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ON TOTAL RESTRAINED DOMINATION IN GRAPHS

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Abstract. In this paper we initiate the study of total restrained domination in graphs. Let $G = (V, E)$ be a graph. A total restrained dominating set is a set $S \subseteq V$ where every vertex in $V - S$ is adjacent to a vertex in S as well as to another vertex in $V - S$, and every vertex in S is adjacent to another vertex in S . The total restrained domination number of G , denoted by $\gamma_r^t(G)$, is the smallest cardinality of a total restrained dominating set of G . First, some exact values and sharp bounds for $\gamma_r^t(G)$ are given in Section 2. Then the Nordhaus-Gaddum-type results for total restrained domination number are established in Section 3. Finally, we show that the decision problem for $\gamma_r^t(G)$ is NP-complete even for bipartite and chordal graphs in Section 4.

Keywords: total restrained domination number, Nordhaus-Gaddum-type results, NP-complete, level decomposition

MSC 2000: 05C69, 05C35

1. INTRODUCTION

Graph theory terminology not presented here can be found in [1]. Let $G = (V, E)$ be a graph with $|V| = n$. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The graph induced by $S \subseteq V$ is denoted by $G[S]$. A set S is a dominating set if for every vertex $u \in V - S$ there exists $v \in S$ such that $uv \in E$. The domination number of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G .

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A set $S \subseteq V$ is a restrained dominating set if every vertex in $V - S$ is adjacent to a vertex in S and to another vertex in $V - S$. Let $\gamma_r(G)$ denote the size of a smallest restrained dominating set. It has been studied by G. S. Domke [1] and M. A. Henning [2].

In this paper we initiate the study of total restrained domination in graphs. The total restrained domination is a particular case of the restrained domination. Let $G = (V, E)$ be a graph. A total restrained dominating set is a set $S \subseteq V$ where every vertex in $V - S$ is adjacent to a vertex in S as well as to another vertex in $V - S$ and every vertex in S is adjacent to another vertex in S . The total restrained domination number of G , denoted by $\gamma_r^t(G)$, is the smallest cardinality of a total restrained dominating set of G . Note that every graph without an isolated vertex has a restrained dominating set, since $S = V$ is such a set. We will call a set S a γ_r^t -set if S is a total restrained dominating set of cardinality $\gamma_r^t(G)$.

One possible application of the concept of total restrained domination is that of prisoners and guards. Here each vertex not in the total restrained dominating set corresponds to a position of a prisoner, and every vertex in the total restrained dominating set corresponds to a position of a guard. Note that each prisoner's position is observed by a guard's position (to effect security) while each prisoner's position is seen by at least one other prisoner's position (to protect the rights of prisoners) and each guard's position is seen by at least one other guard's position (to protect each other). To be cost-effective, it is desirable to place as few guards as possible (in the sense above).

In this paper we give the following results. First, some exact values and sharp bounds for $\gamma_r^t(G)$ are given in Section 2. Then Nordhaus-Gaddum-type results for total restrained domination number are established in Section 3. Finally, we show that the decision problem for $\gamma_r^t(G)$ is NP-complete even for bipartite and chordal graphs in Section 4.

2. SOME EXACT VALUES AND SHARP BOUNDS FOR $\gamma_r^t(G)$

Let K_n , C_n and P_n denote, respectively, the complete graph, the cycle and the path of order n . Also, let K_{n_1, n_2, \dots, n_t} denote the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \dots \cup S_t$ where $|S_i| = n_i$ for $1 \leq i \leq t$. We call $K_{1, n-1}$ a star.

The following four theorems are immediate.

Theorem 1. *If $n \geq 2$ is an integer, then*

$$\gamma_r^t(K_n) = \begin{cases} n & \text{for } n = 2, 3, \\ 2 & \text{for } n \geq 4. \end{cases}$$

Theorem 2. If $n \geq 2$, then $\gamma_r^t(K_{1,n-1}) = n$.

Theorem 3. If n_1 and n_2 are integers such that $\min\{n_1, n_2\} \geq 2$, then

$$\gamma_r^t(K_{n_1, n_2}) = 2.$$

Theorem 4. If $t \geq 3$ is an integer, then

$$\gamma_r^t(K_{n_1, n_2, \dots, n_t}) = \begin{cases} 3 & \text{for } t = 3 \text{ and } n_1 = n_2 = n_3 = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 5. If $n \geq 2$ is an integer, then $\gamma_r^t(P_n) = n - 2 \lfloor \frac{n-2}{4} \rfloor$.

