Geophys. J. R. astr. Soc. (1978) **52**, 397–432

On transient sliding motion

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Received 1977 July 12

Summary. Two kinds of transient sliding motion under a case of idealized dry friction are studied. One concerns uni-directional slip at constant propagation velocity along a strip of constant length in the propagation direction. The other regards extensional slip along a strip expanding symmetrically with constant velocity. The former kind involves one leading and one trailing edge, whereas the latter involves two leading edges. At a leading edge there must be a region of tearing, where sliding is initiated, and at a trailing edge a region of healing, where sliding ceases. The finiteness of these regions follows from the requirement of bounded strains. In the linearized treatment chosen, the edge processes are described by a modulus of tearing and a modulus of healing, both being characteristics of the material. Relations between the applied remote stress, the extension of the sliding region, the amount of slip, the slip propagation velocity and the rate of energy dissipation are given.

1. Introduction

As is well known, forces above a certain level must be applied to initiate a sliding motion between two bodies in contact. However, sliding cannot start *at once* over the whole interface. It ought to be initiated in some small region and then propagate in a wave manner along the two mating surfaces. Such a motion is here referred to as a *transient sliding motion*, as opposed to sliding at constant velocity of the two bodies.

Two different kinds of transient sliding motion will be studied. The first one refers to a fixed grip situation. It is schematically illustrated by the model shown in Fig. 1. The two bodies are assumed to be of equal length in the unloaded state. The left ends are displaced a distance $2\Delta u$ from each other. This displacement has been accompanied by a slip along the interface, from the left end to a point P, where some obstacle to sliding motion is assumed. Thus, to the right of P no slip has occurred. The obstacle P is now removed. Then a wave pulse of sliding motion propagates from P to the right ends of the bodies. It is assumed that the bodies are long enough so that a steady wave pulse develops. This kind of sliding motion will be referred to as *uni-directional slip propagation*. However, in the study the two bodies are taken to be semi-infinite.

The other kind of transient sliding motion to be studied is schematically illustrated by the model shown in Fig. 2. The two bodies are loaded by tangential loads, uniformly

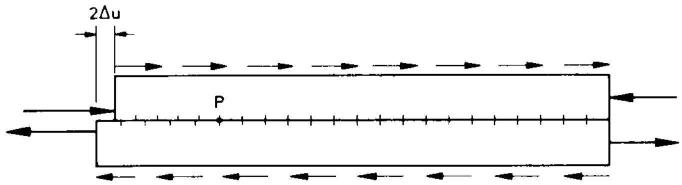

Figure 1. Model illustrating schematically uni-directional slip propagation. When the obstacle P is removed a slip $2\Delta u$ propagates to the right.

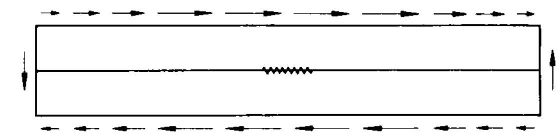

Figure 2. Model illustrating schematically extensional slip propagation. Slip is assumed to start at a central portion and spread symmetrically towards the ends.

distributed except towards the ends, where the loads are small. It is assumed that the 'static friction' in a small central portion is lower than the otherwise constant 'static friction', but larger than the dynamic friction, which is assumed to be constant over the whole contact area. Then, as the magnitude of the loads is increased, a sliding motion will eventually start at some point in the central portion and spread in both directions towards the ends. It is assumed that this motion is antisymmetric with respect to a point in the region of initiation. Further it is assumed that a constant propagation velocity is approached after an initial acceleration period. This kind of sliding motion will be referred to as *extensional slip propagation*. In the study the two bodies are taken to be semi-infinite.

The two cases differ in the respect that the first one concerns one leading edge where *tearing* occurs and one trailing edge where *healing* takes place, whereas the second case concerns two leading edges. Another difference consists in the amount of slip that can be transmitted. In the first case it is limited to a fixed quantity, given by the constraints. In the second case it increases indefinitely with time.

The conditions of the mating surfaces influence the physical mechanism of sliding. The interface is seldom clean – often a layer of contamination or of abraded particles is present. The implications of different conditions of the mating surfaces on the mechanism of sliding can be very substantial. The present study is restricted to an idealized case of dry friction.

2 Some aspects of friction

In a motion where one body is sliding over another one, part of the force required to produce motion is due to the resistance to sliding. The tangential stress at the interface, corresponding to this force (the friction force), can be written

$$\tau = dW/dS \tag{1}$$

where S is the relative tangential displacement between the two mating surfaces and W is the energy per unit of area that is irrecoverably lost in the sense that it does not contribute to the potential energy or to the kinetic energy of the system as a whole. Most part of W is eventually dissipated as heat (although a small part goes to other forms of energy, for instance elastic strain energy due to residual stresses near the sliding surfaces). Some part of W, in almost all cases completely negligible, consists of kinetic energy in the region of sliding (due to translation and rotation of particles between the bodies). This kinetic energy goes eventually to other forms of energy and is mainly dissipated as heat.

The irrecoverable energy W is produced by different types of mechanisms, mainly:

plastic deformation,
viscous flow,
hysteresis,
crack formation, and (generally negligible)
inertia.

Since most part of W turns into heat, the temperature near the sliding surfaces is increased. The temperature increase depends on the amount of heat produced per unit of time and on how fast heat is conducted away from the region of sliding. Therefore, the temperature generally increases with the sliding velocity.

The temperature, in turn, influences W. Thus plastic deformation and viscous flow is generally associated with less energy dissipation at higher temperatures. Also changes of the mechanisms of energy production may occur, for instance due to melting and to acceleration of chemical processes, such as oxidation.

The relative importance of different types of mechanisms producing the energy W depends naturally on the conditions in the sliding region. Two extreme cases can be distinguished, idealized dry friction and idealized hydrodynamic lubrication.

The dependence of τ on the compressive stress σ (the normal stress), by which the bodies are pressed together, and on the relative sliding velocity $\dot{u}$, is fundamentally different in the two extreme cases.

At idealized dry friction τ is proportional to σ and independent on $\dot{u}$. This case should be realized when two metallic bodies, with rate- and temperature-independent perfectly plastic properties, are in direct contact. The usual physical explanation is the following one: when the two bodies are pressed together adhesion ('welding') occurs at the regions of real contact, forming junctions which are sheared off when sliding takes place. At sufficiently small compressive stress σ the area of real contact is substantially smaller than the total area. Then the area of real contact should be proportional to σ (because of the assumption of perfect plasticity) and the force required to shear the junctions should be proportional to this area (again because of the assumption of perfect plasticity). Thus the stress τ should be proportional to σ. The assumption of rate- and temperature-independence of the perfectly plastic properties implies that τ should be independent of $\dot{u}$. (An obvious relaxation of the demand of perfect plasticity can be made without changing this result.)

At idealized hydrodynamic lubrication τ is independent of σ and proportional to $\dot{u}$. This case is realized when a layer of a Newtonian fluid with pressure- and temperature-independent viscosity is separating the two bodies. The explanation is immediately obvious, since Hagen–Poiseuille flow should occur in the layer.

The two cases discussed were, rather arbitrarily, called extreme. This does not imply that the dependence of τ on σ and $\dot{u}$ in real cases should necessarily be something between the dependencies found for the idealized cases. In fact, cases exist where τ decreases when $\dot{u}$ is increased or increases faster than in proportion to σ or $\dot{u}$. Irregular dependence of τ on σ and $\dot{u}$ sometimes found in different ranges and also the time elapsed after initiation of the sliding motion may influence $\tau(\sigma, \dot{u})$.

Real cases where one body (not necessarily metallic) is sliding over another one, without any layer intentionally placed in between (for instance, as a lubricant or an abrasive) or produced during the sliding motion as abraded particles, show approximate independence of τ on $\dot{u}$ and approximate proportionality between τ and σ. The deviations from the idealized case of dry friction can go in different directions and depend on different factors, such as presence of oxide films, hardness variations with the depth from the surface, elastic (Hertzian-type) contact forces instead of plastic ones, large surface roughnesses (causing 'ploughing' and associated hysteresis losses), etc. Of special interest are effects caused by the increase of temperature with the sliding velocity. This increase often contributes to a decrease of τ, because of local softening and even melting, implying decrease of the stresses required to shear off junctions between the bodies.

Real cases where a thick layer of a viscous fluid is separating the bodies show approximate independence of τ on σ and approximate proportionality between τ and $\dot{u}$. By a 'thick' layer is meant one that completely separates the two bodies from contact with each other. The deviations from the idealized case of hydrodynamic lubrication are caused by the pressure- and temperature-dependence of the viscosity. The viscosity generally increases somewhat with the pressure. (Water, with practically seen pressure-independent viscosity, is an exception.) Increases of the viscosity with the pressure implies some increase of τ with σ. An increase of the temperature lowers the viscosity, implying somewhat slower increase of τ than in proportion to $\dot{u}$. A decrease of the thickness of the layer of viscous fluid implies an increase of τ (because of the properties of Hagen–Poiseuille flow). If the thickness is decreased until the solids are no longer completely separated from direct contact with each other, τ can increase considerably. (In some applications, such as oil-lubricated bearings, the layer thickness itself depends upon both velocity and normal force.)

When a layer of a solid material (a homogeneous solid or a powder) or of a mixture of a solid material and a liquid is separating two bodies, the sliding properties depend on the nature of the layer (plastic or viscoelastic, thick or thin, adherent to the surfaces of the sliding bodies or not, soft or hard, high or low melting point, etc.) as well as on the nature of the body materials (soft or hard, ductile or brittle, strong or weak hysteresis, good or poor heat conductivity, high or low melting point, etc.). The structure of the function $\tau(\sigma, \dot{u})$ varies considerably between different cases.

Time also plays a role in the friction behaviour. Thus, for instance, experiments with steel-balls, spinning on a metal surface and decelerated by friction, show different deceleration at the same rotational velocity if the initial rotational velocity has been different. Such phenomena can be explained by differences in temperature and by different amounts of accumulated plastic and viscous work, abrasion, cracking, etc. Another cause of time (or rather displacement) dependence of the friction force can be misalignment between the sliding surfaces.

Size effects may also influence the mechanism of sliding. In an earthquake fault a sizable layer of gouge material is believed to separate two blocks from direct contact. While in engineering applications of sliding motion layer thicknesses are typically of the order of 10^{-4} m, the layer thickness in an earthquake fault is larger by a couple of orders of magnitude. This difference implies different temperature characteristics. Thus heat is, practically seen, not conducted away from the sliding region of an earthquake fault during a motion that lasts less than a few tens of seconds. The temperature rise depends essentially on the time (or, rather, the displacement). Another implication of the comparatively large thickness of the region of sliding in an earthquake fault is that inertia effects cannot *a priori* be neglected.

3 Uni-directional slip propagation

3.1 LINEAR THEORY OF IDEALIZED DRY FRICTION

Consider two semi-infinite, isotropic, homogeneous and linearly elastic solids in contact

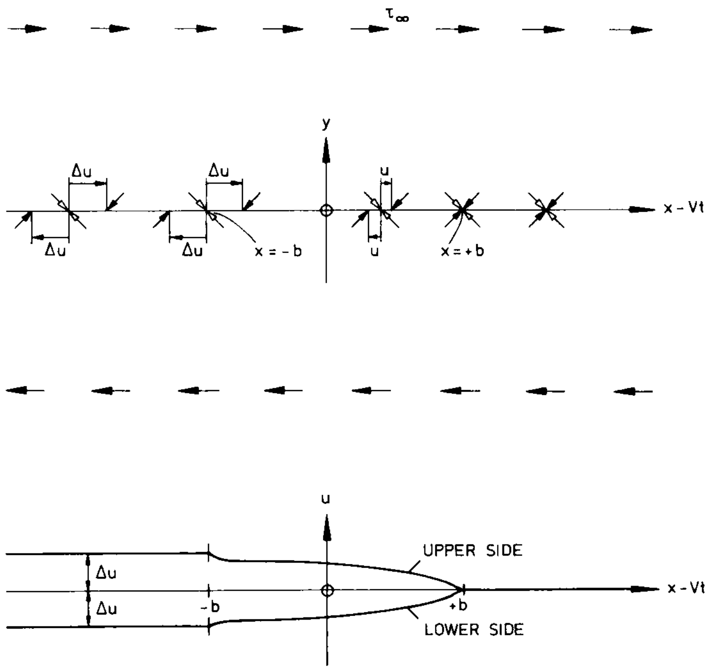

Figure 3. The displacement u at the interface $y = 0$ shown in the xy plane (upper figure) and in the xu plane (lower figure). Empty arrowheads indicate original, filled ones current positions of material points at the interface. It can be seen from the upper figure that a misfit occurs at $x \approx -b$. This can be avoided by taking the Lagrangian length of the sliding region to be $2b + \Delta u$ at the lower side of the interface. Since the analysis is performed for one half-plane, only, this would be perfectly possible. However, this complication is not sensible, since the analysis requires that $\Delta u \ll 2b$.

along the plane $y = 0$. The remote stress is $\tau_{xy} = \tau_\infty > 0$. A slip of total magnitude $2\Delta u$ is propagated as a wave of sliding in a region

$$-b < x - Vt < b, \quad y = 0 \tag{2}$$

see Fig. 3. It is assumed that a shear stress

$$\tau_{xy} = \tau_0(x - Vt), \quad -b < x - Vt < b \tag{3}$$

acts on the mating surfaces in the region of sliding.

