Int. Journal of Math. Analysis, Vol. 4, 2010, no. 49, 2401 - 2414

On (ϵ)-Trans-Sasakian Manifolds

S. S. Shukla and D. D. Singh

Department of Mathematics
University of Allahabad
Allahabad-211002, India
ssshukla_au@rediffmail.com
singh_30dd@rediffmail.com

Abstract. In this article, we introduce (ϵ)-trans-Sasakian manifolds and give an example of such manifolds. Some basic results regarding (ϵ)-trans-Sasakian manifolds have been obtained in this context. Conformally flat and Weyl-semi symmetric (ϵ)-trans-Sasakian manifolds are also studied.

Mathematics Subject Classification: 53C25, 53C40, 53C50

Keywords: (ϵ)-trans-Sasakian manifold, η-Eienstein manifold, conformally flat, quasi-constant curvature, Weyl-semi symmetric

1 Introduction

It is well known that the properties of a manifold depend on the nature of metric defined on it. In Riemannian geometry, we study manifolds with metric which is positive definite. Since manifolds with indefinite metric have significant use in Physics, it is interesting to study such manifolds equipped with different structures. In [1], A.Bejancu and K.L.Duggal introduced the notion of (ϵ)-Sasakian manifolds with indefinite metric. In 1998, Xu Xufeng and Chao Xiaoli proved that every (ϵ)-Sasakian manifold is a hyper surface of an indefinite Käehlerian manifold and established a necessary and sufficient condition for an odd dimensional Remannian manifold to be an (ϵ)-Sasakian manifold [21]. In [6], U.C.De and Avijit Sarkar introduced and studied the notion of (ϵ)-Kenmotsu manifolds with indefinite metric giving an example.
On the other hand in 1985, J.A.Oubiña [16] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. Trans-Sasakian structures of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are cosympletic, α-Sasakian and

β-Kenmotsu respectively. It is known that trans-Sasakian structures are generalized quasi-Sasakian structures [20]. In this paper, we introduce (ϵ)-trans-Sasakian manifolds with indefinite metric , which appear as a natural generalization of both (ϵ)-Sasakian and (ϵ)-Kenmotsu manifolds. The paper is organized as follows:
In section 2, we introduce the notion of (ϵ)-trans-Sasakian manifold with an example and deduce some basic results regarding such type of manifolds. Section 3 deals with conformally flat (ϵ)-trans-Sasakian manifolds. In section 4, we study weyl-semi symmetric (ϵ)-trans-Sasakian manifolds.

2 Preliminaries

A $(2n+1)$-dimensional differentiable manifold (M, g) is said to be an (ϵ)-almost contact metric manifold [7], if it admits a (1,1) tensor field ϕ, a structure vector field ξ, a 1-form η and an indefinite metric g such that

(2.1) $\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$

(2.2) $g(\xi, \xi) = \epsilon, \quad \eta(X) = \epsilon g(X, \xi),$

(2.3) $g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y),$

for all vector fields X, Y on M, where ϵ is 1 or -1 according as ξ is space like or time like and rank ϕ is $2n$. If

(2.4) $d\eta(X, Y) = g(X, \phi Y)$, for all $X, Y \in \Gamma(TM)$,

then $M(\phi, \xi, \eta, g, \epsilon)$ is called an (ϵ)-almost contact metric manifold.
An (ϵ)-almost contact metric manifold is called an (ϵ)-trans-Sasakian manifold if

(2.5) $(\nabla_X \phi)Y = \alpha \{g(X, Y)\xi - \epsilon\eta(Y)X\} + \beta \{g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X\},$

for any $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection with respect to g.
We note that if $\epsilon = 1$, i.e. structure vector field ξ is space like, then an (ϵ)-trans-Sasakian manifold is usual trans-Sasakian manifold [16].
Definition (2.1) [22]. An (ϵ)-trans-Sasakian manifold is called an η-Eienstein manifold if its Ricci tensor S satisfies the condition

(2.6) $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$

where a and b are scalar functions and X, Y are any vector fields on M.
Definition (2.2) [3]. An (ϵ)-trans-Sasakian manifold is called a manifold of quasi-constant curvature if its curvature tensor $\overline{R}$ of type (0,4) satisfies

(2.7) $\overline{R}(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$

$+q[g(X, W)T(Y)T(Z) -$
$g(X, Z)T(Y)T(W)$

$+g(Y, Z)T(X)T(W) -$

$g(Y,W)T(X)T(Z)]$,

where $\overline{R}(X,Y,Z,W) = g(R(X,Y)Z,W), R$ is the curvature tensor of type (1,3), p, q are scalar functions and ρ is a unit vector field defined by

(2.8) $g(X,\rho) = T(X)$.

Definition (2.3) [4]. A Riemannian manifold whose curvature tensor $\overline{R}$ of type (0,4) satisfies the condition

$$\overline{R}(X,Y,Z,W) = F(Y,Z)F(X,W) - F(X,Z)F(Y,W),$$

where F is a symmetric tensor of type (0,2), is called a special manifold with associated tensor F and is denoted by $\psi(F)_n$.

Definition (2.4). An (ϵ)-trans-Sasakian manifold is called Weyl-semi symmetric if it satisfies the condition $R.C = 0$, where $R(X,Y)$ denotes the curvature operator and $C(X,Y)Z$ is the Weyl-conformal curvature tensor.

Lemma (2.1). An (ϵ)-almost contact metric manifold is an (ϵ)-trans-Sasakian manifold if and only if

(2.9) $(\nabla_X \xi) = \epsilon\{-\alpha\phi X + \beta(X - \eta(X)\xi)\}$.