Proof. Let S be a γ_r^t -set of P_n whose vertex set is $V = \{v_1, v_2, \dots, v_n\}$. Note that $v_1, v_2, v_{n-1}, v_n \in S$ and any component of $P_n[V - S]$ is of size exactly two. Suppose there are m components in $P_n[V - S]$. Then $2m + 2(m + 1) \leq n$, that is $m \leq \frac{n-2}{4}$. Thus $|S| = n - 2m \geq n - 2 \lfloor \frac{n-2}{4} \rfloor$. Hence, $\gamma_r^t(P_n) \geq n - 2 \lfloor \frac{n-2}{4} \rfloor$.

Case 1. If $n - 2 \equiv 0 \pmod{4}$, then let $n - 2 = 4t$ for an integer t and

$$|S| \geq n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor = n - 2t = 2t + 2.$$

On the other hand, $\{v_{4i+1}, v_{4i+2} \mid i = 0, 1, \dots, t-1\} \cup \{v_{4t+1}, v_{4t+2}\}$ is a total restrained dominating set of G with cardinality $n - 2 \lfloor \frac{n-2}{4} \rfloor$ and $\gamma_r^t(P_n) \leq n - 2 \lfloor \frac{n-2}{4} \rfloor$. Hence $\gamma_r^t(P_n) = n - 2 \lfloor \frac{n-2}{4} \rfloor$.

Case 2. If $n - 2 \equiv 1 \pmod{4}$, then let $n - 2 = 4t + 1$ for an integer t and

$$|S| \geq n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor = n - 2t = 2t + 3.$$

On the other hand, $\{v_{4i+1}, v_{4i+2} \mid i = 0, 1, \dots, t-1\} \cup \{v_{4t+1}, v_{4t+2}, v_{4t+3}\}$ is a total restrained dominating set of G with cardinality $n - 2 \lfloor \frac{n-2}{4} \rfloor$ and $\gamma_r^t(P_n) \leq n - 2 \lfloor \frac{n-2}{4} \rfloor$. Hence $\gamma_r^t(P_n) = n - 2 \lfloor \frac{n-2}{4} \rfloor$.

Case 3. If $n - 2 \equiv 2 \pmod{4}$, then let $n - 2 = 4t + 2$ for an integer t and

$$|S| \geq n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor = n - 2t = 2t + 4.$$

On the other hand, $\{v_{4i+1}, v_{4i+2} \mid i = 0, 1, \dots, t-1\} \cup \{v_{4t+1}, v_{4t+2}, v_{4t+3}, v_{4t+4}\}$ is a total restrained dominating set of G with cardinality $n - 2 \lfloor \frac{n-2}{4} \rfloor$ and $\gamma_r^t(P_n) \leq n - 2 \lfloor \frac{n-2}{4} \rfloor$. Hence $\gamma_r^t(P_n) = n - 2 \lfloor \frac{n-2}{4} \rfloor$.

Case 4. If $n - 2 \equiv 3 \pmod{4}$, then let $n - 2 = 4t + 3$ for an integer t and

$$|S| \geq n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor = n - 2t = 2t + 5.$$

On the other hand, $\{v_{4i+1}, v_{4i+2} \mid i = 0, 1, \dots, t-1\} \cup \{v_{4t+1}, v_{4t+2}, v_{4t+3}, v_{4t+4}, v_{4t+5}\}$ is a total restrained dominating set of G with cardinality $n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor$ and $\gamma_r^t(P_n) \leq n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor$. Hence $\gamma_r^t(P_n) = n - 2 \left\lfloor \frac{n-2}{4} \right\rfloor$. \square

We omit the proof of the following result as it is similar to that of Theorem 5.

Theorem 6. *If $n \geq 3$, then $\gamma_r^t(C_n) = n - 2 \left\lfloor \frac{n}{4} \right\rfloor$.*

Now using the above theorems, we give a general upper bound for $\gamma_r^t(G)$.

Theorem 7. *For any graph of order $n \geq 2$ and with no isolated vertices we have $2 \leq \gamma_r^t(G) \leq n$ and the bound is sharp.*

We close this section by providing a lower bound for the total restrained domination number of a tree.

Theorem 8. *Let T be a tree of order $n \geq 2$. Then $\gamma_r^t(T) \geq \Delta(T) + 1$. Furthermore, $\gamma_r^t(T) = \Delta(T) + 1$ if and only if T is a star.*

Proof. Let T be a tree of order $n \geq 2$. Since T has at least $\Delta(T)$ leaves and any total restrained dominating set must contain all leaves and their neighbors, we have $\gamma_r^t(T) \geq \Delta(T) + 1$.