The following notations are introduced:

G = modulus of rigidity
c = propagation velocity of irrotational waves
kc = propagation velocity of equivoluminal waves
$\beta = V/c$
$a_1^2 = 1 - \beta^2, a_1 > 0$
$a_2^2 = 1 - \beta^2/k^2, a_2 > 0$
$f(\beta) = 2(1 - k^2)a_2(1 - a_2^2)[4a_1a_2 - (1 + a_2^2)^2]^{-1}$
u = displacement in positive x direction
v = displacement in positive y direction.

Then, for $y = 0$ one obtains (see Appendix 1) with a change in notation, so that $x - Vt \to x$, for brevity:

$$\tau_{xy} = \begin{cases} \tau_0(x) & \text{for } |x| < b \\ \dfrac{1}{\pi} \cdot \dfrac{x(x^2 - b^2)^{1/2}}{|x|} \displaystyle\int_{-b}^{b} \dfrac{\tau_0(s)ds}{(b^2 - s^2)^{1/2}(x - s)} & \text{for } |x| > b \end{cases} \quad (5)$$

$$\frac{\partial u}{\partial x} = \begin{cases} \dfrac{f(\beta)}{2\pi(1 - k^2)G}(b^2 - x^2)^{1/2} \displaystyle\oint_{-b}^{b} \dfrac{\tau_0(s)ds}{(b^2 - s^2)^{1/2}(x - s)} & \text{for } |x| < b \quad (6) \\ 0 \quad \text{for } |x| > b & (7) \end{cases}$$

$$\sigma_x = 4(1 - k^2)G\partial u/\partial x \quad (8)$$

under the condition of stress continuity,

$$\frac{1}{\pi} \int_{-b}^{b} \frac{\tau_0(s)ds}{(b^2 - s^2)^{1/2}} = \tau_\infty \quad (9)$$

C on the integral sign denotes the Cauchy principal value.

The case of *idealized dry friction* will now be studied. To this end it is assumed that $\tau_0(x)$ equals a constant, the dynamic friction stress τ_D; except near the ends of the sliding region. Thus,

$$\tau_0(x) = \tau_D \quad \text{for } -b + d_H < x < b - d_T \quad (10)$$

where $d_T \ll b$ is the length of the region where idealized dry friction sliding is initiated and $d_H \ll b$ is the length of the region where idealized dry friction sliding ceases. These regions will be called the *tearing region* and the *healing region*, respectively: see Fig. 4.

It is convenient to introduce the notation

$$\tau_1(x) = \tau_0(x) - \tau_D, \quad -b < x < b. \quad (11)$$

Then one notices that equation (9) can be written:

$$T - H = \pi(2b)^{1/2}(\tau_\infty - \tau_D) \quad (12)$$

where (since $d_T \ll b, d_H \ll b$):

$$\left.\begin{aligned} T &= \int_{b-d_T}^{b} \frac{\tau_1(s)}{(b - s)^{1/2}} ds \\ H &= -\int_{b}^{-b+d_H} \frac{\tau_1(s)}{(b + s)^{1/2}} ds \end{aligned}\right\}. \quad (13)$$

T will be referred to as the *modulus of tearing* and H as the *modulus of healing*. Both T and H must be positive. Whereas it is quite obvious that T is positive, the sign of H needs some explanation.

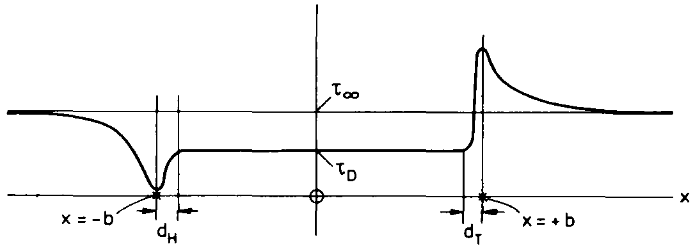

Figure 4. The shear stress τ_{yx} at the interface. No slip occurs at $|x| > b$. Idealized dry friction sliding is fully developed in the region $-b + d_H < x < b - d_T$. The regions $b > x > b - d_T$ (the tearing region) and $-b + d_H > x > -b$ (the healing region) are transition regions where idealized dry friction sliding (x stands for $x - Vt$) develops respectively recedes.

By putting $x = -b + d_H$ and making approximations motivated by the relations $d_T \ll b$, $d_H \ll b$, one obtains:

$$\frac{\partial u}{\partial x} = \frac{f(\beta)}{2\pi(1-k^2)G}\left[-\frac{T}{2b} + \int_{-b}^{-b+d_H} \frac{\tau_1(s)}{(b+s)^{1/2}} \frac{ds}{d_H - b - s}\right]$$

$$= \frac{f(\beta)}{2\pi(1-k^2)G}\left[-\frac{T}{2b} - \theta\frac{H}{d_H}\right] \quad \text{where } \theta > 1.$$

Hence $\partial u/\partial x > 0$ if $H + (d_H/2\theta b)\, T \leqslant 0$. This would imply that $\partial u/\partial x = 0$ somewhere in the interval $-b + d_H < x < b - d_T$. But vanishing $\partial u/\partial x$ means no sliding, i.e. the basic assumptions are violated. Thus, in view of the approximations made, one has to take $H > 0$. Compare also equation (29) in the following.

One observes the analogy with the theory of Barenblatt (1959) for equilibrium cracks. In fact, T and H are given by integrals of exactly the same structure as the modulus of cohesion, introduced by Barenblatt. With the same motivation as used by Barenblatt, T and H are assumed to be *constants characteristic of the material* as well as τ_D. However, it should be noticed that they ought to depend on the propagation velocity V and – as is the case also with τ_D – on the normal stress σ_y and the temperature. (In the present treatment no consideration is taken of the temperature rise during the sliding motion.) Thus, in the present treatment the character of material properties of T and H is manifested by their independence on b and τ_∞.

From equation (12) the length of the sliding region is found:

$$2b = \frac{1}{\pi^2}\left(\frac{T - H}{\tau_\infty - \tau_D}\right)^2. \tag{14}$$

The total amount of slip (on one side) is

$$\Delta u = -\int_{-b}^{b} \frac{\partial u}{\partial x}\, dx = \frac{f(\beta)}{2\pi(1-k^2)G}\int_{-b}^{b} (b^2 - x^2)^{1/2} \oint_{-b}^{b} \frac{\tau_1(s)ds}{(b^2 - s^2)^{1/2}(s - x)}\, dx$$

$$= \frac{f(\beta)}{2\pi(1-k^2)G}\int_{-b}^{b} \frac{\tau_1(s)}{(b^2 - s^2)^{1/2}} \oint_{-b}^{b} \frac{(b^2 - x^2)^{1/2}dx}{s - x}\, ds$$

$$= \frac{f(\beta)}{2(1-k^2)G}\int_{-b}^{b} \frac{s\tau_1(s)}{(b^2 - s^2)^{1/2}}\, ds.$$

Thus, since $d_T \ll b$, $d_H \ll b$,

$$\Delta u = \frac{f(\beta)}{2(1-k^2)G}\left(\frac{b}{2}\right)^{1/2}(T+H). \tag{15}$$

Then, because of equation (14):

$$\Delta u = \frac{f(\beta)}{4\pi(1-k^2)G}\cdot\frac{T^2-H^2}{\tau_\infty-\tau_D} \tag{16}$$

$f(\beta)$, T and H are functions of β. It seems reasonable to assume that $(T^2 - H^2)$ increases with the velocity, at least at higher velocities (*cf.* Broberg 1977). $f(\beta)$ increases monotonically from unity at $\beta = 0$ to infinity at the Rayleigh wave velocity, see Fig. 5. The unknown propagation velocity of the wave of sliding is implicitly given by equation (16). By writing this equation in the form

$$\Delta u(\tau_\infty - \tau_D) = \frac{f(\beta)}{4\pi(1-k^2)G}\left\{[T(\beta)]^2 - [H(\beta)]^2\right\} = m(\beta) \tag{17}$$

it is immediately seen that β increases with Δu or with τ_∞. See also Fig. 6.

From equation (17) (or Fig. 6) it follows that Δu (at given τ_∞) must exceed a certain value

$$\Delta u_0 = \frac{[T(0)]^2 - [H(0)]^2}{4\pi(1-k^2)G(\tau_\infty - \tau_D)} \tag{18}$$

to be propagated indefinitely. If $\Delta u < \Delta u_0$ the slip (see Fig. 1) would extend only some distance beyond the obstacle (P in Fig. 1) and the sliding motion would come to a rest.

The energy dissipation per unit of pulse propagation and per unit of width of the interface will now be calculated. At a propagation distance dS the energy

$$dW = 2\int_{-b}^{b} \tau_0(x)\frac{\partial u}{\partial S}\,dS\,dx \tag{19}$$

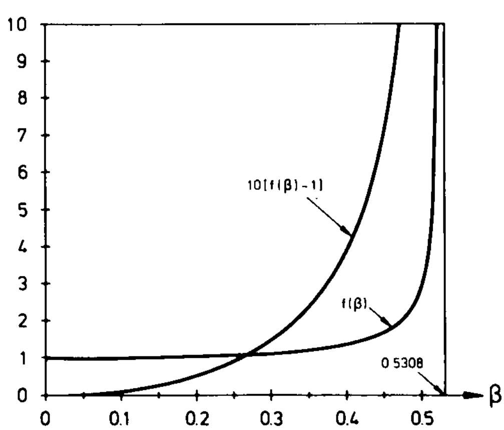

Figure 5. The function $f(\beta)$ for $k^2 = 1/3$. The non-dimensional Rayleigh wave velocity is $\beta = 0.5308$.

