggested by results already in the literature. Almost thirty years ago Baer [**1**] showed that a torsion group whose automorphism group is finite is itself finite. This result was sharpened by Nagrebeckiǐ [**9**] who proved that if the automorphism group Aut G of a group G is finite, then the elements of finite order form a finite subgroup of G. Subsequently it was observed that certain apparently weaker finiteness properties are in fact equivalent to the finiteness of Aut G. Thus Robinson [**14**] showed that if Aut G is a Černikov group, then it is finite. Recently Zimmerman [**17**] has proved that Aut G will also be finite if it is a countable torsion FC-group with no elements of order 2 or 3.

Here we shall consider properties of the automorphism group which are usually weaker than finiteness but which are strong enough to force Sylow subgroups of the group to be small. The most general property that we consider is that of having finite abelian subgroup rank; this property requires that every abelian subgroup A have finite torsion-free rank $r_0(A)$ and finite p-rank $r_p(A)$ for all primes p. Somewhat stronger is the requirement of finite abelian total rank; for this the total rank

$$r_0(A) + \sum_p r_p(A)$$

of each abelian subgroup A must be finite. Stronger still are the maximal and minimal conditions on abelian subgroups, max $-ab$ and min $-ab$.

Of course Nagrebeckiǐ's Theorem is really about abelian-by-finite groups. We shall prove our results for soluble-by-finite groups.

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THEOREM 1: *Let G be a soluble-by-finite group.*

(a) *If* Aut G *has finite abelian subgroup rank, then each Sylow subgroup of G is a Černikov group.*

(b) *If* Aut G *satisfies the maximal condition on abelian subgroups, then the Sylow subgroups of G are finite.*

(c) *If* Aut G *satisfies the minimal condition on abelian subgroups, then* Aut G *is finite and the elements of finite order form a finite subgroup of G.*

There are stronger results for soluble-by-finite groups which have the additional property of being torsion-by-nilpotent.

THEOREM 2: *Let G be a soluble-by-finite group which is also torsion-by-nilpotent.*

(a) *If* Aut G *has finite abelian subgroup rank, then the Sylow subgroups of G are finite.*

(b) *If* Aut G *has finite abelian total rank, then the elements of finite order form a finite subgroup of G.*

These theorems are proved in §2 and §3, where it is also indicated how the hypotheses on G may be weakened from "soluble-by-finite" to "radical-by-finite". It seems difficult to decide what happens in the absence of such a condition.

In §4 we construct three central extensions which show clearly the limitations to results like Theorems 1 and 2.

THEOREM 3.

(i) *There is a soluble minimax group G_1 with an infinite Sylow subgroup such that* Aut G_1 *is a soluble minimax group.*

(ii) *There is a soluble group G_2 with finite Prüfer rank such that* Aut G_2 *is polycyclic and abelian-by-finite, but G_2 contains an infinite abelian torsion subgroup.*

(iii) *There is a nilpotent group G_3 of class 2 with finite Prüfer rank whose automorphism group has finite Prüfer rank and is abelian-by-finite, but G_3 has an infinite abelian torsion subgroup.*

(Recall that a minimax group is a group with a series of finite length whose factors satisfy the maximal or minimal condition).

The first example shows that in the situation of Theorem 1(a) one cannot expect the Sylow subgroups to be finite. The second and third examples demonstrate that under the hypotheses of Theorems 1(b) or

2(a) there may occur infinite torsion subgroups. Furthermore Theorem 1(b) does not hold if max $-ab$ is replaced by "finite abelian total rank".

We shall make frequent use of information about the structure of the automorphism group of a group extension; as general references for this theory we cite [**15**] and [**16**].

NOTATION.

G_{ab} : the derived factor group G/G'.

$M(G)$: the Schur multiplicator of G.

$\tilde{\omega}(G)$: the set of prime divisors of the orders of elements of G.

A_p : the p-component of the abelian group A.

2. Proof of Theorem 1. We begin with a formal characterization of groups whose automorphism group has finite abelian subgroup rank, max $-ab$ or min $-ab$. This will not be used in the proofs of the main theorems.

LEMMA 1. *Let G be a group with centre C and central quotient group Q. Denote the cohomology class of the extension $C \rightarrowtail G \twoheadrightarrow Q$ by Δ. Then* Aut G *has finite abelian subgroup rank (respectively* max $-ab$, min $-ab$*) if and only if* $\mathrm{Hom}(Q_{ab}, C)$ *and* $C_{\mathrm{Aut}\ C \times \mathrm{Aut}\ Q}(\Delta)$ *have finite abelian subgroup rank (respectively* max $-ab$, min $-ab$*).*

PROOF. By [**12**], Corollary 2.3, there is an exact sequence

$$\mathrm{Hom}(Q_{ab}, C) \rightarrowtail \mathrm{Aut}\ G \twoheadrightarrow C_{\mathrm{Aut}\ C \times \mathrm{Aut}\ Q}(\Delta)$$

and so the sufficiency of the condition follows immediately.