Proof. For an (ϵ)-trans-Sasakian manifold, the equation (2.5) implies

(2.10) $\nabla_X \phi Y - \phi\nabla_X Y = \alpha\{g(X,Y)\xi - \epsilon\eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \epsilon\eta(Y)\phi X\}$.

Replacing Y by ξ and using (2.2), above equation gives

(2.11) $-\phi\nabla_X \xi = \epsilon\{\alpha(\eta(X)\xi - X) - \beta\phi X\}$.

Applying ϕ on both sides of above equation and using (2.1) and $(\nabla g)(\xi,\xi) = 0$, we obtain

$$\nabla_X \xi = \epsilon\{-\alpha\phi X + \beta(X - \eta(X)\xi)\}.$$

Conversely, we suppose that the relation (2.9) holds for an (ϵ)-almost contact metric manifold. Since $\eta \wedge \phi$ is upto a constant factor the volume element of the manifold [14], it is parallel with respect to ∇, i.e.

(2.12) $\nabla_X(\eta \wedge \phi) = 0$,

which gives

(2.13) $$(\nabla_X\eta)(Y)\Phi(Z,W) + \eta(Y)(\nabla_X\Phi)(Z,W) \\ +(\nabla_X\eta)(Z)\Phi(W,Y) + \eta(Z)(\nabla_X\Phi)(W,Y) \\ +(\nabla_X\eta)(W)\Phi(Y,Z) + \eta(W)(\nabla_X\Phi)(Y,Z) = 0,$$

where Φ is the fundamental 2-form of the (ϵ)-almost contact metric structure, defined by $\Phi(X,Y) = g(X,\phi Y)$, for all vector fields $X, Y \in \Gamma(TM)$.

By substituting $W = \xi$ in above equation, we get

(2.14) $\epsilon(\nabla_X\phi)Y = -\eta(Y)\phi\nabla_X\xi + \epsilon g(\phi\nabla_X\xi, Y)\xi$.

Now, putting the value of $\nabla_X\xi$ in (2.14), we get

$$(\nabla_X\phi)Y = \alpha\{g(X,Y)\xi - \epsilon\eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \epsilon\eta(Y)\phi X\},$$

which shows the manifold is (ϵ)-trans-Sasakian manifold.

Example (2.1). Let us consider a 3-dimensional manifold $M = \{(x,y,z) \in$

$\mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.
Let $e_1 = e^z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$, $e_2 = e^z\frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$, which are linearly independent vector fields at each point of M. Define a semi-Reimannian metric g on M as

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \epsilon,$$

where $\epsilon = \pm 1$.
Let η be the 1-form defined by $\eta(Z) = \epsilon g(Z, e_3)$, for any $Z \in \Gamma(TM)$ and ϕ be the tensor field of type (1,1) defined by $\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$. Then by applying linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi U) = g(Z, U) - \epsilon\eta(Z)\eta(U),$$

for any $Z, U \in \Gamma(TM)$. Hence for $e_3 = \xi$, $(\phi, \xi, \eta, g, \epsilon)$ defines an (ϵ)-almost contact metric structure on M.
Let ∇ be the Levi-Civita connection with respect to g and R be the curvature tensor of type (1,3), then we have

$$[e_1, e_2] = \epsilon\,(ye^z e_2 - e^{2z}e_3), \quad [e_1, e_3] = -\epsilon e_1, \quad [e_2, e_3] = -\epsilon e_2.$$

By using Koszul's formula for the Levi-Civita connection with respect to g, we obtain

$$\nabla_{e_1} e_3 = -\epsilon e_1 + \tfrac{1}{2}\epsilon e^{2z} e_2, \quad \nabla_{e_2} e_3 = -\epsilon e_2 - \tfrac{1}{2}\epsilon e^{2z} e_1, \quad \nabla_{e_3} e_3 = 0,$$
$$\nabla_{e_1} e_2 = -\tfrac{1}{2}\epsilon e^{2z} e_3, \quad \nabla_{e_2} e_2 = \epsilon e_3 + \epsilon y e^z e_1, \quad \nabla_{e_3} e_2 = -\tfrac{1}{2}\epsilon e^{2z} e_1,$$
$$\nabla_{e_1} e_1 = \epsilon e_3, \quad \nabla_{e_2} e_1 = -\epsilon y e^z e_2 + \tfrac{1}{2}\epsilon e^{2z} e_3, \quad \nabla_{e_3} e_1 = \tfrac{1}{2}\epsilon e^{2z} e_2.$$

Now, for $\xi = e_3$, above results satisfy

$$\nabla_X \xi = \epsilon\,\{-\alpha\phi X + \beta(X - \eta(X)\xi)\},$$

with $\alpha = -\frac{1}{2}e^{2z}$ and $\beta = -1$. Consequently $M(\phi, \xi, \eta, g, \epsilon)$ is a 3-dimensional (ϵ)-trans-Sasakian manifold.

Lemma (2.2). Let M be an (ϵ)-trans-Sasakian manifold. Then, we have

$$(2.15) \qquad (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta\,\{g(X, Y) - \epsilon\eta(X)\eta(Y)\},$$

for any $X, Y \in \Gamma(TM)$.