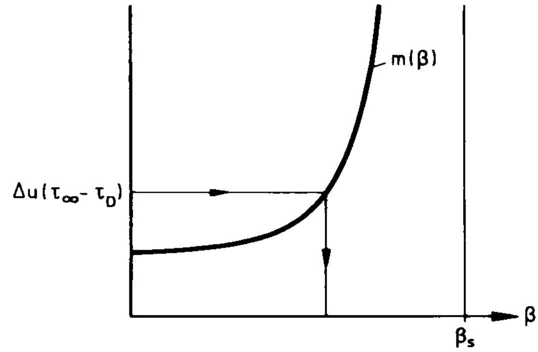

Figure 6. Relation between amount of slip, Δu, remote load τ_∞, and non-dimensional slip propagation velocity β. β_S is the non-dimensional Rayleigh wave velocity. The function $m(\beta)$ depends both on the elastic properties of the material and on the moduli of tearing and healing.

is dissipated per unit of width. (The factor 2 depends, of course, on the fact that there are two mating surfaces at the interface.) Thus, since

$$\partial u/\partial S = -\partial u/\partial x \tag{20}$$

equation (19) can be written

$$\frac{dW}{dS} = -2\int_{-b}^{b} \tau_0 \frac{\partial u}{\partial x}\, dx = 2\tau_D \Delta u - 2\int_{-b}^{b} \tau_1(x)\frac{\partial u}{\partial x}\, dx$$

$$= 2\tau_D \Delta u + \frac{f(\beta)}{\pi(1-k^2)G}\int_{-b}^{b} \tau_1(x)(b^2-x^2)^{1/2} \oint_{-b}^{b} \frac{\tau_0(s)ds}{(b^2-s^2)^{1/2}(s-x)}\, dx. \tag{21}$$

Now, because

$$\oint_{-b}^{b} \frac{ds}{(b^2-s^2)^{1/2}(s-x)} = 0, \quad -b < x < b$$

and $d_T \ll b, d_H \ll b$, one obtains

$$\frac{dW}{dS} = 2\tau_D \Delta u + \frac{f(\beta)}{\pi(1-k^2)G}\left\{\oint_{-b}^{-b+d_H} \oint_{-b}^{-b+d_H} \frac{\tau_1(x)\tau_1(s)}{s-x}\left(\frac{b+x}{b+s}\right)^{1/2} ds dx\right.$$

$$\left. + \oint_{b-d_T}^{b} \oint_{b-d_T}^{b} \frac{\tau_1(x)\tau_1(s)}{s-x}\left(\frac{b-x}{b-s}\right)^{1/2} ds dx\right\}.$$

The double integrals are easily evaluated. The last one can be written:

$$I = \oint_{b-d_T}^{b} \oint_{b-d_T}^{b} \frac{\tau_1(x)\tau_1(s)}{(b-x)^{1/2}(b-s)^{1/2}}\, ds dx + \oint_{b-d_T}^{b} \oint_{b-d_T}^{b} \frac{\tau_1(x)\tau_1(s)}{s-x}\left(\frac{b-s}{b-x}\right)^{1/2} ds dx$$

$$= \left[\oint_{b-d_T}^{b} \frac{\tau_1(s)}{(b-s)^{1/2}}\, ds\right]^2 - \oint_{b-d_T}^{b} \oint_{b-d_T}^{b} \frac{\tau_1(x)\tau_1(s)}{x-s}\left(\frac{b-s}{b-x}\right)^{1/2} dx ds = T^2 - I.$$

Thus

$$I = \tfrac{1}{2} T^2$$

and hence

$$\frac{dW}{dS} = 2\tau_D \Delta u + \frac{f(\beta)}{2\pi(1-k^2)G}(T^2 + H^2). \tag{22}$$

The second term of the right member refers to energy dissipation due to tearing and healing, whereas the first term refers to energy dissipation due to sliding between the regions of tearing and healing. Actually some part of the energy dissipation given by the first term should be referred to the regions of tearing and healing, but because of the assumption of the smallness of these regions this part can be neglected.

Use of equation (16) shows that the first term of the right member of equation (22) can be written

$$2\tau_D \Delta u = \frac{f(\beta)}{2\pi(1-k^2)G} \cdot \frac{\tau_D}{\tau_\infty - \tau_D}(T^2 - H^2).$$

Thus the ratio, κ, between energy dissipation due to processes in the tearing and healing regions and energy dissipations due to sliding in the intermediate region is

$$\kappa = \frac{T^2 + H^2}{T^2 - H^2} \cdot \frac{\tau_\infty - \tau_D}{\tau_D}. \tag{23}$$

This expression shows that the energy dissipation due to tearing and healing is not necessarily small compared to the total energy dissipation.

There is an obvious physical explanation why the modulus of tearing, T, is larger than the modulus of healing, H. In the tearing process the compressive stress σ_y performs a negative work, whereas it performs a positive work in healing. This work is not explicitly accounted for in the mathematical analysis, because of the assumption of plane contact between mating surfaces and uniform material properties in the depth direction from these surfaces. It can be visualized by imagining a saw-tooth structure of the mating surfaces, but naturally other kinds of real surface properties should imply similar influence of the normal stress.

The healing does not necessarily restore the grip between the solids to the same condition as before tearing. The difference between T and H, therefore, can be considerably enlarged if some long-time action, preceding the event of sliding, has strengthened the grip.

Possible mechanisms are, for instance, chemical binding and mechanical consolidation due to creep.

3.2 THE MASS VELOCITIES IN THE SOLIDS

The mass velocities in the solids are

$$\begin{cases} \partial u/\partial t = -V\partial u/\partial x \\ \partial v/\partial t = -V\partial v/\partial x \end{cases}$$

where, as previously, x stands for $x - Vt$. Thus the mass velocities are obtained from $\partial u/\partial x$ and $\partial v/\partial x$.

In Appendix 1, equations (A1.37–38), the exact expressions for $\partial u/\partial x$ and $\partial v/\partial x$ are given. Since $\tau_0(x) = \tau_D + \tau_1(x)$, and $\tau_1(x)$ is assumed to vanish except near the ends of the region of sliding, one obtains, after omission of second-order terms, the following expressions, valid for $y > 0, (x+b)^2 + a_2^2 y^2 \gg d_H^2, (x-b)^2 + a_2^2 y^2 \gg d_T^2$:

$$\frac{\partial u}{\partial x} = \frac{Ta_2}{\pi^2 GR(\beta)} \left(\frac{2}{b}\right)^{1/2} \int_b^\infty [(a-b)u_0(x+a) + (a+b)u_0(x-a)] \frac{da}{(a^2-b^2)^{1/2}}$$

$$- \frac{Ha_2}{\pi^2 GR(\beta)} \left(\frac{2}{b}\right)^{1/2} \int_b^\infty [(a+b)u_0(x+a) + (a-b)u_0(x-a)] \frac{da}{(a^2-b^2)^{1/2}} \quad (24)$$

$$\frac{\partial v}{\partial x} = \frac{(T-H)a_2}{\pi^2 GR(\beta)} \left(\frac{2}{b}\right)^{1/2} \int_{-\infty}^{+\infty} v_0(x-a)da$$

$$+ \frac{Ta_2}{\pi^2 GR(\beta)} \left(\frac{2}{b}\right)^{1/2} \int_b^\infty [(a-b)v_0(x+a) + (a+b)v_0(x-a)] \frac{da}{(a^2-b^2)^{1/2}} \quad (25)$$

$$- \frac{Ha_2}{\pi^2 GR(\beta)} \left(\frac{2}{b}\right)^{1/2} \int_b^\infty [(a+b)v_0(x+a) + (a-b)v_0(x-a)] \frac{da}{(a^2-b^2)^{1/2}}$$

where

$$R(\beta) = 4a_1a_2 - (1+a_2^2)^2$$

$$u_0(x) = \frac{x}{a_1^2 y^2 + x^2} - \frac{1+a_2^2}{2} \cdot \frac{x}{a_2^2 y^2 + x^2}$$

$$v_0(x) = \frac{a_1^2 y}{a_1^2 y^2 + x^2} - \frac{1+a_2^2}{2} \cdot \frac{y}{a_2^2 y^2 + x^2}.$$

Integration of equations (24–25) gives

$$\frac{G}{\tau_\infty - \tau_D} \cdot \frac{\partial u}{\partial x} = \frac{\sqrt{2}a_2}{R(\beta)} \left\{a_1 \eta D_1 - (q+\xi)S_1 - \frac{1+a_2^2}{2}[a_2\eta D_2 - (q+\xi)S_2]\right\} \quad (26)$$

$$\frac{G}{\tau_\infty - \tau_D} \cdot \frac{\partial v}{\partial x} = \frac{1+a_2^2 - 2a_1a_2}{R(\beta)} + \frac{2\sqrt{2}a_2}{R(\beta)} \left\{a_1[(q+\xi)D_1 + a_1\eta S_1] - \frac{1+a_2^2}{2a_2}[(q+\xi)D_2 + a_2\eta S_2]\right\} \quad (27)$$

where

$\xi = x/b$

$\eta = y/b$

$q = (T/H+1)/(T/H-1)$

$D_1 = \frac{\xi}{|\xi|}(R_1 + \xi^2 - 1 - a_1^2\eta^2)^{1/2}/R_1$

$D_2 = (D_1)_{a_1 \to a_2}$

$S_1 = (R_1 - \xi^2 + 1 + a_1^2\eta^2)^{1/2}/R_1$

$S_2 = (S_1)_{a_1 \to a_2}$

$R_1 = [(\xi^2 + a_1^2\eta^2 - 1)^2 + 4a_1^2\eta^2]^{1/2}$.

Equations (26–27) are given in a form suitable for use when, in addition to Δu and $(\tau_\infty - \tau_D)$ the non-dimensional velocity β and the tearing/healing parameter q (which depends only on the ratio T/H) are regarded as known quantities, rather than H and T. Consequently an expression for the length $2b$ of the region of sliding is given in the form:

$$\frac{\tau_\infty - \tau_D}{G} \cdot \frac{2b}{\Delta u} = \frac{4(1-k^2)}{\pi q f(\beta)}. \tag{28}$$

The expression is found by combining equations (14) and (15).

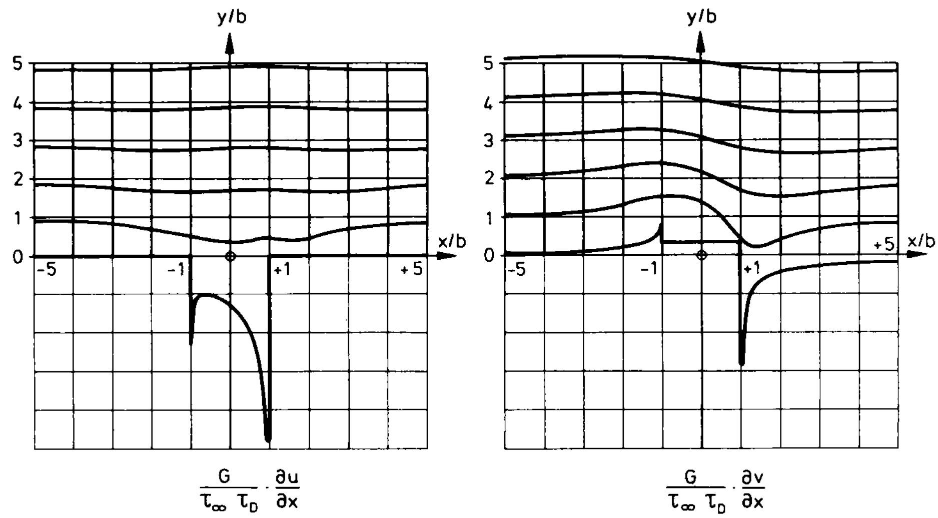

Figure 7. Non-dimensional displacement gradient at $y/b = 0, 1, 2, 3, 4, 5$ for $\beta = 0.3$, $k^2 = 1/3$, $q = 1.5$. The zero-line belonging to each curve is offset the distance y/b from the x/b axis. The gradients are shown at the same scale as the one for y/b. For $|x| \approx b$, $y = 0$ the curve-peaks are rather arbitrarily drawn, since equations (26–27), from which the curves are determined, are not valid near these points. x stands for $(x - Vt)$.

In Fig. 7 the displacement gradients $\partial u/\partial x$ and $\partial v/\partial x$ in the neighbourhood of the region of sliding are shown for $\beta = 0.3$, assuming $q = 1.5$ and $k^2 = 1/3$. Fig. 8 shows the corresponding velocity field in an arrow diagram. Figs 9–10 show the influence of q and β on $\partial u/\partial x$ and $\partial v/\partial x$ for $y = b$. In Fig. 11 the length of the region of sliding is shown as a function of the slip propagation velocity.

A numerical example is shown by Fig. 12. The particle velocities at a fixed point outside the interface are shown as functions of time for some slip propagation velocities.