On the other hand, if Aut G has finite abelian subgroup rank, since $\mathrm{Hom}(Q_{ab}, C)$ is abelian, it follows by a result of Baer and Heineken [**2**] (see also [**11**]) that $C_{\mathrm{Aut}\ C \times \mathrm{Aut}\ Q}(\Delta)$ has finite abelian subgroup rank.

The proofs for max $-ab$ and min $-ab$ are similar.

The proof of Theorem 1 will be approached through a sequence of lemmas. We recall that a group has finite abelian section rank if all its abelian sections have finite torsion-free rank and finite p-rank for all primes p. For radical groups this property is the same as that of having finite abelian subgroup rank, by a theorem of Baer and Heineken [**2**] (see also [**11**]).

LEMMA 2. *Let Q be a soluble-by-finite group of finite abelian section rank. Then $M(Q)$ has finite abelian section rank.*

PROOF. There is a series $1 = Q_0 \lhd \cdots \lhd Q_\ell = Q$ in which Q_{i+1}/Q_i is finite, infinite cyclic or an abelian torsion group with min $-p$ for all p. We can assume that $\ell > 0$. Let $N = Q_{\ell-1}$. Then $M(Q/N) \simeq (Q/N) \otimes (Q/N)$ if Q/N is abelian; otherwise it is finite. Hence $M(Q/N)$ has finite abelian section rank, and so also does $M(N)$ by induction on ℓ. Therefore, by the Lyndon-Hochschild-Serre spectral sequence for homology, it suffices to prove that $H_1(Q/N, N_{ab})$ has finite abelian section rank.

Write $\overline{Q} = Q/N$ and $A = N_{ab}$. If $\overline{Q}$ is finite, the fact that $H_1(\overline{Q}, A)$ is isomorphic with Ker $(A \otimes_{\overline{Q}} I_{\overline{Q}} \to A)$, where $I_{\overline{Q}}$ is the augmentation ideal, gives the result. If $\overline{Q}$ is infinite cyclic, $H_1(\overline{Q}, A)$ is isomorphic with a subgroup of A and all is clear.

Assume now that $\overline{Q}$ is an abelian torsion group with finite p-rank for all p. By the long exact sequence for homology we may assume that A is either torsion-free of finite rank or a torsion group with finite p-rank for all p. Since $H_1(\overline{Q}, A)$ is isomorphic with $\oplus_p H_1(\overline{Q}, A_p)$, we can assume A to be a p-group in the latter case. Then $\overline{Q}/C_{\overline{Q}}(A)$ is finite in both situations by theorems of Schur and Černikov (see [**3**; Theorems 36.14] and [**10**; Part 1, Theorem 3.29.2]). The spectral sequence (or five term homology sequence) allows us to assume that A is a trivial $\overline{Q}$-module. Then $H_1(\overline{Q}, A) \simeq \overline{Q}_{ab} \otimes A$ and the result follows.

COROLLARY. *If G is a group such that $G/Z(G)$ is soluble-by-finite with finite abelian section rank, then G' is soluble-by-finite with finite abelian section rank.*

This follows from Lemma 2 because $G' \bigcap Z(G)$ is an image of $M(G/Z(G))$.

The next two lemmas are technical results designed to produce large abelian direct factors.

LEMMA 3. *Let A be an additive abelian group and let B be an infinite p-subgroup of A with finite exponent such that A/B has finite p-rank. Then A has an infinite direct summand contained in B.*

PROOF. Let P/B be the p-component of A/B; then P/B is a Černikov group. Hence $p^kP + B = p^{k+1}P + B$ for some $k \geq 0$ and $p^{k+r}P = p^{k+r+1}P = L$ say, for some $r \geq 0$, since B has finite exponent. Then L is a divisible p-group of finite rank. The group $P/B+L$ is finite, so we have $P = B^* + L$ where $B \leq B^*$ and B^*/B is finite; therefore

$B^* = B + X$ with X finite, and $P = B + Y$ where $Y = X + L$ has finite rank. Now $B = B_0 \oplus B_1$ where $B \bigcap Y \leq B_0$ and B_0 is finite. Then $P = (B_0 + Y) \oplus B_1$, and since P is a direct summand of A (see [**5**; Vol. 2, Theorem 100.1]), B_1 is an infinite direct summand of A contained in B.