Proof. By using (2.2), we have

$$(2.16) \qquad \begin{aligned}(\nabla_X \eta)Y &= \nabla_X \eta(Y) - \eta(\nabla_X Y)\\ &= \nabla_X\,(\epsilon g(Y, \xi)) - \epsilon g\,(\nabla_X Y, \xi)\\ &= \epsilon g(Y, \nabla_X \xi).\end{aligned}$$

By substituting (2.9) in (2.16), we obtain

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta\,\{g(X, Y) - \epsilon\eta(X)\eta(Y)\}\,.$$

Lemma (2.3). In an (ϵ)-trans-Sasakian manifold,

$$(2.17) \qquad \begin{aligned}R(X, Y)\xi = {}&(\alpha^2 - \beta^2)\,\{\eta(Y)X - \eta(X)Y\}\\ &+2\alpha\beta\,\{\eta(Y)\phi X - \eta(X)\phi Y\}\\ &+\epsilon\,\{(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y\}\,,\end{aligned}$$

$$(2.18) \qquad R(\xi, Y)X = (\alpha^2 - \beta^2)\,\{\epsilon g(X, Y)\xi - \eta(X)Y\}$$

$$+2\alpha\beta\left\{\epsilon g(\phi X,Y)\xi+\eta(X)\phi Y\right\}$$
$$+\epsilon(X\alpha)\phi Y+\epsilon g(\phi X,Y)(grad\alpha)$$
$$-\epsilon g(\phi X,\phi Y)(grad\beta)$$
$$+\epsilon(X\beta)\left\{Y-\eta(Y)\xi\right\},$$

(2.19) $$R(\xi,Y)\xi=\left\{\alpha^2-\beta^2-\epsilon(\xi\beta)\right\}\left\{-Y+\eta(Y)\xi\right\} -\left\{2\alpha\beta+\epsilon(\xi\alpha)\right\}(\phi Y),$$

(2.20) $$2\alpha\beta+\epsilon(\xi\alpha)=0,$$

for any $X,Y\in\Gamma(TM)$.

Proof. By using (2.1) and (2.9), we have

$$R(X,Y)\xi=\nabla_X\nabla_Y\xi-\nabla_Y\nabla_X\xi-\nabla_{[X,Y]}\xi$$
$$=-\epsilon(X\alpha)\phi Y+\alpha^2\left\{\eta(Y)X-\eta(X)Y\right\}$$
$$-\beta^2\left\{\eta(Y)X-\eta(X)Y\right\}+2\alpha\beta\left\{\eta(Y)\phi X-\eta(X)\phi Y\right\}$$
$$+\epsilon(X\beta)\left\{Y-\eta(Y)\xi\right\}+\epsilon(Y\alpha)\phi X-\epsilon(Y\beta)\left\{X-\eta(X)\xi\right\}.$$
$$=(\alpha^2-\beta^2)\left\{\eta(Y)X-\eta(X)Y\right\}+2\alpha\beta\left\{\eta(Y)\phi X-\eta(X)\phi Y\right\}$$
$$+\epsilon\left\{(Y\alpha)\phi X-(X\alpha)\phi Y+(Y\beta)\phi^2X-(X\beta)\phi^2Y\right\}.$$

By using equation (2.17) in $g\left(R(\xi,Y)X,Z\right)=g\left(R(X,Z)\xi,Y\right)$, we obtain (2.18).

The equation (2.19) follows from (2.17) on replacing X by ξ.

Now, putting $X=\xi$ in (2.18), we get

(2.21) $$R(\xi,Y)\xi=\left\{\alpha^2-\beta^2-\epsilon(\xi\beta)\right\}\left\{-Y+\eta(Y)\xi\right\}+\left\{2\alpha\beta+\epsilon(\xi\alpha)\right\}(\phi Y).$$

In view of equations (2.19) and (2.21), we have

$$2\alpha\beta+\epsilon(\xi\alpha)=0,$$

which shows that α and β are not arbitrary functions but related to each other with structure vector field ξ.

If α is constant, then from (2.20), we have $\beta=0$. Hence we can state the following:

Corollary (2.4). An (ϵ)-trans-Sasakian manifold of type (α,β) with a non-zero constant α is always (ϵ)-α-Sasakian manifold.

Lemma (2.5). In an (ϵ)-trans-Sasakian manifold of type (α,β), if $\phi(grad\alpha)=(2n-1)(grad\beta)$, then we have

(2.22) $$\xi\beta=0,$$

Proof. We know that

$$X\beta=g(X,grad\beta)$$
$$=g\left(X,\frac{\phi(grad\alpha)}{2n-1}\right)$$
$$=-g\left(\phi X,\frac{(grad\alpha)}{2n-1}\right)$$
$$=-\frac{1}{2n-1}g(\phi X,grad\alpha),$$

which implies

$$(2n-1)(X\beta)+(\phi X)\alpha=0.$$

Similarly, we have

$$(2n-1)(Y\beta)+(\phi Y)\alpha=0.$$

On putting $Y=\xi$ in above equation, we obtain $\xi\beta=0$.

Lemma (2.6). Let M be a $(2n+1)$-dimensional (ϵ)-trans-Sasakian manifold. Then we have

$$\begin{aligned}(2.23)\qquad \eta(R(X,Y)Z)=\epsilon(\alpha^2-\beta^2)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\\ +2\epsilon\alpha\beta\{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)\}\\ +\{(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)\}\\ +\{(X\beta)g(\phi^2 Y,Z)-(Y\beta)g(\phi^2 X,Z)\},\end{aligned}$$

$$(2.24)\qquad \eta(R(X,Y)\xi)=0,$$

for any $X,Y,Z\in\Gamma(TM)$.