For $y = 0$ equation (26) can be written:

$$\frac{\partial u}{\partial x} = -\frac{f(\beta)}{2\pi(1-k^2)G(2b)^{1/2}}\left[\left(\frac{b+x}{b-x}\right)^{1/2} T + \left(\frac{b-x}{b+x}\right)^{1/2} H\right]. \tag{29}$$

The expression is valid for $(x+b)^2 \gg d_H^2$, $(x-b)^2 \gg d_T^2$, $|x| < b$. It can, of course, also be found from the exact expression, equation (6), by omission of second-order terms. Similarly a simple expression for $\partial v/\partial x$ $(y = 0)$ is easily obtained, either from equation (27)

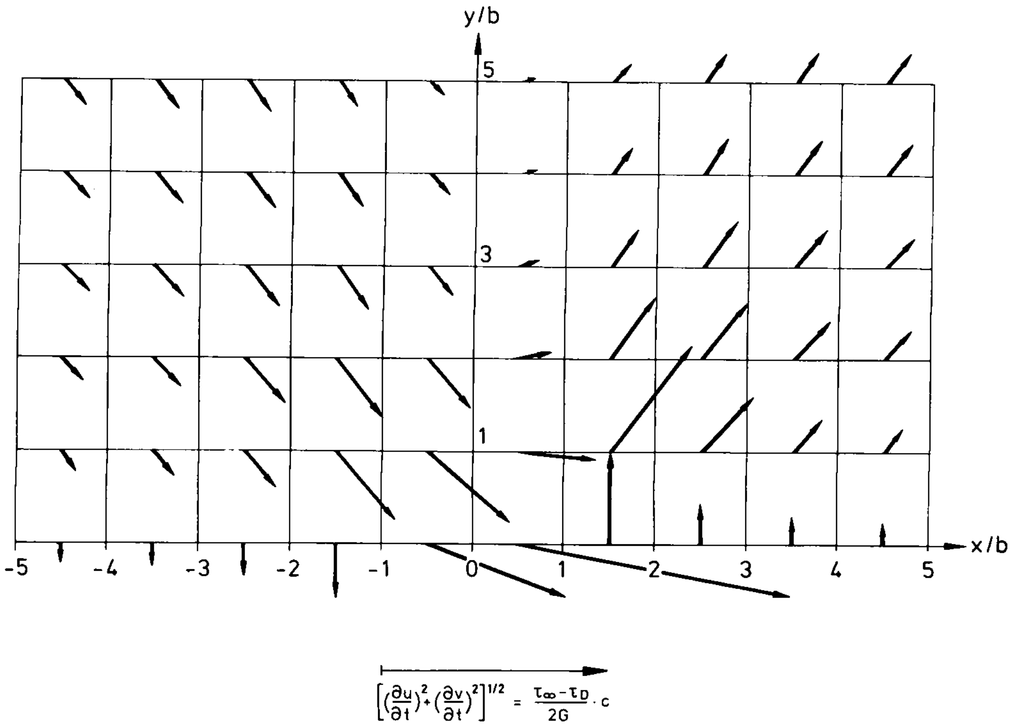

Figure 8. The velocity field in the neighbourhood of the region of sliding. Arrow tails indicate position, arrow lengths the magnitude of the velocity according to the scale shown. $\beta = 0.3$, $k^2 = 1/3$, $q = 1.5$. x stands for $(x - Vt)$.

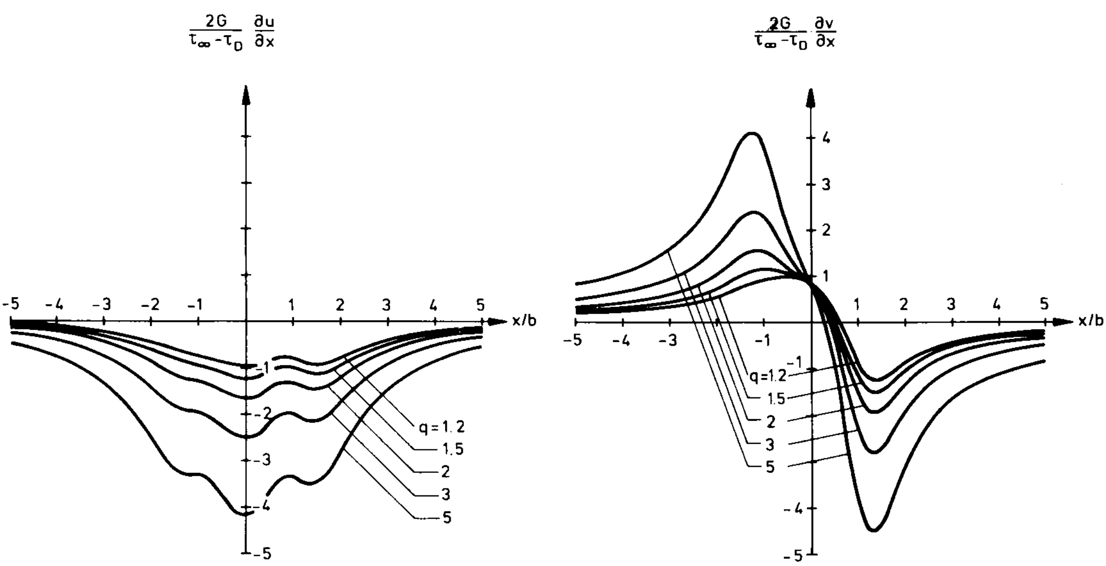

Figure 9. Non-dimensional displacement gradients at $y/b = 1$, for different values of q. $k^2 = 1/3, \beta = 0.3$. x stands for $(x - Vt)$.

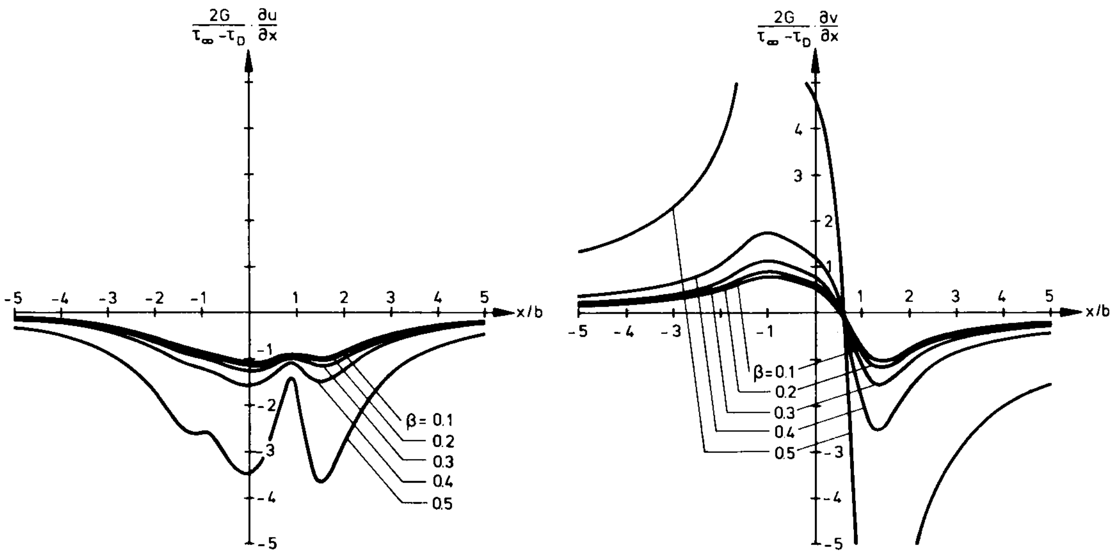

Figure 10. Non-dimensional displacement gradients at $y/b = 1$, for different values of β. $k^2 = 1/3, q = 1.5$. x stands for $(x - Vt)$.

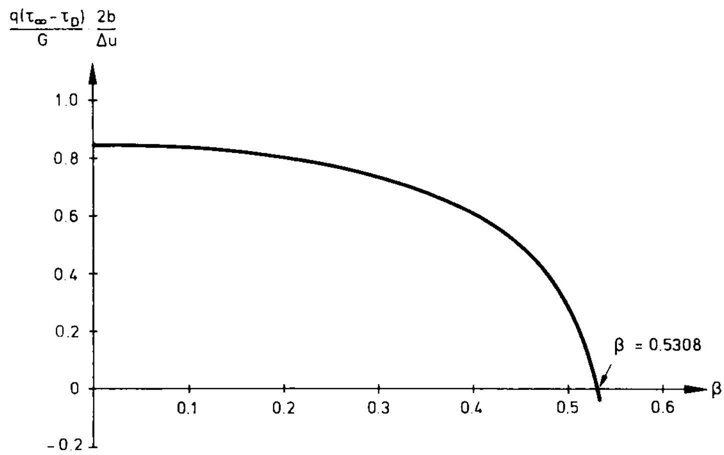

Figure 11. Non-dimensional length of the sliding region as a function of non-dimensional slip propagation velocity β. $k^2 = 1/3$.

or, with omission of second-order terms, from equations (A1.39) and (5). The last-mentioned equation then takes on the form:

$$\tau_{xy} = \tau_\infty + \frac{1}{\pi(2b)^{1/2}} \left\{ \left[\left(\frac{x+b}{x-b} \right)^{1/2} - 1 \right] T - \left[\left(\frac{x-b}{x+b} \right)^{1/2} - 1 \right] H \right\}. \tag{30}$$

It is valid for $y = 0$, $(x+b)^2 \gg d_H^2$, $(x-b)^2 \gg d_T^2$, $|x| > b$.

After integration of equation (27) for $y = 0$ one obtains with use of equation (28):

$$\frac{v(x) - v(\lambda b)}{\Delta u} = -\frac{1 + a_2^2 - 2a_1 a_2}{\pi q a_2 (1 - a_2^2)} \left\{ \lambda - (\lambda^2 - 1)^{1/2} + q \ln [\lambda - (\lambda^2 - 1)^{1/2}] - \xi + A(\xi) \right\} \tag{31}$$

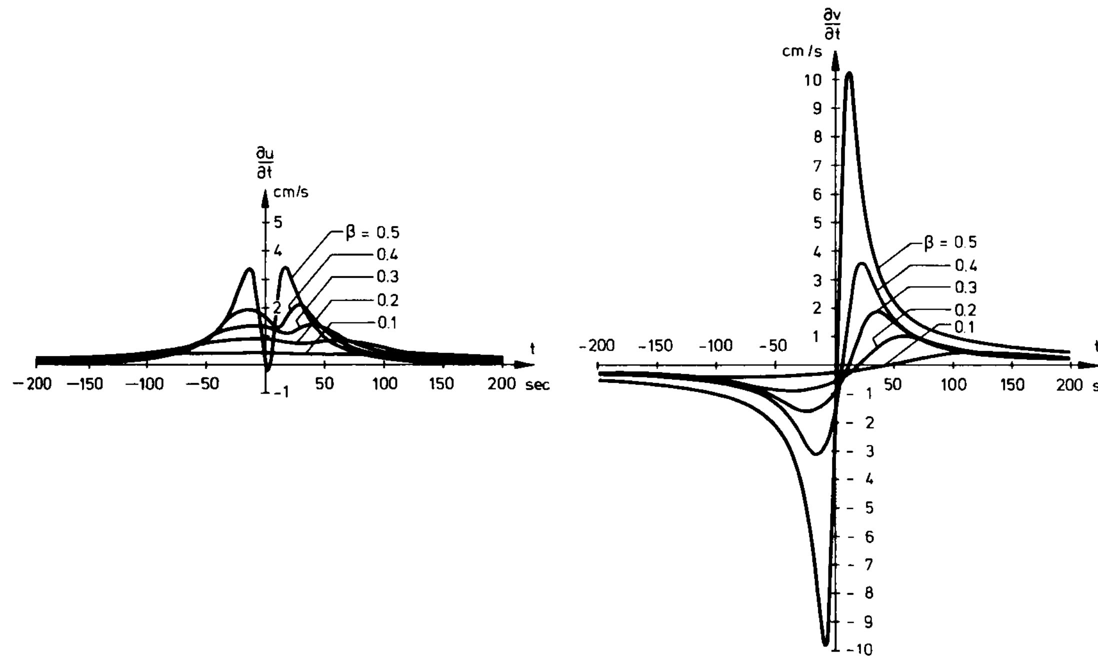

Figure 12. Particle velocity a point situated at 50 km distance from the fault as function of time. It is supposed that $(\tau_\infty - \tau_D)/G = 2 \times 10^{-5}$, $2\Delta u = 3$ m, $c = 5.2$ km/s, $\beta = 0.3$, $k^2 = 1/3$ and $q = 1.5$. Time $t = 0$ is (arbitrarily) chosen to the instant at which the centre of the region of sliding is most close.