LEMMA 4. *Let G be a group and let C be a central subgroup of G. Assume that G/C is soluble-by-finite with finite abelian section rank and C is an infinite p-group of finite exponent. Then G has an infinite direct factor contained in C.*

PROOF. By the Corollary to Lemma 2 it follows that G' has finite abelian section rank, and so $G' \bigcap C$ is finite. Therefore CG'/G' is infinite, and, applying Lemma 3 to the group G_{ab} and its subgroup CG'/G', we obtain $G = C_0C_1$ where $C_0 \bigcap C_1 = G', C_0 \leq CG'$ and C_0/G' is infinite. Then $C_0 = (C_0 \bigcap C)G'$ and $G = (C_0 \bigcap C)C_1$. Since $G' \bigcap C$ is finite, we can write $C_0 \bigcap C = C_2 \times C_3$ where C_2 is finite and $G' \bigcap C \leq C_2$. Finally, $G = C_1C_2C_3 = (C_1C_2) \times C_3$ and C_3 is infinite.

The next lemma extends results of Hallett and Hirsch (see [**5**; Vol. 2, §166]).

LEMMA 5. *Let A and B be abelian groups and suppose that A is torsion-free. Let there be given a ring homomorphism* End$A \to$ End$B$, *so that B becomes a module over* End$A$. *If b is an element of B such that $K = C_{\mathrm{Aut}\ A}$* (b) *is a torsion group, then K has finite exponent dividing* 12 *and every involution in K belongs to the centre. In particular K is locally finite.*

PROOF. It is shown in [**14**, Lemma 6] that K has finite exponent dividing 12.

If α and β are elements of K with $\alpha^2 = 1$, then $\varphi = (1+\alpha)\beta(1-\alpha)$ and $\psi = (1-\alpha)\beta(1+\alpha)$ are endomorphisms of A such that $\varphi^2 = \psi^2 = 0$. Therefore $1 + \varphi$ and $1 + \psi$ are automorphisms of A, and it is easy to see that they belong to K. Hence $(1+\varphi)^n = 1 = (1+\psi)^n$ for some $n > 0$, which leads to $\varphi = 0 = \psi$. Therefore $0 = \varphi - \psi = 2(\alpha\beta - \beta\alpha)$ and $\alpha\beta = \beta\alpha$.

Finally, the group $K/Z(K)$ has finite exponent dividing 6, so it is locally finite by M. Hall's positive solution of the Burnside Problem for exponent 6 (see [**6**]).

We are now in a position to prove Theorem 1.

PROOF OF THEOREM 1(a). Let $C = Z(G)$ and $Q = G/C$. Firstly we will prove that the divisible part of C_p has finite rank. Suppose that this is false. Then we can write $C = P \times R$ where P is a divisible p-group of infinite rank. Note that $G' \bigcap C$ has finite p-rank by the Corollary to Lemma 2. Hence $G' \bigcap C \leq P_0 \times R$ where $P = P_0 \times P_1$ and P_0 has finite rank. Now $P_1 G'/G'$ is a direct factor of G_{ab}, so $G = XP_1$ and $X \bigcap P_1 \leq G'$ for some X. But $X \bigcap P_1$ must be trivial, so P_1 is a direct factor of G. This is impossible since abelian subgroups of Aut G are countable.

The next step is to prove that C_p has finite rank. Suppose that this rank is in fact infinite.

If C/C^p is finite, then $C_p/(C_p)^p$ is finite and C_p is the direct product of a finite group and a divisible group. However this gives the contradiction that C_p has finite rank. Hence C/C^p is infinite. Then, because G' and Q have finite abelian section rank, the group $G/G'C^p$ must have an infinite elementary abelian p-group as a quotient. Since Aut G has no infinite elementary abelian p-subgroups and $\mathrm{Hom}(G/G'C^p, C^p)$ is isomorphic with a subgroup of Aut G, it follows that $(C^p)_p = 1$; thus C_p must have finite exponent. Hence $C = C_p \times E$ for some $E \leq C$. Applying Lemma 4 to the group G/E and its central subgroup $C/E \simeq C_p$, we obtain $G = G_1 G_2$ where $G_1 \bigcap G_2 = E, G_1 \leq C$ and G_1/E is an infinite group. Since $E \leq G_1 \leq C_p \times E$, we have $G_1 = D \times E$ where $D = G_1 \bigcap C_p$. Finally $G = G_1 G_2 = DEG_2 = DG_2 = D \times G_2$, a contradiction since $D \simeq G_1/E$ is an infinite abelian group of finite exponent.

Therefore C_p has finite rank and the Sylow p-subgroups of G are Černikov groups.

PROOF OF THEOREM 1(b). As before, let $C = Z(G)$ and $Q = G/C$. Write T for the torsion subgroup of C and Δ for the cohomology class of the extension $C \rightarrowtail G \twoheadrightarrow Q$. As a soluble-by-finite group with $\max -ab, Q$ is polycyclic-by-finite, by a theorem of Plotkin (see [**10**; Part 1, Theorem 3.31]); it suffices to prove that T cannot have a p^∞-subgroup. Suppose that P is such a subgroup and write $C = P \times R$.