It is easy to verify that if T is a star, then $\gamma_r^t(T) = \Delta(T) + 1$.

Conversely, let T be a tree of order $n \geq 2$ such that $\gamma_r^t(T) = \Delta(T) + 1$. Let S be a γ_r^t -set of T . Clearly, S must contain all leaves and their neighbors. Since T has at least $\Delta(T)$ leaves, $T[S]$ is a star with the same leaves as T . Let t denote the center of the star. If $n > \Delta(T) + 1$, then there exist two vertices $u, v \in V(T) - S$ such that $uv \in E(T)$. Since u and v must be dominated by S , both u and v are adjacent to t . It follows that T has a cycle, which is a contradiction. So, $n = \Delta(T) + 1$ and T is a star. \square

3. NORDHAUS-GADDUM-TYPE RESULTS FOR TOTAL RESTRAINED
DOMINATION NUMBER

Nordhaus and Gaddum provided the best possible bounds on the sum of the chromatic numbers of a graph and its complement in [3]. The corresponding result for the domination number was presented by Jaeger and Payan [4]: if G is a graph of order $n \geq 2$, then $\gamma(G) + \gamma(\overline{G}) \leq n + 1$. A bound on the sum of restrained domination numbers of a graph and its complement was given by G. S. Domke [1]: If G is a graph of order $n \geq 2$ such that both G and \overline{G} are not P_3 , then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$.

Let $\text{diam}(G)$ denote the diameter of G , and let u, v be two vertices of G such that $d(u, v) = \text{diam}(G)$. The family $W = \{V_0, V_1, V_2, \dots, V_{\text{diam}(G)}\}$ is called the level decomposition of G with respect to u if $V_i = \{x : x \in V(G) \text{ and } d(u, x) = i\}$ for $i = 0, 1, 2, \dots, \text{diam}(G)$.

We now prove the best possible bound on the sum of total restrained domination numbers of a graph and its complement.

Theorem 9. *Let G be a graph with no isolated vertices and $\Delta(G) < n - 1$. If the diameter of G or \overline{G} is more than 2, then $\gamma_r^t(G) + \gamma_r^t(\overline{G}) \leq n + 4$.*

Proof. Since G is a graph with no isolated vertices and $\Delta(G) < n - 1$, \overline{G} is a graph with no isolated vertices and $\Delta(\overline{G}) < n - 1$.

If G is disconnected, then $\gamma_r^t(\overline{G}) = 2$. Since $\gamma_r^t(G) \leq n$, the theorem holds. Hence, without loss of generality, we assume both G and \overline{G} are connected. We discuss the following cases.

Case 1. $\text{diam}(G) \geq 5$; then it is clear that $\gamma_r^t(\overline{G}) = 2$ and the theorem holds.

Case 2. $\text{diam}(G) = 4$; then let u, v be two vertices of G such that $d(u, v) = 4$, and let the family $W = \{V_0, V_1, V_2, V_3, V_4\}$ be the level decomposition of G with respect to u . It is obvious that $\{u\} = V_0$ and $v \in V_4$.

Case 2.1. There exists two vertices u and v with $d(u, v) = 4$ such that the level decomposition of G with respect to either u or v , say u , satisfies $|V_4| \geq 2$. Then $\{u, v\}$ is a total restrained dominating set of \overline{G} . So $\gamma_r^t(\overline{G}) = 2$ and the theorem holds.

Case 2.2. For arbitrary two vertices u and v with $d(u, v) = 4$, the level decomposition of G with respect to either u or v , say u , has $|V_4| = 1$. That is to say $V_0 = \{u\}$ and $V_4 = \{v\}$. Let $V_{21} = \{w \in V_2 \mid \text{there exists at least one vertex in } V_1 \cup V_2 \cup V_3, \text{ say } z, \text{ such that } wz \notin E(G)\}$ and let $V_{22} = V_2 - V_{21}$. Then $V_0 \cup V_{22} \cup V_4$ is a total restrained dominating set of \overline{G} . So $\gamma_r^t(\overline{G}) \leq |V_0| + |V_{22}| + |V_4| = |V_{22}| + 2$. Hence, if $|V_{22}| \leq 2$, then $\gamma_r^t(\overline{G}) \leq 4$ and the theorem holds. If $|V_{22}| \geq 3$, then let $t \in V_{22}$ and let $V_0 \cup V_1 \cup V_{21} \cup V_3 \cup V_4 \cup \{t\}$ be a total restrained dominating set of G . Hence,

$$\gamma_r^t(G) \leq |V_0| + |V_1| + |V_{21}| + |V_3| + |V_4| + 1 = n - |V_{22}| + 1.$$