Proof. By using (2.17), we have

$$\begin{aligned}\eta(R(X,Y)Z)&=\epsilon\,(g(R(X,Y)Z,\xi)\\ &=-\epsilon\,(g(R(X,Y)\xi,Z)\\ &=\epsilon(\alpha^2-\beta^2)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\\ &\quad+2\epsilon\alpha\beta\{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)\}\\ &\quad+\{(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)\}\\ &\quad+\{(X\beta)g(\phi^2 Y,Z)-(Y\beta)g(\phi^2 X,Z)\}.\end{aligned}$$

Replacing Z by ξ in above equation, we have (2.24).

Remark. For an (ϵ)-trans-Sasakian manifold, using (2.2), (2.20) and (2.21) in equation (2.23), we obtain

$$(2.25)\qquad \epsilon\eta(R(\xi,Y)X)=(\alpha^2-\beta^2)\{g(X,Y)-\epsilon\eta(X)\eta(Y)\}.$$

Lemma (2.7). In a $(2n+1)$-dimensional (ϵ)-trans-Sasakian manifold,

$$(2.26)\qquad S(X,\xi)=\{2n(\alpha^2-\beta^2)-\epsilon(\xi\beta)\}\eta(X)-\epsilon(\phi X)\alpha-\epsilon(2n-1)(X\beta),$$

$$(2.27)\qquad Q\xi=\epsilon\left[\{2n(\alpha^2-\beta^2)-\epsilon\xi\beta\}\xi+\phi(grad\alpha)-(2n-1)(grad\beta)\right],$$

for any $X\in\Gamma(TM)$.

Proof. By using Lemma (2.6), we have

$$\begin{aligned}\epsilon g(R(X,Y)Z,\xi)&=\eta(R(X,Y)Z)\\ &=\epsilon(\alpha^2-\beta^2)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\\ &\quad+2\epsilon\alpha\beta\{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)\}\\ &\quad+\{(X\alpha)g(\phi Y,Z)-(Y\alpha)g(\phi X,Z)\}\\ &\quad+\{(X\beta)g(\phi^2 Y,Z)-(Y\beta)g(\phi^2 X,Z)\}.\end{aligned}$$

On putting $Y=Z=e_i$ and taking summation over $i, 1\le i\le 2n+1$, where $\{e_i\}$ is an orthonormal basis of tangent space at each point of the manifold, we obtain

$$S(X,\xi)=\{2n(\alpha^2-\beta^2)-\epsilon(\xi\beta)\}\eta(X)-\epsilon(\phi X)\alpha-\epsilon(2n-1)(X\beta).$$

The equation (2.27) follows from (2.26).

If $(2n-1)(grad\beta) - \phi(grad\alpha) = (2n-1)(\xi\beta)\xi$, then (2.26) and (2.27) respectively reduce to

(2.28) $$S(X,\xi) = 2n\left\{\alpha^2 - \beta^2 - \epsilon\xi\beta\right\}\eta(X)$$

and

(2.29) $$Q\xi = 2n\epsilon\left\{\alpha^2 - \beta^2 - \epsilon\xi\beta\right\}\xi.$$

Again, if $\phi(grad\alpha) = (2n-1)(grad\beta)$, then we have $\xi\beta = 0$. In this case, we obtain

(2.30) $$S(X,\xi) = 2n(\alpha^2 - \beta^2)\eta(X)$$

and

(2.31) $$Q\xi = 2n\epsilon(\alpha^2 - \beta^2)\xi.$$

3 Conformally flat (ϵ)-trans-Sasakian manifolds

In this section, we study conformally flat (ϵ)-trans-Sasakian manifolds.

At first we recall that the Weyl-conformal curvature tensor C of a $(2n+1)$-dimensional $(n > 1)$ Riemannian manifold is given by

(3.1) $$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right] + \frac{r}{2n(2n-1)}\left[g(Y,Z)X - g(X,Z)Y\right].$$

If the manifold is conformally flat, i.e. $C = 0$, then from (3.1), we obtain

(3.2) $$R(X,Y)Z = \frac{1}{2n-1}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right] - \frac{r}{2n(2n-1)}\left[g(Y,Z)X - g(X,Z)Y\right].$$

Now, taking scalar product on both sides of above equation with W, we have

(3.3) $$g(R(X,Y)Z,W) = \frac{1}{2n-1}\left[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\right] - \frac{r}{2n(2n-1)}\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right].$$

On putting $W = \xi$, the above equation becomes

(3.4) $$g(R(X,Y)Z,\xi) = \frac{1}{2n-1}\left[\epsilon S(Y,Z)\eta(X) - \epsilon S(X,Z)\eta(Y) + g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi)\right] - \frac{r}{2n(2n-1)}\left[\epsilon g(Y,Z)\eta(X) - \epsilon g(X,Z)\eta(Y)\right].$$

If $(2n-1)grad\beta - \phi(grad\alpha) = (2n-1)(\xi\beta)\xi$, then using (2.28), the equation (3.4) gives

(3.5) $$\epsilon\eta(R(X,Y)Z) = \frac{1}{2n-1}[\epsilon S(Y,Z)\eta(X) - \epsilon S(X,Z)\eta(Y) + 2n(\alpha^2 - \beta^2 - \epsilon\xi\beta)g(Y,Z)\eta(X) - 2n(\alpha^2 - \beta^2 - \epsilon\xi\beta)g(X,Z)\eta(Y)] - \frac{r\epsilon}{2n(2n-1)}\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right].$$