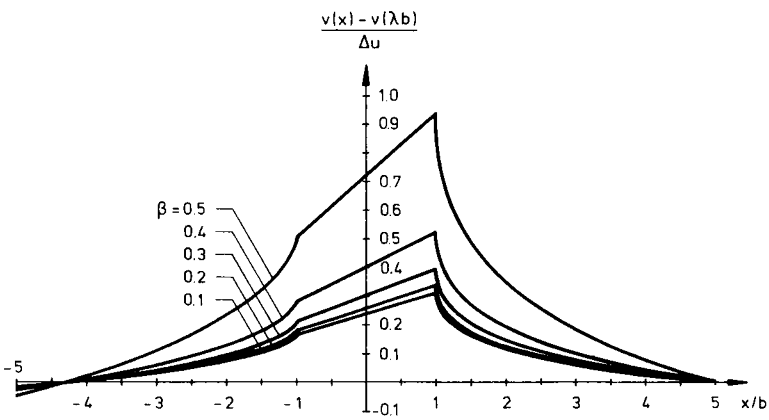

Figure 13. Non-dimensional lateral displacement of the fault. The displacement at $x = 5b$ is (arbitrarily) put equal to zero. x stands for $(x - Vt)$.

where

$$A(\xi) = \begin{cases} 0 & \text{for } \xi^2 < 1 \\ \dfrac{\xi}{|\xi|}(\xi^2 - 1)^{1/2} - q \ln\left[|\xi| - (\xi^2 - 1)^{1/2}\right] & \text{for } \xi^2 > 1 \end{cases}$$

and $\xi = \lambda$ is a reference point which can be arbitrarily chosen. Fig. 13 shows the lateral displacement with respect to this point for some slip propagation velocities, assuming $q = 1.5$ and $k^2 = 1/3$.

For $\xi^2 + \eta^2 \to \infty$ one obtains from equations (26–27):

$$\frac{G}{\tau_\infty - \tau_D} \cdot \frac{\partial u}{\partial x} \to -\frac{q}{R(\beta)} \left[\frac{2a_1 a_2}{\xi^2 + a_1^2 \eta^2} - \frac{a_2^2(1 + a_2^2)}{\xi^2 + a_2^2 \eta^2} \right] \eta \tag{32}$$

$$\frac{G}{\tau_\infty - \tau_D} \cdot \frac{\partial v}{\partial x} \to \frac{q}{R(\beta)} \left[\frac{2a_1 a_2}{\xi^2 + a_1^2 \eta^2} - \frac{1 + a_2^2}{\xi^2 + a_2^2 \eta^2} \right] \xi. \tag{33}$$

Equations (32–33) illustrate the significance of the tearing/healing parameter q.

The exact expressions for τ_{xy} and $\partial u/\partial x$, equations (5) and (6), show that τ_{xy} is continuous and non-singular at $|x| = b$ and that $\partial u/\partial x \to 0$ as $|x| = b$ is approached. The latter property can be referred to as 'smooth closing' (*cf.* Barenblatt 1959). It should also be noted that $|\partial u/\partial x|$ possesses maxima at $x = b - d_T$ and $x = b + d_H$ or in the neighbourhood of these points.

3.3 INSPECTION OF THE VALIDITY OF THE LINEAR THEORY

The derivation of equations (14) and (16) was made under the assumption of elastic behaviour of the semi-infinite solids. Thus all energy dissipation should be due to processes in the interface. The validity of this assumption will now be inspected. To this end the stress σ_x near the mating surfaces will be studied.

From equations (8) and (6) one obtains for $x \approx b - d_T$, the position of maximum $|\sigma_x|$:

$$|\sigma_x| \approx \frac{4}{\pi} f(\beta) d_T^{1/2} \int_{b-d_T}^{b} \frac{\tau_1(s)ds}{(b-s)^{1/2}(s-b+d_T)}.$$

Thus,

$$|\sigma_x| > \frac{4}{\pi} f(\beta)\, T d_T^{-1/2} \qquad \text{for } x \approx b - d_T.$$

Similarly

$$|\sigma_x| > \frac{4}{\pi} f(\beta)\, H d_H^{-1/2} \qquad \text{for } x \approx b - d_H.$$

Due to the autonomous character of the tearing and healing regions, σ_x does not depend on Δu or τ_∞ near these regions. If $|\sigma_x|$ exceeds a certain value, say σ_0, plastic flow or cracking will take place. Thus, if the conditions

$$\frac{4}{\pi} f(\beta)\, T d_T^{-1/2} < \sigma_0$$

$$\frac{4}{\pi} f(\beta)\, H d_H^{-1/2} < \sigma_0$$

are not fulfilled, the treatment leading to equations (14) and (16) may seem to be invalid. However, if calculated stresses $|\sigma_x|$ exceed σ_0 but are confined to a close neighbourhood of the tearing and healing regions, no principle deviation from the linear theory has occurred. It should be borne in mind that the mathematical formulation of the problem is idealized in the respect that energy dissipation is assumed to be due only to processes in the interface, notably the action of τ_{yx}. In reality energy dissipation is initiated in a layer of finite thickness. It is immaterial whether this initiation is caused by the action of σ_x or of τ_{yx} as long as the layer thickness is small compared to the length $(2b)$ of the sliding region. In the mathematical treatment it is introduced as a dissipation due to the stress τ_{yx} at the interface. This is possible by using appropriate values of T and H, *cf.* equation (22). Thus, supposing the energy dissipation per unit of slip propagation and per unit of width, due to tearing to be $(dW/dS)_T$, one takes the modulus of tearing to be

$$T(\beta) = \left[\frac{2\pi(1-k^2)G}{f(\beta)}\left(\frac{dW}{dS}\right)_T\right]^{1/2}. \tag{34}$$

It is essential to note that T and H are functions of β but *not* of Δu and τ_∞.

The physical appearance of the surface of the solid $y < 0$ after passage of the pulse of sliding may be influenced by large values of σ_x. Thus cracking may occur near the surface. Since in addition to σ_x also the stress $\tau_0(x) > 0$ acts, the direction of cracks should be inclined forwards.

The situation becomes different if calculated stresses $|\sigma_x|$ exceeding σ_0 are present at the mating surfaces *along a major part of the region of sliding.* This situation prevails if (see equation (8)):

$$|\sigma_x|_{\text{average}} = 2(1-k^2)G\Delta u/b > \sigma_0, \tag{35}$$

i.e. if

$$\Delta u > \Delta u_1 = \frac{\sigma_0(T-H)^2}{4\pi^2(1-k^2)G(\tau_\infty - \tau_D)^2}. \tag{36}$$

Naturally, if the energy dissipation due to the stress σ_x occurs mainly in a layer which is thin compared to the length $(2b)$ of the sliding region, a linearized treatment is still possible if τ_D is appropriately adjusted. However, this adjustment depends on Δu and τ_∞ and hence the adjusted τ_D is not a material property. (The variation along the sliding region is neglected in the discussion for simplicity.) Thus, for each fixed τ_∞, one can make a change

$$\tau_D \to \tau_U(\Delta u;\quad \tau_\infty) \tag{37}$$

where τ_U increases with Δu if $\Delta u > \Delta u_1$. But then equation (17) reads

$$\Delta u\,[\tau_\infty - \tau_U(\Delta u;\quad \tau_\infty)] = m(\beta). \tag{38}$$

The left member should possess a maximum at some value of Δu, say $\Delta u = \Delta u_2$. This maximum corresponds to a maximum attainable (non-dimensional) velocity β_2. A rough estimate of this velocity can be found by using condition (35) together with equations (14) and (16):

$$f(\beta_2) \approx \frac{1}{2\pi}\cdot\frac{T+H}{T-H}\cdot\frac{\sigma_0}{\tau_\infty - \tau_D} \tag{39}$$

where the value $|\sigma_x|_{\text{average}} \approx \sigma_0$ has been assumed.

If $|\sigma_x|$ exceeds σ_0 over a major portion of the sliding region, non-linear effects, especially cracking, may occur to rather large depths from the sliding surfaces.

3.4 THE SIGNIFICANCE OF THE 'STATIC FRICTION'

In the preceding theory the 'static friction', i.e. the stress τ_s at which sliding is usually presumed to start, does not enter into the expressions found. Thus, for instance, equations

(14) and (16) are formally valid even for $\tau_\infty > \tau_s$. This seemingly inconsistent result is explained by the assumption made that the sliding motion (with prescribed total slip Δu on each side) by some mechanism is nucleated at $x = -\infty$ and propagated towards $x = +\infty$ from this site. Such a mechanism could, for instance, be due to a local pile up of stresses of the kind schematically shown in Fig. 1. However, if completely uniform conditions are assumed the remote stress τ_∞ could be increased to τ_s. But, as soon as it reaches τ_s, sliding would be nucleated at many sites along $y = 0$. The model therefore simply breaks down when $\tau_\infty \geqslant \tau_s$.

It would be tempting to assume that (for $\tau_\infty < \tau_s$) the shear stress at $x = b$ equals τ_s, i.e. $\tau_D + \tau_1(b) = \tau_s$. However, the stress at which sliding starts at the leading edge could be considerably larger than τ_s if rate effects are present. At each position along $y = 0$ the shear stress $\tau_{yx}(x)$ does not deviate very much from τ_∞ until the leading edge comes rather close. The shear stress could then rise from approximately τ_∞ to a maximum value in a very short time.

It seems reasonable to assume that a rate sensitivity, if it exists, would be of the same character as the one of delayed yielding. This would mean that practically no 'damage' takes place as long as $\tau_{yx}(x) < \tau_s$ and that the stress at which tearing starts is fairly insensitive to changes of the time duration by a factor of the order of 10. Then the postulate that T (as well as H) is a material property would still be reasonable, except for $\tau_s - \tau_\infty \ll \tau_s - \tau_D$, since then $\tau_{yx}(x)$ may exceed τ_s at a very large distance ahead of the region of sliding.

It should also be noted that the actual shear stress at the tearing region is not necessarily identical to the stress $\tau_1(x)$ used in the mathematical formulation. $\tau_1(x)$ is a stress which does not need to be specified closer than by the value it imposes on the integrals in equation (13). The actual shear stress in the symmetry plane, if used in the place of $\tau_1(x)$, may impose other values than T and H on these integrals.

4 Extensional slip propagation

Consider two semi-infinite, homogeneous, isotropic and linearly elastic solids in contact along the plane $z = 0$. The remote stress is $\tau_{xz} = \tau_\infty > 0$. A sliding motion starts at $x = 0$ and extends symmetrically with constant velocity $V = \beta c$, where c, as in Section 3, is the propagation velocity of irrotational waves. It is assumed that a shear stress

$$\tau_{zx} = \tau_0(\tau^2/x^2), \qquad |x| < \beta\tau \tag{40}$$

where $\tau = ct$, t being the time, acts on the mating surfaces in the region of sliding, see Figs 14–15.

For $z = 0$ the displacement u_x in the positive x direction and the shear stress $\tau_{zx}(\tau/|x|)$ are given by the expressions (see Appendix 2):

$$\frac{\partial^2 u_x}{\partial \tau^2} = \frac{1}{2\pi k^2 G|x|}\left(\frac{\tau^2}{x^2} - \frac{1}{\beta^2}\right)^{-1/2} I\left(\frac{\tau^2}{x^2}\right) U\left(\frac{\tau}{|x|} - \frac{1}{\beta}\right) \tag{41}$$

$$\tau_{zx} = \begin{cases} \tau_0(\tau^2/x^2) & \text{for } \tau > |x|/\beta \\ \tau_\infty - \dfrac{2}{\pi}\,\mathrm{Re}\displaystyle\int_1^{\tau/|x|} M(v^2)\,dv & \text{for } |x| < \tau < |x|/\beta \\ \tau_\infty & \text{for } \tau < |x| \end{cases} \tag{42}$$

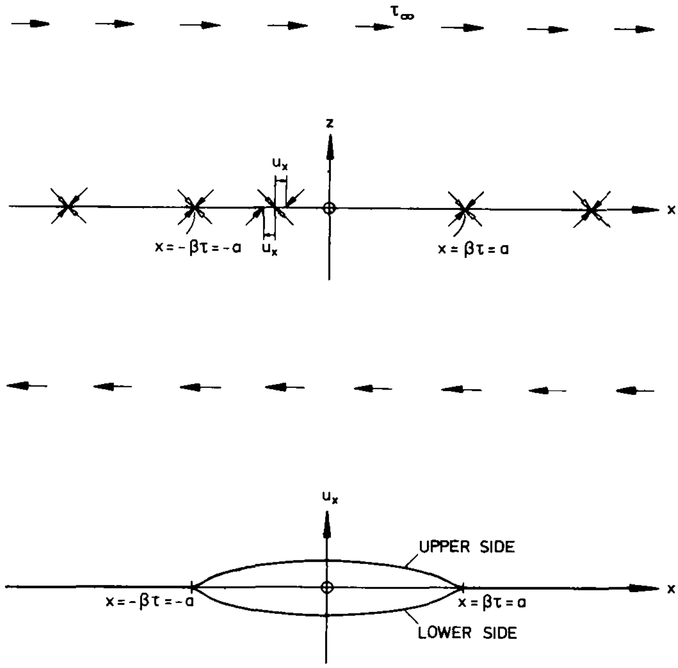

Figure 14. The displacement u_x at the interface $y = 0$ shown in the xz plane (upper figure) and in the xu_x plane (lower figure). Empty arrowheads indicate original, filled ones current positions of material points at the interface.

