ON TRANSIENT MARKOV PROCESSES WITH A COUNTABLE NUMBER OF STATES AND STATIONARY TRANSITION PROBABILITIES[1]

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1. Summary. We consider a Markov process $x_0, x_1, \cdots$ with a countable set S of states and stationary transition probabilities $p(t \mid s) = P\{x_{n+1} = t \mid x_n = s\}$. Call a set C of states *almost closed* if (a) $P\{x_n \varepsilon C$ for an infinite number of $n\} > 0$ and (b) $x_n \varepsilon C$ infinitely often implies $x_n \varepsilon C$ for all sufficiently large n, with probability one. It is shown that there is a set $(C_1, C_2, \cdots)$ essentially unique, of disjoint almost closed sets such that (a) all except at most one of the C_i are atomic, that is, C_i does not contain two disjoint almost closed subsets, (b) the non-atomic C_i, if present, contains no atomic subsets, (c) the process is certain to enter and remain in some set C_i. A relation between the sets C_i and the bounded solutions of the system of equations

$$\alpha(s) = \sum_t \alpha(t)p(t \mid s) \tag{1}$$

is obtained; in particular there is only one atomic C_i and no non-atomic C_i if and only if the only bounded solution of (1) is $\alpha(t) =$ constant. This condition is shown to hold if the process is the sum of independent identical (numerical or vector) variables; whence, for such a process, the probability of entering a set J infinitely often is zero or one. The results are new only if the process has transient components. The main tool is the martingale convergence theorem.

2. The structure theorem.

Theorem 1. *Let $x_0, x_1, \cdots$ be a Markov process with a countable set S of states (we restrict S to those states with a positive probability of being entered) and stationary transition probabilities. For any subset I of S, denote by $L(I)$, $U(I)$ the events* lim inf $\{x_n \varepsilon I\}$, lim sup $\{x_n \varepsilon I\}$ *respectively, by $\mathfrak{M}$ the class of I with $P(U(I)) = 0$, and by $\mathcal{C}$ the class of I with $L(I) = U(I)$ a.e. If $C \varepsilon \mathcal{C}$ and $C \not\varepsilon \mathfrak{M}$, C will be called almost closed.*

(1) *Call a Borel measurable function f on the space Ω of all infinite sequences $\omega = (x_0, x_1, \cdots)$, $x_n \varepsilon S$, invariant if for every ω, $f(\omega) = f(T\omega)$, where $T(x_0, x_1, \cdots) = (x_1, x_2, \cdots)$, and call an event* invariant *if its characteristic function is invariant (so that, for any $J \subset S$, $L(J)$ and $U(J)$ are invariant). For any invariant event V there is a $C \varepsilon \mathcal{C}$ such that $U(C) = V$ a.e., so that the Borel field of invariant events is identical, up to events of probability zero, with the (Borel) field $\mathfrak{D}$ of events of the form $D = U(J)$* a.e., $J \subset S$.

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(2) *There is a finite or countable collection* $\{C_1, C_2, \cdots\}$ *of disjoint almost closed sets with the following properties:*

(a) *every* C_i *except at most one is atomic, that is, does not contain two disjoint almost closed subsets,*

(b) *the non-atomic* C_i, *if present, contains no atomic subsets,*

(c) $\sum_i P(L(C_i)) = 1$.

The collection $\{C_1, C_2, \cdots\}$ *is essentially unique, that is, if* $\{C_1', C_2', \cdots\}$ *has properties (a), (b), (c), then each* C_i' *differs from some* C_j *by a set in* $\mathfrak{M}$.