Again, taking $X = \xi$ in (3.5) and using (2.25), we obtain

$(\alpha^2-\beta^2)\{g(Y,Z)-\epsilon\eta(Y)\eta(Z)\} = \frac{\epsilon}{2n-1}S(Y,Z)+\frac{1}{2n-1}[2n(\alpha^2-\beta^2-\epsilon\xi\beta)-\frac{r\epsilon}{2n}]g(Y,Z)$

$$-\frac{1}{2n-1}[2\epsilon\{2n(\alpha^2-\beta^2-\epsilon\xi\beta)\}-\frac{r\epsilon}{2n}]\eta(Y)\eta(Z),$$

which on simplification gives

$$S(Y,Z)=\left\{\frac{r}{2n}-\epsilon(\alpha^2-\beta^2)+2n\xi\beta\right\}g(Y,Z) - \left\{\frac{r\epsilon}{2n}-(2n+1)(\alpha^2-\beta^2)+4n\epsilon\xi\beta\right\}\eta(Y)\eta(Z). \tag{3.6}$$

Thus we can state the following theorem:

Theorem (3.1). A conformally flat $(2n+1)$-dimensional (ϵ)-trans-Sasakian manifold is η-Eienstein manifold, if $(2n-1)grad\beta-\phi(grad\alpha)=(2n-1)(\xi\beta)\xi$.

Now, applying (3.6) in (3.3), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) = \frac{1}{2n-1}[&\left\{\frac{r}{2n}-\epsilon(\alpha^2-\beta^2)+2n\xi\beta\right\}g(X,W)g(Y,Z)\\ &-\left\{\frac{r\epsilon}{2n}-(2n+1)(\alpha^2-\beta^2)+4n\epsilon\xi\beta\right\}g(X,W)\eta(Y)\eta(Z)\\ &-\left\{\frac{r}{2n}-\epsilon(\alpha^2-\beta^2)+2n\xi\beta\right\}g(Y,W)g(X,Z)\\ &+\left\{\frac{r\epsilon}{2n}-(2n+1)(\alpha^2-\beta^2)+4n\epsilon\xi\beta\right\}g(Y,W)\eta(X)\eta(Z)\\ &+\left\{\frac{r}{2n}-\epsilon(\alpha^2-\beta^2)+2n\xi\beta\right\}g(Y,Z)g(X,W)\\ &-\left\{\frac{r\epsilon}{2n}-(2n+1)(\alpha^2-\beta^2)+4n\epsilon\xi\beta\right\}g(Y,Z)\eta(X)\eta(W)\\ &-\left\{\frac{r}{2n}-\epsilon(\alpha^2-\beta^2)+2n\xi\beta\right\}g(X,Z)g(Y,W)\\ &+\left\{\frac{r\epsilon}{2n}-(2n+1)(\alpha^2-\beta^2)+4n\epsilon\xi\beta\right\}g(X,Z)\eta(Y)\eta(W)]\\ &-\frac{r}{2n(2n-1)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]. \end{aligned} \tag{3.7}$$

On simplification the above relation gives

$$\begin{aligned} &g(R(X,Y)Z,W)\\ &=\left\{\frac{r-4n(\epsilon(\alpha^2-\beta^2)-2n\xi\beta)}{2n(2n-1)}\right\}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ &+\left\{\frac{r\epsilon-2n((2n+1)(\alpha^2-\beta^2)+4n\epsilon\xi\beta)}{2n(2n-1)}\right\}[g(Y,W)\eta(X)\eta(Z)\\ &-g(X,W)\eta(Y)\eta(Z)+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)]. \end{aligned} \tag{3.8}$$

Thus in view of definition (2.2), we have the following:

Theorem (3.2). A $(2n+1)$-dimensional $(n>1)$ conformally flat (ϵ)-trans-Sasakian manifold is of quasi-constant curvature, if $(2n-1)grad\beta-\phi(grad\alpha)=(2n-1)(\xi\beta)\xi$.

Remark. In [6], it is proved that a manifold of quasi-constant curvature is a $\psi(F)_n$.

Hence in view of theorem (3.2), we have

Corollary (3.3). A conformally flat (ϵ)-trans-Sasakian manifold is a $\psi(F)_n$, if $(2n-1)grad\beta-\phi(grad\alpha)=(2n-1)(\xi\beta)\xi$.

4 Weyl-semi symmetric (ϵ)-trans-Sasakian manifolds

In this section, we study Weyl-semi symmetric (ϵ)-trans-Sasakian manifolds. Let M be a $(2n+1)$-dimensional (ϵ)-trans-Sasakian manifold which is Weyl-semi symmetric, i.e. $R(X,Y).C = 0$, where $R(X,Y)$ is the curvature operator and $C(X,Y)Z$ is Weyl-conformal curvature tensor.
Taking scalar product with ξ on both sides of equation (3.1), we have

$$g\left(C(X,Y)Z,\xi\right) = g\left(R(X,Y)Z,\xi\right) - \tfrac{1}{2n-1}[S(Y,Z)g(X,\xi) - S(X,Z)g(Y,\xi) + g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi)] + \tfrac{r}{2n(2n-1)}[g(Y,Z)g(X,\xi) - g(X,Z)g(Y,\xi)],$$

which implies

$$\epsilon\eta\left(C(X,Y)Z\right) = \epsilon\eta\left(R(X,Y)Z\right) - \tfrac{1}{2n-1}[\epsilon S(Y,Z)\eta(X) - \epsilon S(X,Z)\eta(Y) + g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi)] + \tfrac{r}{2n(2n-1)}[\epsilon g(Y,Z)\eta(X) - \epsilon g(X,Z)\eta(Y)]. \tag{4.1}$$