By the Universal Coefficients Theorem $H^2(Q, C) \simeq \mathrm{Hom}(M(Q), P) \oplus H^2(Q, R)$. Now $\mathrm{Hom}(M(Q), P)$ is a torsion group since $M(Q)$ is finitely generated. Write $\Delta = \Delta_0 + \Delta_1$ where $\Delta_0 \in \mathrm{Hom}(M(Q), P)$ and $\Delta_1 \in H^2(Q, R)$. Then $e\Delta_0 = 0$ for some $e > 0$. For each p-adic integer $\alpha \equiv 0 \pmod p$, the map $x \mapsto x^{1+e\alpha}$ extends from P to an automorphism of G operating trivially on G/C and C/P. If Γ

is the subgroup of Aut G generated by such automorphisms, then Γ acts trivially on G/C and C/P. Hence Γ' acts trivially on G/P, and obviously also on P, so that Γ is metabelian. Therefore, since it satisfies max $-ab$, Γ is countable, a contradiction.

PROOF OF THEOREM 1(c). We shall employ the notation of the preceding proof. The group Q is soluble-by-finite with min $-ab$ and so it is Černikov by a result of Plotkin (see [**10**; Part 1, Theorem 3.32]). Obviously Aut G is a torsion group and so, if $\Gamma_1 = C_{\text{Aut }G}(Q)$, the group Aut G/Γ_1 is Černikov, being a torsion group of automorphisms of Q. Each C_p is Černikov by Theorem 1(a), and it is easy to see that if $p \notin \tilde{\omega}(Q)$, then C_p is a direct factor of G (see, e.g., [**4**]). If $\tilde{\omega}(T)$ were infinite, it would follow that Aut G contained an infinite elementary abelian 2-group, contradicting min $-ab$. Therefore $\tilde{\omega}(T)$ is finite and T is Černikov. If $\Gamma_2 = C_{\Gamma_1}(T)$, then Γ_1/Γ_2 is finite.

Now write $\overline{C} = C/T$ and $\overline{G} = G/T$, and let $\overline{\Delta}$ denote the cohomology class of the extension $\overline{C} \rightarrowtail \overline{G} \twoheadrightarrow Q$. Consider the short exact sequence of Q-modules $T \rightarrowtail C \twoheadrightarrow \overline{C}$; in the long exact cohomology sequence

$$\cdots \to H^2(Q,T) \to H^2(Q,C) \xrightarrow{\varepsilon} H^2(Q,\overline{C}) \to \cdots$$

the mappings are Aut C-homomorphisms by naturality, and $\Delta\varepsilon = \overline{\Delta}$. Therefore $C_{\text{Aut }C}(\Delta) \leq C_{\text{Aut }C}(\overline{\Delta})$.

Since Q is a Černikov group, $M(Q)$ is finite [**13**], and it follows via the Universal Coefficients Theorem that $H^2(Q,T)$ is finite. Therefore $|C_{\text{Aut }C}(\overline{\Delta}) : C_{\text{Aut }C}(\Delta)|$ is finite and $C_{\text{Aut }C}(\overline{\Delta})$ has min $-ab$ because $C_{\text{Aut }C}(\Delta)$ is isomorphic with a section of Aut G.

Since T is a direct factor of C, the natural map $\varphi : C_{\text{Aut }C}(\overline{\Delta}) \to C_{\text{Aut }C}(\overline{\Delta})$ is surjective; moreover Kerφ induces a finite group of automorphisms in T, so it is abelian-by-finite. Hence, by [**10**; Part 1, p.88], the group $\Gamma = C_{\text{Aut }\overline{C}}(\overline{\Delta})$ has min $-ab$. Now apply Lemma 5 with $A = \overline{C}$, $B = H^2(Q,\overline{C})$, $b = \overline{\Delta}$; it follows that Γ is locally finite with finite exponent. Since Γ has min $-ab$, it is finite by a well-known theorem of Hall-Kulatilaka and Kargapolov (see [**7**] or [**10**]).

If Γ_3 denotes $C_{\Gamma_2}(\overline{C})$, then Γ_2/Γ_3 is isomorphic with a subgroup of Γ, so Γ_2/Γ_3 is finite. Finally, Γ_3 stabilizes the series $1 \triangleleft T \triangleleft C \triangleleft G$; hence Γ_3 is nilpotent and in view of min $-ab$ it must be a Černikov group.

Therefore Aut G is Černikov, and by Theorem A of [**14**] it is finite.