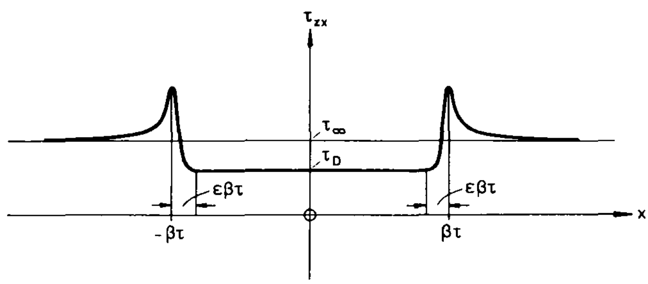

Figure 15. The shear stress τ_{zx} at the interface. No slip occurs at $|x| > \beta\tau$. Idealized dry friction sliding is fully developed in the region $|x| < (1 - \epsilon)\beta\tau$. The regions $(1 - \epsilon)\beta\tau < |x| < \beta\tau$ (the tearing regions) are transition regions where idealized dry friction sliding is developed.

where U is the unit step function and

$$I(u) = \int_{1/\beta^2}^{\infty} \frac{s^{1/2}(s - 1/\beta^2)^{1/2} L(s)\tau_0'(s)ds}{s - u} \tag{43}$$

$$M(u) = [L(u)]^{-1}(1/\beta^2 - u)^{-1/2} I(u) \tag{44}$$

$$L(u) \doteq (u - 1/k^2)^{1/2}[u(u-1)^{1/2}(u - 1/k^2)^{1/2} - (1/2k^2 - u)^2]^{-1}. \tag{45}$$

The condition of stress continuity reads:

$$\tau_\infty = \tau_0(1/\beta^2) + \frac{2}{\pi}\mathrm{Re}\int_1^{1/\beta} M(v^2)dv. \tag{46}$$

The case of *idealized dry friction* will now be studied. To this end it is assumed that $\tau_0(\tau^2/x^2)$ equals a constant, the dynamic friction stress τ_D, except near the ends of the region of sliding. Thus:

$$\tau_0(\tau^2/x^2) = \tau_D \quad \text{for } |x|/\tau < \beta(1-\epsilon) = \delta \tag{47}$$

where $\epsilon\beta\tau \ll \beta\tau$ is the length of the regions where idealized dry friction sliding is initiated, i.e. the tearing regions.

It is convenient to introduce the notation

$$\tau_1(|x|;\ \tau) = \tau_0(\tau^2/x^2) - \tau_D, \quad |x|/\tau < \beta. \tag{48}$$

Then (see Appendix 3) equation (46) yields:

$$(\tau_\infty - \tau_D)(2a)^{1/2} = \frac{2}{\pi} h(\beta)\left(\frac{a}{a_*}\right)^{1/2} T \tag{49}$$

where

$$T = \int_{a_*-d_*}^{a_*} \frac{\tau_1(|x|;\ \tau_*)}{(a_* - |x|)^{1/2}}\, d|x| \tag{50}$$

$$a_* = \beta\tau_*$$

$$d_* = (\beta - \delta)\tau_*$$

$$a = \beta\tau$$

$$h(\beta) = \frac{L(1/\beta^2)g_2(\beta)}{4k^2(k^2-\beta^2)\beta} \to 1 \quad \text{as } \beta \to 0 \tag{51}$$

$$g_2(\beta) = \frac{8k^2(k^2-\beta^2)}{\beta^2} E(\sqrt{1-\beta^2}) - 4k^2(k^2-\beta^2)K(\sqrt{1-\beta^2})$$
$$- \frac{8k^4 - 8k^2\beta^2 + \beta^4}{\beta^2} E(\sqrt{1-\beta^2/k^2}) + (4k^2 - 3\beta^2)K(\sqrt{1-\beta^2/k^2}). \tag{52}$$

K and E are the complete elliptic integrals of the first and the second kind. τ_* is an arbitrary time measure. a is obviously the current half-length of the region of sliding and d_* is the length of the tearing region at $a = a_*$. $h(\beta)$ increases monotonically from unity at $\beta = 0$ to infinity at $\beta = c_s/c$, where c_s is the Rayleigh wave velocity, see Fig. 16.

One observes that T changes with τ_* because of the relation (48), i.e. the assumption of self-similarity is not consistent with the concept of a modulus of tearing as a material property. However, the smallness of the tearing region, compared to the length of the region of sliding (except in the early stages) enables a quasi-dynamic treatment, such that the concept of a modulus of tearing can be used. From the structure of the right member of equation (50) follows that such a consideration implies that equations (49) and (50) change to

$$(\tau_\infty - \tau_D)(2a)^{1/2} = \frac{2}{\pi} h(\beta) T \tag{53}$$

$$T = \int_{a-d}^{a} \frac{\tau_1(|x|)}{(a - |x|)^{1/2}}\, d|x| \tag{54}$$

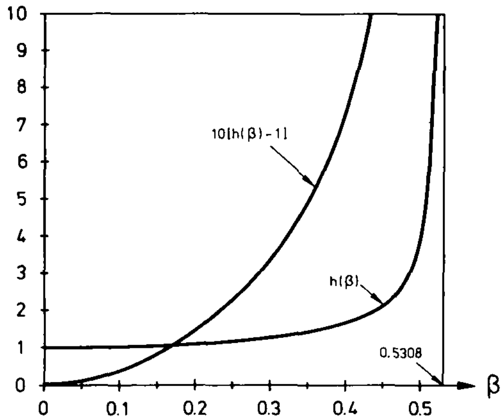

Figure 16. The function $h(\beta)$ for $k^2 = 1/3$. The non-dimensional Rayleigh wave velocity is $\beta = 0.5308$.

where d is a constant length and $\tau_1(x)$ is a function of $(a - |x|)$ such that $\tau_1(|x|) = 0$ for $|x| < a - d$. Naturally, as in Section 3, T should depend on the (non-dimensional) velocity β and on the normal stress σ_z.

Equations (53) and (54) obviously parallel equations (14) and (13) in Section 3. The right member of equation (53), which can be written $2h(\beta)T(\beta)/\pi$, for clarity, is a monotonically increasing function of β (at least if T is not decreasing at increasing β). Thus the velocity β increases with a, i.e. the edges of the sliding region are accelerated during the extension. It is assumed that this acceleration can also be described in a quasi-dynamic treatment, so that equation (53) is still (approximately) valid even if β varies with a.

One observes that a certain minimum length, $2a_0$, of the region of slip nucleation (for instance a region where the 'static friction' is lower than τ_∞) is required to initiate sustained extensional sliding. Thus from equation (53):

$$2a_0 = \frac{1}{\pi^2}\left[\frac{2T(0)}{\tau_\infty - \tau_D}\right]^2. \tag{55}$$

From equation (41) one obtains for $\beta - |x|/\tau \gg \beta - \delta$:

$$\frac{\partial^2 u_x}{\partial \tau^2} = \frac{L(1/\beta^2)}{4\pi k^2 G\beta^2 |x|}\left(\frac{2}{a_*}\right)^{1/2} T\left(\frac{\tau^2}{x^2} - \frac{1}{\beta^2}\right)^{-3/2}.$$

Then use of equation (49) and (51) gives

$$\frac{\partial^2 u_x}{\partial \tau^2} = \frac{k^2 - \beta^2}{\beta g_2(\beta) G|x|}(\tau_\infty - \tau_D)\left(\frac{\tau^2}{x^2} - \frac{1}{\beta^2}\right)^{-3/2}.$$

Since u_x should be much smaller in the tearing regions than in the central part of the region of sliding one obtains after two integrations:

$$u_x = \frac{(k^2 - \beta^2)(\tau_\infty - \tau_D)}{g_2(\beta)G}(a^2 - x^2)^{1/2} \quad \text{for } a - |x| \gg d. \tag{56}$$

This expression holds also in the quasi-dynamic formulation, with the proper relation between β and a found from equation (53).

The maximum amount of slip is

$$\Delta u = \frac{(k^2 - \beta^2)(\tau_\infty - \tau_D)a}{g_2(\beta)G} \tag{57}$$

on each side. In view of equation (53) this equation can also be written

$$\Delta u = \frac{f(\beta)}{\pi^2(1-k^2)G} \cdot \frac{T^2}{\tau_\infty - \tau_D} \tag{58}$$

where $f(\beta) = (1-k^2)L(1/\beta^2)/(2k^2\beta)$ is the same function as used in equation (16).

The energy dissipation per unit of area of the interface after the tearing region has passed by is given by the expression (see Appendix 4):

$$\frac{dW}{dS} = 2\tau_D u_x(\tau/|x|) + \frac{f(\beta)}{2\pi(1-k^2)G} T^2 . \tag{59}$$

One observes the similarity between this equation and equation (22).

5 Summary and conclusions

By introducing two material constants (though dependent on slip propagation velocity and normal stress), the moduli of tearing (T) and healing (H), transient sliding motion can be described in a rather simple way.

The length of a pulse of uni-directional sliding is found to be

$$2b = \frac{1}{\pi^2}\left(\frac{T-H}{\tau_\infty - \tau_D}\right)^2$$

where τ_∞ is the remote shear stress and τ_D the dynamic friction stress.

The length of a region of symmetrical extensional sliding is

$$2a = \frac{h(\beta)}{\pi^2}\left(\frac{2T}{\tau_\infty - \tau_D}\right)^2$$

where $h(\beta)$ is a function of the non-dimensional slip propagation velocity β. At very low velocities $h(\beta)$ equals unity. The equation gives β as a function of a. One observes that a minimum initial length

$$2a_0 = \frac{1}{\pi^2}\left[\frac{2T(0)}{\tau_\infty - \tau_D}\right]^2$$

is required to initiate sustained extensional slip propagation.

The amount of slip (on each side) produced in uni-directional propagation is

$$\Delta u = \frac{f(\beta)}{4\pi(1-k^2)G} \cdot \frac{T^2 - H^2}{\tau_\infty - \tau_D}$$

where $f(\beta)$ is a function of the slip propagation velocity. At very low velocities $f(\beta)$ equals unity. The equation gives β as a function of Δu. One observes that a minimum amount of dislocation at each side,

$$\Delta u_0 = \frac{[T(0)]^2 - [H(0)]^2}{4\pi(1-k^2)G(\tau_\infty - \tau_D)}$$

is required to initiate sustained uni-directional slip propagation.

The amount of slip (on each side) during extensional slip propagation is, except near the tearing regions:

$$u_x = \frac{(k^2 - \beta^2)(\tau_\infty - \tau_D)}{g_2(\beta)G}(a^2 - x^2)^{1/2}$$

where $g_2(\beta)$ is a function of the slip propagation velocity. For comparison with uni-directional slip propagation this expression can also be written, for $x = 0$:

$$\Delta u = \frac{f(\beta)}{\pi^2(1 - k^2)G} \cdot \frac{T^2}{\tau_\infty - \tau_D}.$$

The energy dissipation per unit of area of the interface, after a pulse of uni-directional sliding has passed by, is

$$\frac{dW}{dS} = 2\tau_D \Delta u + \frac{f(\beta)}{2\pi(1 - k^2)G}(T^2 + H^2)$$

where the second term of the right member represents dissipation due to tearing and healing and the first term represents dissipation due to sliding between these regions. The ratio between the second and the first term of the right member is

$$\kappa = \frac{T^2 + H^2}{T^2 - H^2} \cdot \frac{\tau_\infty - \tau_D}{\tau_D}$$

showing that the energy dissipation due to tearing and healing is not necessarily small compared to the total energy dissipation.