The C_i *may be chosen so that all states in a given* C_i *are of the same type: either all states s in* C_i *are transient, that is,* $\{s\} \varepsilon \mathfrak{M}$ *or all states in* C_i *are recurrent, that is, nontransient. The non-atomic* C_i, *if present, consists entirely of transient states.*

PROOF. We represent $x_0, x_1, \cdots$, as usual, as coordinate variables on the space Ω. For any bounded measurable f on Ω, we have

$$E(T^n f \mid x_0 = s_0, \cdots, x_n = s_n) = E(f \mid x_0 = s_n), \tag{4}$$

where $T^n f$ denotes the function g defined by $g(\omega) = f(T^n\omega)$. For, since $T^n f$ depends only on $x_n, x_{n+1}, \cdots$, the left side of (4) equals $E(T^n f \mid x_n = s_n)$ which, because of the stationarity of transition probabilities, is easily shown to equal $E(f \mid x_0 = s_n)$. Thus if f is invariant,

$$E(f \mid x_0 = s_0, \cdots, x_n = s_n) = E(f \mid x_0 = s_n). \tag{5}$$

For any invariant event V and any state s, define $\phi(s) = E(v \mid x_0 = s)$, where v is the characteristic function of V. From the forward martingale convergence theorem ([2], p. 319) and (5), $\phi(x_n) \to v$ with probability one as $n \to \infty$. Thus if I is the set of all states s with $\phi(s) > \frac{1}{2}$, $\omega \varepsilon V$ implies $\omega \varepsilon L(I)$ with probability one, while $\omega \not\varepsilon V$ implies $\omega \not\varepsilon U(I)$ with probability one, that is, $U(I) \subset V \subset L(I)$ a.e. Since always $U(I) \supset L(I)$, we have $I \varepsilon \mathfrak{C}$ and $U(I) = V$ a.e. This establishes part (1) of the theorem.

For part (2), we decompose the measure P on the Borel field of invariant sets into atoms and a completely non-atomic part (see for instance [1], p. 565). Let $V_1, V_2, \cdots$ be the sets of this decomposition, and choose $I_n \varepsilon \mathfrak{C}$ such that $U(I_n) = V_n$ a.e. Since V_i, V_j are disjoint for $i \neq j$, $I_i \cap I_j \varepsilon \mathfrak{M}$ for $i \neq j$, so that, with $C_n = I_n - \bigcup_{j<n} I_j$, $C_n \varepsilon \mathfrak{C}$ $U(C_n) = V_n$ a.e., and $C_1, C_2, \cdots$ are disjoint. Properties (a), (b), (c) of part (2) and the essential uniqueness are immediate. The final assertion of part (1) is a consequence of the known ([3], p. 322) facts that if $\{s\} \not\varepsilon \mathfrak{M}$, that is, if it is possible for the process to enter s infinitely often, then if the system ever enters s it is certain to enter infinitely often s and all states which can be entered from s. The latter class C of states is almost closed (in fact is closed, that is, if the system ever enters C it remains in C), consists entirely of recurrent states, and has no almost closed proper subsets. This completes the proof of the theorem.

Any collection of sets $\{C_1, C_2, \cdots\}$ of the form described in part (2) of Theorem 1 will be called a *decomposition* of the Markov process. A process will be called *simple* if its decomposition consists of a single set C_1, and a simple process will be called non-atomic or atomic according to the type of C_1.

3. The equation $\alpha(s) = \sum \alpha(t)p(t \mid s)$. For a process of the type considered in Section 1, write $p(t \mid s) = P\{x_{n+1} = t \mid x_n = s\}$, so that $p(t \mid s) \geqq 0$, $\sum_t p(t \mid s) = 1$ for all s. The structure of the process is closely related to the nature of the bounded solutions of the equation

$$\alpha(s) = \sum_t \alpha(t)p(t \mid s). \tag{6}$$

This relation is most simply stated in terms of the bounded invariant functions of the process, as follows:

THEOREM 2. *For any bounded invariant function f, $\alpha(s) = E(f \mid x_0 = s)$ satisfies* (6), *and every bounded solution of* (6) *may be represented in this form.*

PROOF. That any $\alpha(s) = E(f \mid x_0 = s)$ satisfies (6) is clear, since $E(f \mid x_0 = s, x_1 = t) = \alpha(t)$ from (5), and (6) then follows from the formula $E(f \mid x_0) = E(E(f \mid x_0, x_1) \mid x_0)$. Conversely if $\alpha(t)$ is any bounded solution of (6), the sequence $z_n = \alpha(x_n)$ is a bounded martingale and hence converges to a limiting bounded f. Since $z_n(T\omega) = z_{n+1}(\omega)$, $f(T\omega) = (f\omega)$ a.e., that is, f is invariant. The martingale convergence theorem also yields $E(f \mid x_0) = z_0 = \alpha(x_0)$, so that the solution $\alpha(t)$ has the required form.