3. Proof of Theorem 2. The proof of Theorem 2 is also accomplished through a chain of lemmas.

LEMMA 6. *Let Q be a soluble-by-finite group and let p be a prime. Assume that the Sylow p-subgroups of Q are Černikov and that Q has no p^∞-quotients. Assume also that there is a normal torsion subgroup T with finite Sylow p-subgroups such that Q/T is nilpotent. Then $M(Q)$ is an extension of a finitely generated subgroup by a p'-group. Thus $M(Q)$ has no p^∞-sections.*

PROOF. Let $\overline{Q} = Q/T$. Since $\overline{Q}$ has no p^∞-quotients, it has finite torsion-free rank. In addition it is easy to see that the Sylow p-subgroup of $\overline{Q}$ is Černikov. From this it follows readily that $\overline{Q}_{ab}$ is finitely generated-by-p'. The latter property is preserved by tensor products, so it is possessed by every lower central factor of $\overline{Q}$. Induction on the nilpotent class and use of the spectral sequence lead us to conclude that $M(\overline{Q})$ is finitely generated-by-p'.

Now form a Q-invariant series in T of finite length whose factors are either finite or else abelian torsion groups with finite p-components. Let N be the least non-trivial term of this series. Then $M(Q/N)$ has the required structure by induction, as does $M(N)$. It therefore suffices to consider the group $H_1(Q^*, N^*)$ where $Q^* = Q/N$ and $N^* = N_{ab}$. Now $H_1(Q^*, N^*_{p'})$ is a p'-group, so we need only consider $H_1(Q^*, N^*_p)$. Since $P \equiv N^*_p$ is finite, we can further reduce to the situation where P is a trivial Q^*-module. Finally, $H_1(Q^*, P) \simeq Q^*_{ab} \otimes P$, and this is certainly finitely generated-by-p'.

LEMMA 7. *Let Q be a torsion-by-nilpotent, soluble-by-finite group and let p be a prime. Assume that Q has finite Sylow p-subgroups, and let A be a Q-module which is a divisible abelian p-group of finite rank. If $H_0(Q, A) = 0$, then $H^2(Q, A)$ is a torsion group.*

PROOF. The group $Q/C_Q(A)$ is finite-by-nilpotent and by Theorem G of [**8**], the group $H^0(Q/C_Q(A), A)$ is finite; thus $H^0(Q, A)$ is finite. Let T be the subgroup of all elements of finite order in Q and put $C = C_T(A)$; then $\overline{Q} = Q/C$ is finite-by-nilpotent and by Theorem H of [**8**] the group $H^2(\overline{Q}, A)$ is torsion.

The locally finite group C is a finite extension of a p'-group (see [**7**, Theorem 3.17]), so the groups $H^2(C, A)$ and $H^1(\overline{Q}, H^1(C, A))$ are torsion. The spectral sequence for $C \rightarrowtail Q \twoheadrightarrow \overline{Q}$ now shows that $H^2(Q, A)$ is torsion group.

LEMMA 8. *Let Q be a torsion-by-nilpotent, soluble-by-finite group and* *let p be a prime. Assume that Q has no p^∞-quotients and the Sylow p-subgroups of Q are finite. If A is a Q-module which is a divisible abelian p-group of finite rank, then $H^2(Q, A)$ is a torsion group.*

PROOF. Since A satisfies min, there is an $r > 0$ such that $[A,_r Q] = [A,_{r+1} Q]$ is divisible. By appealing to the exact cohomology sequence we may reduce to two cases:

(a) A is a trivial Q-module;

(b) $A = [A, Q]$.

In case (a) $H^2(Q, A) \simeq \mathrm{Hom}(M(Q), A)$, which is a torsion group by Lemma 6. Henceforth we assume that $A = [A, Q]$.

Let $C = C_Q(A)$. Then $\overline{Q} = Q/C$ is finite-by-nilpotent and $H_0(Q, A) = 0$. By Theorem H of [8], the groups $H^n(\overline{Q}, A)$ are torsion for all $n \geq 0$. Now consider $H^1(\overline{Q},\mathrm{Hom}(C, A))$. As in the proof of Lemma 6 we see that the group C_{ab} is finitely generated-by-p'. Hence $B \equiv \mathrm{Hom}(C, A)$ is a p-group of finite rank. Once again the exact cohomology sequence shows that we may suppose either B a trivial $\overline{Q}$-module or $H_0(\overline{Q}, B) = 0$. In the first case $H^1(\overline{Q}, B)$ is torsion since $\overline{Q}$ has no p^∞-quotients; in the second case the same conclusion holds by Theorem H of [8]. Finally, $H^2(C, A) \simeq \mathrm{Hom}(M(C), A)$, which is torsion, by Lemma 6 again. The spectral sequence for $C \rightarrowtail Q \twoheadrightarrow \overline{Q}$ shows that $H^2(Q, A)$ is a torsion group.

We can now prove Theorem 2.