On putting $X = Y$, the equation (4.1) gives

$$\epsilon\eta\left(C(X,X)Z\right) = \epsilon\eta\left(R(X,X)Z\right) = 0. \tag{4.2}$$

On the other hand, by substituting $Z = \xi$ in equation (4.1), we get

$$\epsilon\eta\left(C(X,Y)\xi\right) = \epsilon\eta\left(R(X,Y)\xi\right) - \tfrac{1}{2n-1}[\epsilon S(Y,\xi)\eta(X) - \epsilon S(X,\xi)\eta(Y) + g(Y,\xi)S(X,\xi) - g(X,\xi)S(Y,\xi)] + \tfrac{r}{2n(2n-1)}[\epsilon g(Y,\xi)\eta(X) - \epsilon g(X,\xi)\eta(Y)]. \tag{4.3}$$

If $(2n-1)(grad\beta) - \phi(grad\alpha) = (2n-1)(\xi\beta)\xi$, then by using (2.2), (2.24), (2.28) in above equation, we get

$$\eta\left(C(X,Y)\xi\right) = 0. \tag{4.4}$$

Also, in above case, for $X = \xi$, in view of (2.25) and (2.28), the equation (4.1) gives

$$\eta\left(C(\xi,Y)Z\right) = \tfrac{1}{2n-1}[\left\{\tfrac{r}{2n} - (\epsilon(\alpha^2-\beta^2) - 2n\xi\beta)\right\}g(Y,Z) - \left\{\tfrac{r\epsilon}{2n} - (2n+1)(\alpha^2-\beta^2-\epsilon\xi\beta)\right\}\eta(Y)\eta(Z) - S(Y,Z)]. \tag{4.5}$$

Since the manifold is Weyl-semi symmetric, i.e. $R.C = 0$, we have

$$R(X,Y)C(U,V)W - C\left(R(X,Y)U,V\right)W - C\left(U,R(X,Y)V\right)W - C(U,V)R(X,Y)W = 0. \tag{4.6}$$

Taking scalar product with ξ on both sides of above equation and putting $X = \xi$, we have

$$g\left[R(\xi,Y)C(U,V)W,\xi\right] - g\left[C\left(R(\xi,Y)U,V\right)W,\xi\right] - g\left[C\left(U,R(\xi,Y)V\right)W,\xi\right] - g\left[C(U,V)R(\xi,Y)W,\xi\right] = 0. \tag{4.7}$$

Now, in view of (2.25), the expression (4.7) gives

$$(\alpha^2-\beta^2)\left[g\left(C(U,V)W,Y\right) - \epsilon\eta(Y)\eta\left(C(U,V)W\right)\right] \tag{4.8}$$

$$-(\alpha^2-\beta^2)\left[g(Y,U)\eta\left(C(\xi,V)W\right)-\epsilon\eta(U)\eta\left(C(Y,V)W\right)\right]$$
$$-(\alpha^2-\beta^2)\left[g(V,Y)\eta\left(C(U,\xi)W\right)-\epsilon\eta(V)\eta\left(C(U,Y)W\right)\right]$$
$$-(\alpha^2-\beta^2)\left[g(W,Y)\eta\left(C(U,V)\xi\right)-\epsilon\eta(W)\eta\left(C(U,V)Y\right)\right]=0.$$

The above expression can be written as

$$\widetilde{C}(U,V,W,Y)-\epsilon\eta(Y)\eta\left(C(U,V)W\right) -g(Y,U)\eta\left(C(\xi,V)W\right)+\epsilon\eta(U)\eta\left(C(Y,V)W\right) -g(Y,V)\eta\left(C(U,\xi)W\right)+\epsilon\eta(V)\eta\left(C(Y,U)W\right) -g(Y,W)\eta\left(C(U,V)\xi\right)+\epsilon\eta(W)\eta\left(C(U,V)Y\right)=0, \tag{4.9}$$

where $\widetilde{C}(U,V,W,Y)=g\left(C(U,V)W,Y\right)$.

On putting $Y=U$ in equation (4.9) and using (4.2), (4.4), we get

$$\widetilde{C}(U,V,W,U)-g(U,U)\eta\left(C(\xi,V)W\right) -g(U,V)\eta\left(C(U,\xi)W\right)+\epsilon\eta(W)\eta\left(C(U,V)U\right)=0. \tag{4.10}$$

Again, putting $U=e_i$ and taking summation over i, $1\le i\le 2n+1$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we have

$$\sum_{i=1}^{2n+1}[\widetilde{C}(e_i,V,W,e_i)-g(e_i,e_i)\eta\left(C(\xi,V)W\right) -g(e_i,V)\eta\left(C(e_i,\xi)W\right)+\epsilon\eta(W)\eta\left(C(e_i,V)e_i\right)]=0. \tag{4.11}$$

Now, by using (2.23) and (2.28) in (4.1), we obtain

$$\sum_{i=1}^{2n+1}\epsilon\eta\left(C(e_i,V)e_i\right)=\tfrac{2n}{2n-1}[(\alpha^2-\beta^2-2n\epsilon\xi\beta)-\epsilon(\alpha^2-\beta^2-\epsilon\xi\beta)]\eta(V), \tag{4.12}$$

$$\sum_{i=1}^{2n+1}g(e_i,V)\eta\left(C(e_i,\xi)W\right)=-\tfrac{1}{2n-1}[\left\{\tfrac{r}{2n}-(\epsilon(\alpha^2-\beta^2)-2n\xi\beta)\right\}g(V,W) -\{\tfrac{r\epsilon}{2n}-((\alpha^2-\beta^2)-2n\epsilon\xi\beta) -2n(\alpha^2-\beta^2-\epsilon\xi\beta)\}\eta(V)\eta(W)- S(V,W)]. \tag{4.13}$$