The inequality

$$\alpha(s) \geqq \sum_t \alpha(t)p(t \mid s) \tag{6'}$$

has been studied by Kendall [7] and Foster [4], who related the existence of solutions of (6′) such that $\alpha(j) \to \infty$ as $j \to \infty$ (enumerating the states by the positive integers) to the existence of a finite closed set of states.

COROLLARY. *The process is simple and atomic if and only if the only bounded solution of* (6) *is $\alpha(t)$ = constant.*

PROOF. If the only bounded solution of (6) is constant, then any bounded invariant f is constant, since, with $\alpha(s) = E(f \mid x_0 = s)$, $\alpha(x_n) \to f$ a.e. as $n \to \infty$. Thus any invariant set has probability zero or one, and the process is simple and atomic. Conversely, if the process is simple and atomic, every invariant set has probability zero or one, every bounded invariant function f, being measurable with respect to the class of invariant sets, is constant a.e., so that every solution of (6), having the form $E(f \mid x_0 = s)$, is constant.

As an application of the corollary, we consider processes which are sums of independent identically distributed variables.

THEOREM 3. *Let V be a finite or countable set of vectors $v_1, v_2, \cdots$ in N-space, and let $p_1, p_2, \cdots$ be positive numbers with sum one. Let S consist of the origin and all vectors representable as $w_1 + \cdots + w_n$, $n = 1, 2, \cdots$, $w_j \varepsilon V$. The only bounded solution of the equation*

$$\alpha(S) = \sum_j \alpha(s + v_j)p_j, \qquad s \varepsilon S \tag{7}$$

is $\alpha(s) = $ *constant. Consequently if* $y_1, y_2, \cdots$ *are independent variables with* $P\{y_n = v_j\} = p_j$, *the Markov process* $x_k = y_1 + \cdots + y_k$, $x_0 = (0, \cdots, 0)$ *is simple and atomic.*

PROOF. Repeated use of (7) yields

$$\alpha(s) = \sum_{j_1, \cdots, j_k} \alpha(s + v_{j_1} + \cdots + v_{j_k})\, p_{j_1} \cdots p_{j_k} \tag{8}$$
$$= \sum_{r \varepsilon R_k} \alpha(s + r_1 v_1 + r_2 v_2 + \cdots) q_k(r),$$

where R_k consists of all sequences $r = (r_1, r_2, \cdots)$ of nonnegative integers whose sum is k, and

$$q_k(r) = k! \textstyle\prod_j (p_j^{r_j} / r_j\, !).$$

Replacing s by $s + v_1$ and k by $k - 1$ in (8) yields

$$\alpha(s + v_1) = \sum_{r \varepsilon R_{k-1}} \alpha(s + v_1 + r_1 v_1 + \cdots) q_{k-1}(r) \tag{9}$$
$$= \sum_{r \varepsilon R_k} \alpha(s + r_1 v_1 + r_2 v_2 + \cdots) q_k(r)(r_1/kp_1).$$

Subtracting (9) from (8) yields

$$\alpha(s) - \alpha(s + v_1) = \sum_{r \varepsilon R_k} f(r) q_k(r)(1 - (r_1/kp_1)), \tag{10}$$

where $f(r)$ is for fixed s and k a bounded function of r and is uniformly bounded in s, k as well, say $|f(r)| \leqq M$ for all s, k, r. For fixed $\epsilon > 0$, let T_k denote that subset of R_k for which $|1 - (r_1/kp_1)| < \epsilon$. Then

$$\sum_{r \not\varepsilon T_k} |\, f(r)\, |\, q_k(r)\, |\, 1 - (r_1/kp_1)\, | \leqq M p_1 \sum_{r \not\varepsilon T_k} q_k(r),$$