PROOF OF THEOREM 2(a). By Theorem 1(a), the Sylow subgroups of G are Černikov. The first step is to show that Sylow subgroups of $Z(G)$ are actually finite. In order to obtain a contradiction, suppose that B is a p^∞-subgroup of $Z(G)$. If G has a p^∞-quotient G/N, then $B \not\leq N$, otherwise the additive group of p-adic integers would occur as a subgroup of Aut G. Therefore $G = BN$ and $B_1 = B \bigcap N$ is finite. The group $\Gamma = C_{\mathrm{Aut}\ B}(B_1)$ has finite index in Aut B, so it is an uncountable abelian group. However each γ in Γ extends to an automorphism $\overline{\gamma}$ of G such that $(bx)^{\overline{\gamma}} = b^\gamma x, (b \in B, x \in N)$. Thus Γ is isomorphic with an uncountable abelian subgroup of Aut G, which is impossible. Therefore G has no p^∞-quotients if $Z(G)$ has p^∞-subgroups.

Let T be the maximal normal torsion subgroup of G; then G/T is torsion-free and nilpotent. Also by [7, Corollary 3.18], there is a characteristic divisible abelian subgroup A of T such that T/A has finite Sylow subgroups. The p-component P of A is non-trivial since

$B \leq A$. Let $Q = G/P$. Then Q has finite Sylow p-subgroups. We see from Lemma 8 that $H^2(Q, P)$ is a torsion group.

Let Δ be the cohomology class of the extension $P \rightarrowtail G \twoheadrightarrow Q$. Then $e\Delta = 0$ for some $e > 0$, and so, for each p-adic integer α satisfying $\alpha \equiv 0 \pmod p$, the map $x \mapsto x^{1+e\alpha}$ is a Q-automorphism of P which extends to an automorphism of G acting trivially on Q. The subgroup Δ generated by all such automorphisms is metabelian, and hence countable; but this is impossible because Δ is uncountable. Thus far it has been shown that the Sylow subgroups of $Z(G)$ are finite.

Denoting again by A, the divisible abelian subgroup of T such that the Sylow subgroups of T/A are finite, we suppose in order to obtain a contradiction that $A \neq 1$. Let p be a prime such that $P \equiv A_p$ is non-trivial. If $Q = G/P$, then $H^0(Q, P) = P \bigcap Z(G)$, so that $H^0(Q, P)$ is finite. If $C = C_Q(P)$, the group Q/C is finite-by-nilpotent, and since $H^0(Q/C, P) = H^0(Q, P)$ is finite, by [**8**, Theorem G], it follows that $H_0(Q, P) = H_0(Q/C, P) = 0$. Therefore, by Lemma 7, the group $H^2(Q, P)$ is torsion and we reach a contradiction as before.

PROOF OF THEOREM 2(b). By Theorem 2(a) the Sylow subgroups of G are finite. Let $C = Z(G), Q = G/C$ and let T denote the torsion subgroup of G. Then $\overline{T} = TC/C$ is a Černikov group since Q has finite total rank. Hence $\overline{T}$ is finite. Since G/TC is nilpotent, Q is finite-by-nilpotent, so that by the well-known theorem of P. Hall (see [**10**; Part 1, Theorem 4.25]) we have $|Q : Z_n(Q)| < \infty$ for some $n \geq 0$, which implies that $|G : Z_{n+1}(G)| < \infty$. By a theorem of Baer (see [**10**; Part 1., p. 113]), G is finite-by-nilpotent.

Let N be a finite normal subgroup of G such that G/N is nilpotent, and let T^* be the torsion subgroup of C. For each prime p in $\tilde{\omega}(T^*)$ such that $(|N|, p) = 1$, the group G cannot be p-radicable; for G/N is nilpotent and has finite Sylow p-subgroup. Therefore G has a quotient G/K_p of order p. For every such p let S_p be a subgroup of T_p^* of order p; let π_1 be the set of primes p such that $S_p \leq K_p$ and π_2 the set of primes p such that $S_p \not\leq K_p$. For each $p \in \pi_1$, we have $S_p \leq K_p \bigcap S$ where $S = \mathrm{Dr}_{p \in \pi_1} S_p$, and so $S \leq K_p$ because $S_q \leq K_p$ for all $q \neq p$. Thus we have $S_p \simeq \mathrm{Hom}(G/K_p, S_p) \rightarrowtail \mathrm{Hom}(G/S, S) \rightarrowtail \mathrm{Aut}\ G$ for each $p \in \pi_1$. It follows that π_1 is finite since $\mathrm{Hom}(G/S, S)$ has finite total rank. Finally, if $p \in \pi_2$, then $G = K_p \times S_p$ and $\mathrm{Dr}_{p \in \pi_2} \mathrm{Aut}\ S_p$ is isomorphic with an abelian subgroup of Aut G, which shows that π_2 is finite. Therefore $\tilde{\omega}(T^*)$ is finite, and consequently T^* is finite. The result is now clear.

REMARK. Theorems 1 and 2 still hold if the group G is radical-by-finite. The proof is essentially the same; we merely replace Lemma 2 by the following easily established generalization.