Also, from (4.5) we have

$$\eta\left(C(\xi,V)W\right)=\tfrac{1}{2n-1}[\left\{\tfrac{r}{2n}-(\epsilon(\alpha^2-\beta^2)-2n\xi\beta)\right\}g(V,W) -\{\tfrac{r\epsilon}{2n}-((\alpha^2-\beta^2)-2n\epsilon\xi\beta) -2n(\alpha^2-\beta^2-\epsilon\xi\beta)\}\eta(V)\eta(W)-S(V,W)]. \tag{4.14}$$

Hence, from (4.13) and (4.14) it follows that

$$\sum_{i=1}^{2n+1}g(e_i,V)\eta\left(C(e_i,\xi)W\right)=-\eta\left(C(\xi,V)W\right). \tag{4.15}$$

Therefore by substituting (4.12) and (4.15) in equation (4.11), we get

$$\sum_{i=1}^{2n+1}\widetilde{C}(e_i,V,W,e_i)-(2n+\epsilon-1)\,\eta\left(C(\xi,V)W\right) +\tfrac{2n}{2n-1}[(\alpha^2-\beta^2-2n\epsilon\xi\beta)-\epsilon(\alpha^2-\beta^2-\epsilon\xi\beta)]\eta(V)\eta(W)=0. \tag{4.16}$$

Now, from (3.1), we have

$$g\left(C(X,Y)Z,U\right)=g\left(R(X,Y)Z,U\right) \tag{4.17}$$

$$-\frac{1}{2n-1}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) + g(Y,Z)S(X,U) - g(X,Z)S(Y,U)] + \frac{r}{2n(2n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Putting $X = U = e_i$ in above equation and taking summation over $i, 1 \leq i \leq 2n+1$, we have

$$\sum_{i=1}^{2n+1} \widetilde{C}(e_i, V, W, e_i) = \frac{(1-\epsilon)}{2n-1}[S(V,W) + \{(2n-1)(\alpha^2-\beta^2) - \tfrac{r}{2n}\}g(V,W) - \{\epsilon(2n-1)(\alpha^2-\beta^2) - 4n\epsilon(\alpha^2 - \beta^2 - \epsilon\xi\beta) + \tfrac{r}{2n}\}\eta(V)\eta(W)] \tag{4.18}$$

Thus using (4.18),(4.14) in equation (4.16), we obtain

$$S(V,W) = \left[\tfrac{r}{2n} - \{\alpha^2 - \beta^2 - (2n+\epsilon-1)\xi\beta\}\right] g(V,W) - \left[\tfrac{r\epsilon}{2n} - (2n+\epsilon)\{\alpha^2 - \beta^2 - \epsilon\xi\beta\}\right]\eta(V)\eta(W). \tag{4.19}$$

Thus we can state the following:

Theorem (4.1). A Weyl-semi symmetric $(2n+1)$-dimensional (ϵ)-trans-Sasakian manifold is an η-Eienstein manifold, if $(2n-1)grad\beta - \phi(grad\alpha) = (2n-1)(\xi\beta)\xi$.

Now, if $\phi(grad\alpha) = (2n-1)(grad\beta)$, then in view of (2.22), the equation (4.19) becomes

$$S(V,W) = \left\{\tfrac{r}{2n} - (\alpha^2 - \beta^2)\right\} g(V,W) - \left\{\tfrac{r\epsilon}{2n} - (2n+\epsilon)(\alpha^2-\beta^2)\right\}\eta(V)\eta(W). \tag{4.20}$$

Also in this case, using (2.23) and (2.30) in equation (4.1), we obtain

$$\eta(C(X,Y)Z) = \frac{1}{2n-1}[\left\{\tfrac{r}{2n} - \epsilon(\alpha^2-\beta^2)\right\}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} - \{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}]. \tag{4.21}$$

On putting $X = \xi$ in (4.21), we get

$$\eta(C(\xi,Y)Z) = \frac{1}{2n-1}[\left\{\tfrac{r}{2n} - \epsilon(\alpha^2-\beta^2)\right\}\{g(Y,Z) - \epsilon\eta(Y)\eta(Z)\} - \{S(Y,Z) - S(\xi,Z)\eta(Y)\}]. \tag{4.22}$$

Again, putting $Y = \xi$ in (4.21), we obtain

$$\eta(C(X,\xi)Z) = \frac{1}{2n-1}[\left\{\tfrac{r}{2n} - \epsilon(\alpha^2-\beta^2)\right\}\{\epsilon\eta(X)\eta(Z) - g(X,Z)\} - \{S(\xi,Z)\eta(X) - S(X,Z)\}]. \tag{4.23}$$

On the other hand, from the expression (4.9), we have

$$\widetilde{C}(U,V,W,Y) = \epsilon\eta(Y)\eta(C(U,V)W) + g(Y,U)\eta(C(\xi,V)W) - \epsilon\eta(U)\eta(C(Y,V)W) + g(Y,V)\eta(C(U,\xi)W) - \epsilon\eta(V)\eta(C(Y,U)W) - \epsilon\eta(W)\eta(C(U,V)Y). \tag{4.24}$$