It follows that

$$\gamma_r^t(G) + \gamma_r^t(\overline{G}) \leq n - |V_{22}| + 1 + |V_{22}| + 2 \leq n + 3.$$

Case 3. $\text{diam}(G) = 3$; then let u, v be two vertices of G such that $d(u, v) = 3$, and let the family $W = \{V_0, V_1, V_2, V_3\}$ be the level decomposition of G with respect to u . It is obvious that $\{u\} = V_0$ and $v \in V_3$.

Case 3.1. There exists two vertices u and v with $d(u, v) = 3$ such that the level decomposition of G with respect to either u or v , say u , satisfies $|V_3| \geq 2$. Then let $t \in V_3$, $V_{21} = \{w \in V_2 \mid \text{there exists at least one vertex in } V_1 \cup V_2 \cup V_3 \setminus \{t\}, \text{ say } z, \text{ such that } wz \notin E(G)\}$ and let $V_{22} = V_2 - V_{21}$. Then $V_0 \cup V_{22} \cup \{t\}$ is a total restrained dominating set of \overline{G} . So $\gamma_r^t(\overline{G}) \leq |V_0| + |V_{22}| + 1 = |V_{22}| + 2$. Hence, if $|V_{22}| \leq 2$, then $\gamma_r^t(\overline{G}) \leq 4$ and the theorem holds. If $|V_{22}| \geq 3$, then assume $s \in V_2$ and $st \in E(G)$. If $s \in V_{22}$, then $V_0 \cup V_1 \cup V_{21} \cup V_3 \cup \{s\}$ is a total restrained dominating set of G . Hence,

$$\gamma_r^t(G) \leq |V_0| + |V_1| + |V_{21}| + |V_3| + 1 = n - |V_{22}| + 1.$$

It follows that

$$\gamma_r^t(G) + \gamma_r^t(\overline{G}) \leq n - |V_{22}| + 1 + |V_{22}| + 2 \leq n + 3.$$

If $s \notin V_{22}$, then let $w \in V_{22}$. Hence, $V_0 \cup V_1 \cup V_{21} \cup V_3 \cup \{s, w\}$ is a total restrained dominating set of G . Consequently,

$$\gamma_r^t(G) \leq |V_0| + |V_1| + |V_{21}| + |V_3| + 2 = n - |V_{22}| + 2.$$

It follows that

$$\gamma_r^t(G) + \gamma_r^t(\overline{G}) \leq n - |V_{22}| + 2 + |V_{22}| + 2 \leq n + 4.$$

Case 3.2. For arbitrary two vertices u and v with $d(u, v) = 3$, the level decomposition of G with respect to either u or v , say u , has $|V_3| = 1$. That is to say $V_0 = \{u\}$ and $V_3 = \{v\}$. Let $V_{11} = \{w \in V_1 \mid \text{there exists at least one vertex in } V_1 \cup V_2, \text{ say } z, \text{ such that } wz \notin E(G)\}$ and let $V_{12} = V_1 - V_{11}$.

Let $V_{21} = \{w \in V_2 \mid \text{there exists at least one vertex in } V_1 \cup V_2, \text{ say } z, \text{ such that } wz \notin E(G)\}$ and let $V_{22} = V_2 - V_{21}$. Then $V_0 \cup V_{12} \cup V_{22} \cup V_3$ is a total restrained dominating set of \overline{G} . So $\gamma_r^t(\overline{G}) \leq |V_0| + |V_{12}| + |V_{22}| + |V_3| = |V_{12}| + |V_{22}| + 2$. Hence, if $|V_{12}| + |V_{22}| \leq 2$, then $\gamma_r^t(\overline{G}) \leq 4$ and the theorem holds. If $|V_{12}| + |V_{22}| \geq 3$, then assume $s \in V_2$ and $sv \in E(G)$. If $s \in V_{22}$, then $V_0 \cup V_{11} \cup V_{21} \cup V_3 \cup \{s\}$ is a total restrained dominating set of G . Hence,

$$\gamma_r^t(G) \leq |V_0| + |V_{11}| + |V_{21}| + |V_3| + 1 = n - |V_{12}| - |V_{22}| + 1.$$