and the sum on the right, being the probability that, in k independent trials with an event of probability p_1, the actual success ratio differs from p_1 by at least ϵp_1, approaches zero as $k \to \infty$ by the law of large numbers. Since

$$\sum_{r \varepsilon T_k} |f(r)|\, q_k(r)\, |1 - (r_1/kp_1)| \leqq M\epsilon,$$

we find from (10), summing separately for $r \varepsilon T_k$ and $r \not\varepsilon T_k$ and letting $k \to \infty$, that $|\alpha(s) - \alpha(s + v_1)| \leqq M\epsilon$. Since ϵ is arbitrary, $\alpha(s) = \alpha(s + v_1)$. Clearly the same proof yields $\alpha(s) = \alpha(s + v_j)$ for any j, and $\alpha(s) = $ constant.

COROLLARY. *If* $y_1, y_2, \cdots$ *are independent identically distributed (vector or scalar) variables with a finite or countable set of values, for any set* J *the probability that an infinite number of sums* $y_1 + \cdots + y_k = x_k$ *are in* J *is zero or one.*

An example of a simple nonatomic process is $x_n = \frac{1}{2} + \sum_1^n \epsilon_y / 2^{y+1}$, where $\epsilon_1, \epsilon_2, \cdots$ are independent and assume the values ± 1 with probability $\frac{1}{2}$ each. The states are all rational numbers $m/2^n$, $m = 1, 3, \cdots, 2^n - 1$, $n = 1, 2, \cdots$. Writing $m/2^n = (m, n)$, the system (7) becomes

$$\alpha(m, n) = \tfrac{1}{2}(\alpha(2m - 1, n + 1) + \alpha(2m + 1, n + 1)). \tag{11}$$

If $x^* = \lim x_n$, x^* is a.e. invariant and has a uniform distribution on (0, 1). Each x_n is a function of x^* a.e., so that every function of $x_0, x_1, x_2, \cdots$ is a.e. a function of x^* and hence invariant a.e. For any bounded function $f(x^*)$,

$$E(f \mid x_0 = (m, n)) = 2^{n-1} \int_{(m-1)/2^n}^{(m+1)/2^n} f(t)\, dt = \alpha(m, n).$$

It is easily verified by substitution that $\alpha(m, n)$ satisfies (11).

The Corollary actually holds without the restriction to random variables with a finite or countable set of values; this is an immediate consequence of an unpublished theorem of Edwin Hewitt and L. J. Savage, which they communicated to the writer. The Hewitt-Savage Theorem, an improved version of the zero-one law, asserts that any event depending on a sequence of independent identically distributed random variables which is invariant under all permutations of every finite set of the random variables has probability zero or one; the event that an infinite number of sums $y_1 + \cdots + y_k$ are in J is clearly of this type. The conclusion of the Corollary, under different hypotheses on the y_i, has also been obtained by Chung and Derman in an unpublished manuscript which they communicated to the writer.

As a second application of the Corollary of Theorem 2, we obtain an interesting result of Foster [4], Harris [5], and Hodges and Rosenblatt [6] concerning the random walk on the nonnegative integers, with $p(0 \mid 0) = 1$, $p(i + 1 \mid i) = p_i$, $p(i - 1 \mid i) = q_i = 1 - p_i$, $0 < p_i < 1$, $i > 0$. The equation (6) becomes

$$\alpha(s) = p_s \alpha(s + 1) + q_s \alpha(s - 1), \qquad s > 0. \tag{12}$$

The general solution of (12) is $\alpha(s) = A + Bz_s$, where $z_0 = 0$, $z_1 = 1$, $z_s = c_1 + \cdots + c_{s-1}$ for $s > 1$, where $c_1 = 1/p_1$, $c_j = (q_1 \cdots q_j / p_1 \cdots p_j)$ for $j > 1$.

Thus (12) has a bounded nonconstant solution if and only if the series

$$\textstyle\sum_j (q_1 \cdots q_j) / (p_1 \cdots p_j)$$

converges, which is the condition obtained in [4], [5], and [6] for passage to the origin to be uncertain.

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