The energy dissipation per unit of area of the interface, after the tearing region of an extensional sliding motion has passed by, is given by the expression

$$\frac{dW}{dS} = 2\tau_D u_x + \frac{f(\beta)}{2\pi(1 - k^2)G} T^2$$

where u_x is the current amount of slip (on each side) at the position regarded.

The mass-velocities at points outside the slip plane can be described by reasonably simple expressions with the use of the tearing–healing parameter

$$q = \frac{T/H + 1}{T/H - 1}$$

in the case of uni-directional sliding motion.

The introduction of the moduli T and H is a way to linearize the problem, completely analogous to the introduction of the modulus of cohesion that Barenblatt, with deep understanding of the physical processes, suggested for equilibrium cracks (Barenblatt 1959). The modulus of tearing is related to the energy dissiaption per unit of slip propagation and per unit of width, due to tearing, $(dW/dS)_T$, through the equation

$$T(\beta) = \left[\frac{2\pi(1 - k^2)G}{f(\beta)}\left(\frac{dW}{dS}\right)_T\right]^{1/2}.$$

Naturally, for the modulus of healing, the same relation is valid after the change $T \rightarrow H$.

The modulus of tearing is larger than the modulus of healing. One reason is that the compressive normal stress at the interface counteracts tearing but cooperates in healing. This mechanism is thus implicitly though not explicitly accounted for in the linearized analysis.

The linearized theory becomes invalid when substantial non-linear effects are present not only near the tearing and healing regions but also at the sides of the intermediate region. Cracks, inclined forwards, can be opened at the side where the mass velocity is opposite to the propagation velocity. Then, in uni-directional slip propagation, a very large amount of slip should be propagated in successive fast-travelling pulses rather than in one single, slower travelling pulse.

The importance of regions of tearing and healing in transient sliding motion has previously been recognized by Knopoff, Mouton & Burridge (1973).

No attention has been paid to the fact that frictional sliding is usually a very irregular process, due to inhomogeneous conditions at the sliding surfaces. Such effects could, in principle, be incorporated in the theory by ascribing each fault its characteristic signature, given in terms of quantities used to describe stochastic processes. However, complications would arise, because the simple Galilean transformation or the self-similarity could no longer be taken advantage of.

Acknowledgment

The present work has been carried out during the author's stay at the California Institute of Technology as a Fairchild Scholar, 1976 March 18–September 17.

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Appendix 1

The following problem is considered:

A linearly elastic, homogeneous and isotropic semi-infinite solid, $y > 0$, is subjected to a remote stress $\tau_{xy} = \tau_\infty > 0$ and to the stress $\tau_{yx} = \tau_0(x - Vt)$ on $y = 0$, $-b < x - Vt < b$, where V is a constant velocity and t the time. The displacement in the positive x direction, u, is zero on $y = 0, x - Vt > b$ and equals a constant on $y = 0, x - Vt < -b$. The stress σ_y is constant and can be taken to be zero. The stresses τ_{yx} and σ_x on $y = 0$ are sought as well as the displacements u and v (the displacement in the positive y direction) in the solid.

A simpler problem will first be solved. In this problem the semi-infinite solid is subjected to the stresses $\tau_{yx} = P\delta(x - Vt)$, $\sigma_y = 0$ on $y = 0$, where $\delta(x)$ is the Dirac delta-function. The displacement u is sought.

If one puts

$$\begin{cases} u = \dfrac{\partial \phi}{\partial x} + \dfrac{\partial \psi}{\partial y} & \text{(A1.1)} \\[2ex] v = \dfrac{\partial \phi}{\partial y} - \dfrac{\partial \psi}{\partial x} & \text{(A1.2)} \end{cases}$$

then the potential functions ϕ and ψ satisfy the equations

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2\phi}{\partial t^2} & \text{(A1.3)} \\ \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = \dfrac{1}{k^2c^2} \cdot \dfrac{\partial^2\psi}{\partial t^2} & \text{(A1.4)} \end{cases}$$

where c is the propagation velocity of irrotational waves and kc the propagation velocity of equivoluminal waves. The stresses can be written in the following way:

$$\sigma_x = \rho c^2 \left[\frac{\partial^2\phi}{\partial x^2} + (1 - 2k^2) \frac{\partial^2\phi}{\partial y^2} + 2k^2 \frac{\partial^2\psi}{\partial x \partial y} \right] \quad \text{(A1.5)}$$

$$\sigma_y = \rho c^2 \left[\frac{\partial^2\phi}{\partial y^2} + (1 - 2k^2) \frac{\partial^2\phi}{\partial x^2} - 2k^2 \frac{\partial^2\psi}{\partial x \partial y} \right] \quad \text{(A1.6)}$$

$$\tau_{xy} = \rho k^2 c^2 \left[2 \frac{\partial^2\phi}{\partial x \partial y} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \right]. \quad \text{(A1.7)}$$

The boundary conditions are

$$\begin{cases} \tau_{yx} = P\delta(x - Vt) & \text{(A1.8)} \\ \sigma_y = 0 & \text{(A1.9)} \end{cases}$$

for $y = 0$.

It is immediately obvious that the field quantities in the problem are functions of $x - Vt$, only. Therefore the transformation

$$\begin{cases} x - Vt = x' \\ y = y'. \end{cases}$$

is introduced. For simplicity, in the following treatment the prime sign is dropped so that x actually stands for $x - Vt$. Equations (A1.1–2, 5–7, 9) then remain unchanged, whereas equations (A1.3–4) are changed to

$$\begin{cases} a_1^2 \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0 & \text{(A1.10)} \\ a_2^2 \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0 & \text{(A1.11)} \end{cases}$$

where

$a_1^2 = 1 - V^2/c^2, \quad a_1 > 0$

$a_2^2 = 1 - V^2/k^2c^2, \quad a_2 > 0.$

Further, equation (A1.8) is changed to

$$\tau_{yx} = P\delta(x) \quad \text{for } y = 0. \quad \text{(A1.12)}$$

A solution to equations (A1.10–11), satisfying the symmetry conditions of the problem and the condition of boundedness of field quantities as $y \to \infty$, is given by

$$\begin{cases} \phi = \int_0^\infty A \exp(-a_1 \alpha y) \sin \alpha x \, d\alpha & \text{(A1.13)} \\ \psi = \int_0^\infty C \exp(-a_2 \alpha y) \cos \alpha x \, d\alpha & \text{(A1.14)} \end{cases}$$

where A and C are functions of α. Use of the boundary conditions (A1.9) and (A1.12) gives

$$\begin{cases} \int_0^\infty \alpha^2 [(1 + a_2^2) A - 2a_2 C] \sin \alpha x \, d\alpha = 0 \\ G \int_0^\infty \alpha^2 [-2a_1 A + (1 + a_2^2) C] \cos \alpha x \, d\alpha = P\delta(x) \end{cases}$$

where $G = \rho k^2 c^2$ is the modulus of rigidity.

Inversion gives

$$\begin{cases} (1 + a_2^2) A - 2a_2 C = 0 \\ -2a_1 A + (1 + a_2^2) C = \dfrac{P}{\pi G \alpha^2}. \end{cases}$$

Thus

$$\begin{cases} A = -\dfrac{P}{\pi G} \cdot \dfrac{2a_2}{R(\beta)} \cdot \dfrac{1}{\alpha^2} & \text{(A1.15)} \\ C = -\dfrac{P}{\pi G} \cdot \dfrac{1 + a_2^2}{R(\beta)} \cdot \dfrac{1}{\alpha^2} & \text{(A1.16)} \end{cases}$$

where

$$R(\beta) = 4a_1 a_2 - (1 + a_2^2)^2, \quad \beta = V/c.$$

From equations (A1.1–2) and (A1.13–16) one obtains for $y > 0$:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{2P}{\pi G} \cdot \dfrac{a_2}{R(\beta)} \int_0^\infty \left[\exp(-a_1 \alpha y) - \dfrac{1 + a_2^2}{2} \exp(-a_2 \alpha y) \right] \sin \alpha x \, d\alpha \\ \dfrac{\partial v}{\partial x} = \dfrac{2P}{\pi G} \cdot \dfrac{a_2}{R(\beta)} \int_0^\infty \left[a_1 \exp(-a_1 \alpha y) - \dfrac{1 + a_2^2}{2a_2} \exp(-a_2 \alpha y) \right] \cos \alpha x \, d\alpha. \end{cases}$$

Integration gives

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{2P}{\pi G} \cdot \dfrac{a_2}{R(\beta)} \cdot u_0(x) & \text{(A1.17)} \\ \dfrac{\partial v}{\partial x} = \dfrac{2P}{\pi G} \cdot \dfrac{a_2}{R(\beta)} \cdot v_0(x) & \text{(A1.18)} \end{cases}$$

where

$$\begin{cases} u_0(x) = \dfrac{x}{a_1^2 y^2 + x^2} - \dfrac{1 + a_2^2}{2} \cdot \dfrac{x}{a_2^2 y^2 + x^2} & \text{(A1.19)} \\ v_0(x) = \dfrac{a_1^2 y}{a_1^2 y^2 + x^2} - \dfrac{1 + a_2^2}{2} \cdot \dfrac{y}{a_2^2 y^2 + x^2}. & \text{(A1.20)} \end{cases}$$

Thus $v_0(x) \to -\pi(1 + a_2^2 - 2a_1 a_2)\delta(x)/2a_2$ as $y \to 0$.

Now the original problem will be treated. Since the solutions (A1.17–18) were obtained for given stresses $\tau_{yx} = P\delta(x)$, $\sigma_y = 0$ on $y = 0$ then for the stresses $\tau_{yx} = \tau(x)$, $\sigma_y = 0$ on $y = 0$ one obtains

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{2}{\pi G} \cdot \dfrac{a_2}{R(\beta)} \cdot \displaystyle\int_{-\infty}^{+\infty} \tau(a) u_0(x - a) da & \text{(A1.21)} \\ \dfrac{\partial v}{\partial x} = \dfrac{2}{\pi G} \cdot \dfrac{a_2}{R(\beta)} \cdot \displaystyle\int_{-\infty}^{+\infty} \tau(a) v_0(x - a) da & \text{(A1.22)} \end{cases}$$

since

$$\tau(x) = \int_{-\infty}^{+\infty} \tau(a)\delta(x - a) da.$$

As $y \to 0$

$$\frac{\partial u}{\partial x} \to \frac{1}{\pi G} \cdot \frac{a_2(1 - a_2^2)}{R(\beta)} \cdot \int_{-\infty}^{+\infty} \frac{\tau(a)}{x - a} da = \frac{f(\beta)}{2\pi(1 - k^2)G} \int_{-\infty}^{+\infty} \frac{\tau(a)}{x - a} da \qquad \text{(A1.23)}$$

where $f(\beta) = 2(1 - k^2)a_2(1 - a_2^2)/R(\beta) \to 1$ as $\beta \to 0$.

The boundary conditions of the problem are

$$\begin{cases} \tau_{xy} = \tau_\infty & \text{as } x^2 + y^2 \to \infty & \text{(A1.24)} \\ \tau_{yx} = \tau_0(x) & \text{for } |x| < b, \ y = 0 & \text{(A1.25)} \\ \sigma_y = 0 & \text{for } y = 0 & \text{(A1.26)} \\ \partial u/\partial x = 0 & \text{for } |x| > b, \ y = 0. & \text{(A1.27)} \end{cases}$$

Introduce the sectionally holomorphic function

$$F(z) = \begin{cases} F_*(z) & \text{for } y > 0 \\ -\overline{F_*(\bar{z})} & \text{for } y < 0 \end{cases}$$

where $z = x + iy$ and

$$F_*(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\tau(a)}{a - z} da, \quad y > 0.$$

The bar-sign denotes complex conjugation.