LEMMA 2*. *Let Q be a radical-by-finite group of finite abelian section rank. Then $M(Q)$ has finite abelian section rank.*

4. Counterexamples.

PROOF OF THEOREM 3.

(i) *There exists a soluble minimax group G_1 with an infinite Sylow 2-subgroup such that Aut G_1 is a soluble minimax group.*

Construction. We begin by producing a soluble minimax group Q with the following properties:

(1) Q has no 2^∞-quotients;

(2) $Z(Q) = 1$;

(3) Aut Q is a soluble minimax-group.

Let θ_2 denote the ring of rational numbers whose denominators are powers of 2. Consider the two elements of $\mathrm{SL}(2, \theta_2)$:

$$s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } t = \begin{bmatrix} 0 & 2 \\ -2^{-1} & 0 \end{bmatrix}.$$

Then $s^2 = t^2 = u$, where $u = -1_2$. If $H =< s, t >$, then $Z(H) =< u >$, which has order 2, and $H/Z(H)$ is infinite dihedral. Thus H is polycyclic. Direct matrix calculations reveal that $C_{\mathrm{SL}(2,Q_2)}(H) = < u >$.

There is an obvious exact sequence

$$\mathrm{Hom}(H/Z(H), Z(H)) \rightarrowtail \mathrm{Aut}\ H \to \mathrm{Aut}\ Z(H) \times \mathrm{Aut}\ (H/Z(H))).$$

But $\mathrm{Hom}(H/Z(H), Z(H))$ is finite, Aut $Z(H) = 1$ and Aut $(H/Z(H))$ is infinite dihedral; hence Aut H is polycyclic, and it follows that $N_{\mathrm{SL}(2,Q_2)}(H)$ is polycyclic.

Let $A = Q_2 \oplus Q_2$ be the natural module over H and write Q for the semidirect product $H \ltimes A$. Obviously Q is a soluble minimax group. Also $A = 2A = [A, u]$, so $A \leq Q'$ and Q_{ab} is finite.

Since A is a characteristic self-centralizing subgroup of Q, there is a natural exact sequence

$$\mathrm{Der}(H, A) \rightarrowtail \mathrm{Aut}\ Q \to N_{\mathrm{GL}(2,\theta_2)}(H),$$

(see [**15**,4.4]). Now $H^1(H, A) = 0$ since $H^i(\langle u\rangle, A) = 0, i = 0, 1$, and the 5-term cohomology sequence may be applied. Thus $\mathrm{Der}(H, A) = \mathrm{Inn}(H, A)$ is minimax. Since $N_{\mathrm{GL}(2,Q_2)}(H)$ is polycyclic, Aut Q is a soluble minimax group.

Let $A = B \oplus D$, where $B \simeq Q_2$ is generated by $b_i, i = 0, 1, 2 \cdots$, with $b_{i+1}^2 = b_i$; let d_i be corresponding generators of $D \simeq Q_2$. Let C be a group of type 2^∞ with generators $1 = c_0, c_1, c_2, \cdots$ and $c_{i+1}^2 = c_i$. We can define a central extension

$$C \rightarrowtail G_1 \twoheadrightarrow Q$$

by $G_1 = \langle c_k, b_i, d_j, s, t | i, j, k = 0, 1, 2, \cdots\rangle$ subject to the c_k being central and relations $[b_i, d_j] = c_{i+j}$, together with other relations of Q. Then $Z(G_1) = C$ since $Z(Q) = 1$.

Every automorphism of G_1 which acts trivially on Q acts trivially on C, too, and thus $C_{\mathrm{Aut}\ G_1}(Q) \simeq \mathrm{Hom}(Q_{ab}, C)$, which is finite. Therefore Aut G_1 is soluble minimax.

(ii) *There exists a soluble group G_2 with finite Prüfer rank such that Aut G_2 is polycyclic and abelian-by-finite, but G_2 has an infinite abelian torsion subgroup.*

Construction. Let $A = \langle a\rangle \times \langle b\rangle$, a free abelian group of rank 2, and let x be the automorphism of A which acts via the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then $x^6 = 1$ and the semidirect product $Q = \langle x\rangle \ltimes A$ is abelian-by-finite.

Define $T = \oplus_{p>3}\langle t_p\rangle$ where t_p has order p, and put $T^* = \mathrm{Cr}_{p>3}\langle t_p\rangle$, the cartesian sum. For each $p > 3, i \geq 0$ consider the element $x_{p,i}$ of T^* defined by $(x_{p,i})_p = 0$ and $(x_{p,i})_q = \frac{1}{p^i}t_q$ if $q \neq p$, and let $C = \langle T, x_{p,i} | p\rangle 3, i \geq 0 >$. Then $T < C < T^*, x_{p,0} \equiv x_{q,0} \pmod T$, for each p, q, and C/T is isomorphic with the additive group $Q_{6'}$ of rational numbers with denominators prime to 6. Let u be the element of T^* defined by $u_p = t_p$ for all p; then $x_{p,o} \equiv u \pmod{T}$ and hence $u \in C$.