In view of (4.21), (4.22) and (4.23) ,the expression (4.24) becomes

$$\widetilde{C}(U,V,W,Y) = \frac{1}{2n-1}[\left\{\tfrac{r}{2n} - \epsilon(\alpha^2-\beta^2)\right\} g(V,W)g(Y,U) \tag{4.25}$$

$$+2n(\alpha^2-\beta^2)g(Y,U)\eta(V)\eta(W)-$$
$g(Y,U)S(V,W)$
$$-\left\{\tfrac{r}{2n}-\epsilon(\alpha^2-\beta^2)\right\}g(U,W)g(Y,V)$$
$$-2n(\alpha^2-\beta^2)g(Y,V)\eta(U)\eta(W)+$$
$g(Y,V)S(U,W)$
$$+\epsilon S(Y,V)\eta(U)\eta(W)-\epsilon S(Y,U)\eta(V)\eta(W)].$$

Now, using (4.20) in equation (4.25) and simplifying, we obtain

(4.26) $\widetilde{C}(U,V,W,Y)=\frac{(1-\epsilon)(\alpha^2-\beta^2)}{2n-1}\{g(Y,U)g(V,W)-g(Y,V)g(U,W)\}.$

Hence we have the following theorem:

Theorem (4.2). In a $(2n+1)$-dimensional $(n>1)$ Weyl-semi-symmetric (ϵ)-trans-Sasakian manifold, if $\phi(grad\alpha)=(2n-1)(grad\beta)$, then

$$\widetilde{C}(U,V,W,Y)=\frac{(1-\epsilon)(\alpha^2-\beta^2)}{2n-1}\{g(Y,U)g(V,W)-g(Y,V)g(U,W)\}.$$

From (4.26), we observe that if $\epsilon=1$, then $R.C=0$ implies $C=0$. Thus in view of theorem (3.2), we conclude the following :

Corollary (4.3). A $(2n+1)$-dimensional $(n>1)$ Weyl-semi-symmetric trans-Sasakian manifold is of quasi constant curvature, if $\phi(grad\alpha)=(2n-1)(grad\beta)$.

References

[1] A.Bejancu and K.L.Duggal, Real hypersurfaces of indefinite Kaehler manifolds, *Int.J.Math.and Math.Sci.*, 16 no.3 (1993), 545-556.

[2] D.E.Blair, Contact manifolds in Riemannian Geometry, *Lecture notes in Math. Springer Verlag*, (1976) 509.

[3] B.Y.Chen and K.Yano, Hypersurfaces of a conformally flat space, *tensor, N.S.*, 26 (1972), 318-322.

[4] S.S.Chern, On the curvature and characteristic classes of a Riemannian manifold, *Abh.Math.Sem.Univ.Hamburg*, 20 (1956), 117-126.

[5] U.C.De and M.M.Tripathi, Ricci tensor in 3-dimensional trans-Sasakian manifolds, *Kyungpook Math.J.*, 43 (2003), 247-255.

[6] U.C.De and Avijit Sarkar, On (ϵ)- Kenmotsu manifolds, *Hadronic Journal*, 32 (2009), 231-242.

[7] K.L.Duggal, Spacetime manifold and contact structures, *Int.J.Math. and Math.Sci.*, 16, No.3 (1990), 545-553.

[8] K.L.Duggal and R.Sharma, Symmetries of space time and Riemannian manifolds, *Kluwer Academic Publishers Drodrecht-Boston-London.*

[9] K.L.Duggal and B.Sahin, Lightlike submanifolds of indefinite Sasakian manifolds, *Int.J.Math. and Math.Sci.*, 2007, Art.ID 57585, 21pp.

[10] C.Gherghe, Harmonicity on nearly trans-Sasakian manifolds, *Demonstratio Math.*, 33 (2000), 151-157.

[11] K.Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Math.J.*, 24 (1972), 93-103.

[12] R.Kumar, R.Rani and R.K.Nagaich, On sectional curvature of (ϵ)-Sasakian manifolds, *Int.J.Math. and Math.Sci.*, (2007), ID 93562.

[13] J.C.Marrero, The local structure of trans-Sasakian manifolds, *Ann.Mat.Pure Appl.*, 162(4), (1992), 77-86.

[14] Z.Olszak, Normal almost contact metric manifolds of dimension three, *Annals Polonica Mathematici*, XLVII, (1986), 41-50.

[15] B.O'Neill, Semi-Riemannian geometry with applications to relativity, *Academic Press* 1983.

[16] J.A.Oubiña, New classes of almost contact metric structures, *Pub.Math.Debrecen*, 32 (1985), 187-193.

[17] R.Prasad, Pankaj, M.M.Tripathi and S.S.Shukla, On some special stype of trans-Sasakian manifolds, *Riv.Mat.Uni.Parma (8) 2* (2009), 1-17.

[18] A.A.Shaikh and S.K.Hui, On weak symmetries of trans-Sasakian manifolds, *Proceedings of the Estonian Academy of Sciences*, (2009), 58, 4, 213-223.

[19] T.Takahashi, Sasakian ϕ-symmetric spaces, *Tohoku Math.J.*, 29 (1977), 91-113.

[20] M.M.Tripathi, Trans-Sasakian manifolds are generalized quasi-Sasakian, *Nepali Math. Sci. Rep.*, 18 (2001), no.1-2, 11-14.

[21] X.Xufeng and C.Xiaoli, Two theorems on (ϵ)-Sasakian manifolds, *Int.J. Math. and Math.Sci.*, Vol.21 No.2 (1998), 249-254.

[22] K.Yano and M.Kon, Structures on Manifolds, *Series in Pure Math.*, 3, *World Scientific, Singapore*, (1984).

Received: July, 2010