It follows that

$$\gamma_r^t(G) + \gamma_r^t(\overline{G}) \leq n - |V_{12}| - |V_{22}| + 1 + |V_{12}| + |V_{22}| + 2 \leq n + 3.$$

If $s \notin V_{22}$, then let $w \in V_{22}$. Hence, $V_0 \cup V_{11} \cup V_{21} \cup V_3 \cup \{s, w\}$ is a total restrained dominating set of G . Consequently,

$$\gamma_r^t(G) \leq |V_0| + |V_{11}| + |V_{21}| + |V_3| + 2 = n - |V_{12}| - |V_{22}| + 2.$$

It follows that

$$\gamma_r^t(G) + \gamma_r^t(\overline{G}) \leq n - |V_{12}| - |V_{22}| + 2 + |V_{12}| + |V_{22}| + 2 \leq n + 4.$$

□

4. COMPLEXITY ISSUE FOR γ_r^t

To show that the decision problem for arbitrary graphs is NP-complete, we need to use a well known NP-completeness result, called Exact Three Cover(X3C), which is defined as follows.

Exact cover by 3-sets (X3C).

Instance. A finite set X with $|X| = 3q$ and a collection ψ of 3-element subsets of X .

Question. Does ψ contain an exact cover for X , that is, a subcollection $\psi' \subseteq \psi$ such that every element of X occurs in exactly one member of ψ' ? Note that if ψ' exists, then its cardinality is precisely q .

Lemma 1 ([5]). X3C is NP-complete.

Total restrained dominating set (TRDS).

Instance. A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

Question. Is there a total restrained dominating set of cardinality at most k ?

Theorem 10. TRDS is NP-complete, even for bipartite graphs.

Proof. It is clear that TRDS is in NP.

To show that TRDS is an NP-complete problem, we will establish a polynomial transformation from X3C. Let $X = \{x_1, x_2, \dots, x_{3q}\}$ and $\psi = \{C_1, C_2, \dots, C_m\}$ be an arbitrary instance of X3C.

We will construct a bipartite graph G and a positive integer k such that this instance of X3C will have an exact three cover if and only if G has a total restrained dominating set of cardinality at most k .

We now describe the construction of G . With each $x_i \in X$ associate with a path P_4 with vertices x_i, y_i, z_i, t_i . With each C_j associate with a path P_3 with vertices c_j, d_j, s_j . The construction of the bipartite graph G is completed by joining x_i and c_j if and only if $x_i \in C_j$. Finally, set $k = 2m + 7q$.

Suppose ψ has an exact 3-cover, say ψ' . Then $\bigcup_{1 \leq i \leq 3q} \{z_i, t_i\} \cup \bigcup_{1 \leq j \leq m} \{d_j, s_j\} \cup \{c_j \mid C_j \in \psi'\}$ is a total restrained dominating set of cardinality $2m + 7q$. This construction can clearly be accomplished in polynomial time.

Suppose, conversely, that D is a total restrained dominating set of cardinality at most $2m + 7q$. Then the vertices in the set L , defined by $\bigcup_{1 \leq i \leq 3q} \{z_i, t_i\} \cup \bigcup_{1 \leq j \leq m} \{d_j, s_j\}$, are all end vertices and their neighbors and have to be in D . Hence, $|D| - |L| \leq (2m + 7q) - (2m + 6q) = q$. Let $I = \{i \in \{1, 2, \dots, 3q\} \mid x_i \in D \text{ or } y_i \in D\}$ and let $J = \{j \in \{1, 2, \dots, m\} \mid c_j \in D\}$. Then, since D is a dominating set of G , $\bigcup_{i \in I} \{x_i, y_i\} \cup \bigcup_{j \in J} N[c_j] \supseteq \{x_1, x_2, \dots, x_{3q}\}$. We conclude that $|I| + 3|J| \geq 3q$. Also, $|I| + |J| \leq |D| - |L| \leq q$. Hence, $|3I| + 3|J| \leq |I| + 3|J|$, so that $I = \emptyset$. We conclude that $x_i, y_i \notin D$ for $i = 1, 2, \dots, 3q$. Since $x_i, i = 1, 2, \dots, 3q$, is dominated by D , we conclude that $|J| = q$ and that $\psi' = \{C_j \mid j \in J\}$ is an exact cover for X . \square

Theorem 11. TRDS is NP-complete, even for chordal graphs.

Proof. The proof is similar to the proof of Theorem 10, except that edges are added so that c_1, c_2, \dots, c_m form a clique. \square

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