Now we can define a central extension

$$C \rightarrowtail G_2 \twoheadrightarrow Q,$$

where $G_2 = \langle C, a, b, x \rangle$ subject to $[G_2, C] = 1$ and relations $x^6 = 1, a^x = b^{-1}, b^x = ab, [a, b] = u$. Thus $Z(G_2) = C$, since $Z(Q) = 1$.

Let γ be an automorphism of G_2 which acts trivially on Q; then γ acts trivially on G_2' and so $u\gamma = u$. Since $u \notin T$, it is clear that γ acts trivially on C/T.

For any prime p we have $x_{p,1}(\gamma - 1) \equiv a(p) \in T$ and also $px_{p,1} = u - t_p$, so that $(u - t_p)(\gamma - 1) = pa(p)$. Since $u\gamma = u$, it follows that $t_p(\gamma - 1) = -pa(p)$, which shows that the element $t_p(\gamma - 1)$ must be 0. Therefore γ operates trivially on T.

Since C/T is divisible by primes greater than 3 and $|Q_{ab}| = 6$, we have $\mathrm{Hom}(C/T, T) = \mathrm{Hom}(Q_{ab}, C) = 0$, showing that $\gamma = 1$. Consequently the natural homomorphism Aut $G_2 \to$ Aut Q is injective.

Finally Aut Q is polycyclic and abelian-by-finite. For it is easy to see that $N_{\mathrm{GL}(2,\mathbf{Z})}(\langle x \rangle)$ is a dihedral group of order 12. Also, $H^1(\langle x \rangle, A) = 0$. By the exact sequence for Aut Q in (i) we deduce that Aut G_2 has the structure claimed.

REMARK. It may be worthwhile noting that if G is soluble-by-finite and Aut G has max$-ab$, and if there is an infinite torsion subgroup of G, then Q_{ab} must be finite and $M(Q)$ infinite (here as usual $Q = G/Z(G)$).

Of course it is clear that Q_{ab} is finite. If also $M(Q)$ were finite, $H^2(Q, Z(G))$ would have finite exponent, which implies that $Z(G)_p$ is a direct factor of G for infinitely many p, obviously impossible.

(iii) *There exists a nilpotent group G_3 of class 2 with finite Prüfer rank such that Aut G_3 is abelian-by-finite and soluble with finite Prüfer rank, but G_3 has an infinite abelian torsion subgroup.*

Construction. Let $\mathbf{Q}u \oplus \mathbf{Q}v$ be a rational vector space of dimension 2, and let Q be the additive subgroup generated by all elements of the form

$$u_i = \frac{u}{p^i},\ v_j = \frac{v}{q^j},\ w_k = \frac{u+v}{r^k},\ (i,j,k \geq 0),$$

where p, q, r are three distinct fixed primes. Obviously Q is a minimax group and it well-known that $|\mathrm{Aut}\ Q| = 2$.

Define $T = \mathrm{Dr}_{s\neq p,q,r}\langle t_s\rangle$ where t_s has prime order s, and put $T^* = \mathrm{Cr}_{s\neq p,q,r}\langle t_s\rangle$. For each prime $s \neq p,q,r$ and $i \geq 0$, we define an element $x_{s,i}$ of T^* by $(x_{s,i})_s = 0$ and $(x_{s,i})_k = \frac{1}{s^i}t_k$ if $k \neq s$. Consider the group C generated by T and all $x_{s,i}/p^i q^j r^k$ where $i,j,k \geq 0$ and $s \neq p,q,r$. The element c of T^* defined by $c_s = t_s$ for all s belongs to C.

Now we can define a central extension

$$C \rightarrowtail G_3 \twoheadrightarrow Q,$$

where $G_3 = \langle C, u_i, v_j, w_k | i,j,k \geq 0\rangle$ with C central in G_3 and relations $[u_i, v_j] = c^{1/p^i q^j}, [u_i, w_k] = c^{1/p^i r^k}, [v_j, w_k] = c^{-1/q^j r^3}, U_{i+1}^p = u_i, v_{j+1}^q = v_j, w_{k+1}^r = w_k, w_0 = u_0 v_0$.

If γ is an automorphism of G_3 which acts trivially on Q, we argue, as in (ii), that it must operate trivially on C, and so $C_{\mathrm{Aut}\ G_3}(Q) \simeq \mathrm{Hom}(Q, C)$; this last is isomorphic with a subgroup of $C \oplus C$. Therefore $C_{\mathrm{Aut}\ G_3}(Q)$ is an abelian group with finite Prüfer rank. It follows that Aut G_3 is abelian-by-finite cyclic with finite Prüfer rank.

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