Further the following substitution is made:

$$2i(1 - k^2)Gu/f(\beta) = U.$$

Then, using the notations

$$\begin{cases} F_+(x) = \lim\limits_{y \to +0} F(x+iy) \\ F_-(x) = \lim\limits_{y \to -0} F(x+iy) \end{cases}$$

one obtains (by the Plemelj formulae):

$$\begin{cases} F_+(x) - F_-(x) = \tau(x) & \text{(A1.28)} \\ F_+(x) + F_-(x) = \partial U/\partial x. & \text{(A1.29)} \end{cases}$$

Thus, considering the boundary conditions (A1.25) and (A1.27), one can write

$$F_+(x) = G(x) F_-(x) + g(x) \qquad \text{(A1.30)}$$

where

$$g(x) = \begin{cases} \tau_0(x) & \text{for } |x| < b \\ 0 & \text{for } |x| < b \end{cases}$$

$$G(x) = \begin{cases} 1 & \text{for } |x| < b \\ -1 & \text{for } |x| > b. \end{cases}$$

By equation (A1.30) a Hilbert problem is formulated. The solution (*cf.* Muskhelishvili 1953) is

$$F(z) = \frac{1}{2\pi i}\left(\frac{b+z}{b-z}\right)^{1/2} \int_{-b}^{b} \frac{\tau_0(s)(b-s)^{1/2}}{(b+s)^{1/2}(s-z)}\,ds + \left(\frac{b+z}{b-z}\right)^{1/2} P(z)$$

for $y > 0$. $P(z)$ is a polynomial that will be determined by making use of condition (A1.24).

One obtains for $|x| < b$:

$$\begin{cases} F_+(x) = \dfrac{1}{2\pi i}\left(\dfrac{b+x}{b-x}\right)^{1/2} \displaystyle\oint_{-b}^{b} \dfrac{\tau_0(s)(b-s)^{1/2}}{(b+s)^{1/2}(s-x)}\,ds + \left(\dfrac{b+x}{b-x}\right)^{1/2} P(x) + ½\tau_0(x) \\ F_-(x) = \dfrac{1}{2\pi i}\left(\dfrac{b+x}{b-x}\right)^{1/2} \displaystyle\oint_{-b}^{b} \dfrac{\tau_0(s)(b-s)^{1/2}}{(b+s)^{1/2}(s-x)}\,\mathrm{ds} - \left(\dfrac{b+x}{b-x}\right)^{1/2} \bar{P}(x) - ½\tau_0(x) \end{cases}$$

where C on the integral sign denotes the Cauchy principal value. Similarly, for $|x| > b$:

$$\begin{cases} F_+(x) = \dfrac{1}{2\pi}\left|\dfrac{b+x}{b-x}\right|^{1/2} \displaystyle\int_{-b}^{b} \dfrac{\tau_0(s)(b-s)^{1/2}}{(b+s)^{1/2}(s-x)}\,ds + i\left|\dfrac{b+x}{b-x}\right|^{1/2} P(x) \\ F_-(x) = -\dfrac{1}{2\pi}\left|\dfrac{b+x}{b-x}\right|^{1/2} \displaystyle\int_{-b}^{b} \dfrac{\tau_0(s)(b-s)^{1/2}}{(b+s)^{1/2}(s-x)}\,ds + i\left|\dfrac{b+x}{b-x}\right|^{1/2} \bar{P}(x). \end{cases}$$

Condition (A1.24) gives

$$P(x) = -i\tau_\infty/2.$$

Thus, making use of equations (A1.28–29), one obtains for $y = +0$:

$$\tau_{yx} = \begin{cases} \tau_0(x) & \text{for } |x| < b \\ -\dfrac{1}{\pi}\left|\dfrac{b+x}{b-x}\right|^{1/2} \displaystyle\int_{-b}^{b} \dfrac{\tau_0(s)\,(b-s)^{1/2}}{(b+s)^{1/2}\,(x-s)}\,ds + \left|\dfrac{b+x}{b-x}\right|^{1/2} \tau_\infty & \text{for } |x| > b \end{cases}$$

$$\frac{\partial u}{\partial x} = \begin{cases} \dfrac{f(\beta)}{2(1-k^2)G}\left\{\dfrac{1}{\pi}\left(\dfrac{b+x}{b-x}\right)^{1/2} \displaystyle\oint_{-b}^{b} \dfrac{\tau_0(s)\,(b-s)^{1/2}}{(b+s)^{1/2}\,(s-x)}\,ds + \left(\dfrac{b+x}{b-x}\right)^{1/2}\tau_\infty\right\} & \text{for } |x| < b \\ 0 \quad \text{for } |x| > b. \end{cases}$$

By putting $x = b + \epsilon$, $\epsilon > 0$, one finds that

$\tau_{yx} \to \tau_0(b-0)$ as $\epsilon \to 0$ if

$$\frac{1}{\pi}\int_{-b}^{b} \frac{\tau_0(s)}{(b^2 - s^2)^{1/2}}\,ds = \tau_\infty. \tag{A1.31}$$

Thereby continuity of the stresses is obtained at $|x| = b$. Use of equation (A1.31) gives for $y = +0$:

$$\tau_{yx} = \begin{cases} \tau_0(x) \quad \text{for } |x| < b & \text{(A1.32)} \\ \dfrac{1}{\pi}\cdot\dfrac{x(x^2-b^2)^{1/2}}{|x|}\displaystyle\int_{-b}^{b} \dfrac{\tau_0(s)ds}{(b^2-s^2)^{1/2}(x-s)} \quad \text{for } |x| > b & \text{(A1.33)} \end{cases}$$

$$\frac{\partial u}{\partial x} = \begin{cases} \dfrac{f(\beta)}{2\pi(1-k^2)G}(b^2-x^2)^{1/2}\displaystyle\oint_{-b}^{b} \dfrac{\tau_0(s)ds}{(b^2-s^2)^{1/2}(s-x)} \quad \text{for } |x| < b & \text{(A1.34)} \\ 0 \quad \text{for } |x| > b. & \text{(A1.35)} \end{cases}$$

Use of equations (A1.5, 13–16) gives, after comparison with equation (A1.23):

$$\sigma_x = 4(1-k^2)G\,\partial u/\partial x \qquad \text{for } y = 0. \tag{A1.36}$$

Equations (A1.34–35) give $\partial u/\partial x$ for $y = +0$. Since $\tau_{yx} = \tau(x)$ is now known for all values of x on $y = 0$, through equations (A1.32–33) one obtains from equation (A1.21) for $y \geqslant 0$:

$$\frac{\partial u}{\partial x} = \frac{2a_2}{\pi G R(\beta)}\int_{-b}^{b} \tau_0(a)u_0(x-a)da + \frac{2a_2}{\pi^2 G R(\beta)}\int_{b}^{\infty}(a^2-b^2)^{1/2}$$

$$\left\{\int_{-b}^{b} \frac{\tau_0(s)ds}{(b^2-s^2)^{1/2}(a+s)}u_0(x+a) + \int_{-b}^{b}\frac{\tau_0(s)ds}{(b^2-s^2)^{1/2}(a-s)}u_0(x-a)\right\} da. \tag{A1.37}$$

From equation (A1.22) it follows that $\partial v/\partial x$ for $y \geqslant 0$ equals the right member of equation (A1.37) after the change $u_0(x \pm a) \to v_0(x \pm a)$ has been made. However, $\partial v/\partial x$

may contain a term corresponding to a rigid body rotation. In order to get rid of this term one shall subtract $(\partial v/\partial x)_{|x|\to\infty}$ from the result found. Thus

$$\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_{u_0\to v_0} - \left[\left(\frac{\partial u}{\partial x}\right)_{u_0\to v_0}\right]_{|x|\to\infty}. \tag{A1.38}$$

For $y = 0$ one finds

$$\frac{\partial v}{\partial x} = -\frac{1 + a_2^2 - 2a_1a_2}{G[4a_1a_2 - (1 + a_2^2)^2]}(\tau_{xy} - \tau_\infty). \tag{A1.39}$$

Appendix 2

The following problem is considered:

A homogeneous, isotropic and linearly elastic semi-infinite solid, $z \geqslant 0$, is subjected to a remote shear stress $\tau_{zx} = \tau_\infty > 0$ and to a stress $\tau_{zx} = \tau_0(\tau^2/x^2)$ on $z = 0$, $|x| < \beta\tau = \beta ct$, where β is a constant, c the propagation velocity of irrotational waves and t the time. It is assumed that $\beta < k$, where k is the ratio between the propagation velocities of equivoluminal and irrotational waves. The displacement u_x in the positive x direction is zero on $z = 0$, $|x| > \beta\tau$. The stress σ_z is constant and can be taken to zero. The stress τ_{zx} on $z = 0$, $|x| > \beta\tau$ and the displacement u_x on $z = 0$, $|x| < \beta\tau$ are sought.

The first part of the treatment contains some frequently used relations, the derivation of which are not given. It parallels an earlier treatment (Broberg 1960) where details can be found.

Due to the self-similarity of the problem, the stress τ_{zx} on $z = 0$ must be of the form

$$\tau_{zx} = \tau_{zx}(\tau/x) = \tau_{zx}(\tau/|x|). \tag{A2.1}$$

It is assumed that $\tau_{zx} - \tau_\infty$ is integrable.

The displacement vector $\bar{u}$ can be related to two potential functions, ϕ and $\bar{\psi}$, through the equations

$$\frac{\partial^2 \bar{u}}{\partial \tau^2} = \text{grad}\,\phi - k^2\,\text{curl}\,\bar{\psi} \tag{A2.2}$$

where ϕ and $\bar{\psi}$ satisfy the equations

$$\text{div grad}\,\phi = \partial^2\phi/\partial\tau^2, \quad \phi = \phi(x, z) \tag{A2.3}$$

$$-\text{curl curl}\,\bar{\psi} = \frac{1}{k^2}\partial^2\bar{\psi}/\partial\tau^2, \quad \bar{\psi} = \psi(x, z)\hat{y} \tag{A2.4}$$

where $\hat{y}$ is the unit vector in the positive y direction.

Since $\sigma_z = 0$, the stress vector on $z = 0$ is

$$\tau_{zx}(\tau/|x|)\hat{x} = \rho c^2[-(1 - 2k^2)\phi\hat{z} + k^2\bar{\psi} \times \hat{z} - 2k^2\partial\bar{u}/\partial z] \tag{A2.5}$$

where ρ is the density and $\hat{x}$ and $\hat{z}$ the unit vectors in the positive x and z directions.

After Laplace-transformation,

$$\Phi = p\int_0^\infty \exp(-p\tau)\phi d\tau$$

$$\Psi = p\int_0^\infty \exp(-p\tau)\psi d\tau$$

$$\bar{w} = p\int_0^\infty \exp(-p\tau)\bar{u} d\tau$$

$$S = p\int_0^\infty \exp(-p\tau)\tau_{zx}(\tau/|x|)d\tau = S(p|x|)$$

one obtains

$$\begin{cases}\Delta\Phi = p^2\Phi & \text{(A2.6)}\\ \Delta\Psi = \dfrac{p^2}{k^2}\Psi & \text{(A2.7)}\end{cases}$$

$$\frac{S}{\rho c^2}\hat{x} = -\frac{2k^2}{p^2}\left(\frac{\partial^2\Phi}{\partial x\partial z} + k^2\frac{\partial^2\Psi}{\partial x^2} - \frac{p^2}{2}\Psi\right)\hat{x} - \frac{1}{p^2}\left[(1-2k^2)p^2\Phi + 2k^2\frac{\partial^2\Phi}{\partial z^2} - 2k^4\frac{\partial^2\Psi}{\partial x\partial z}\right]\hat{z} \qquad \text{(A2.8)}$$

$$p^2\bar{w} = \left(\frac{\partial\Phi}{\partial x} + k^2\frac{\partial\Psi}{\partial z}\right)\hat{x} + \left(\frac{\partial\Phi}{\partial z} - k^2\frac{\partial\Psi}{\partial x}\right)\hat{z}. \qquad \text{(A2.9)}$$

The solution to equations (A2.6–7), satisfying the symmetry conditions of the problem and the condition of boundedness at infinity, can be written:

$$\begin{cases}\Phi = \displaystyle\int_{-\infty}^{\infty} A\exp[-z(p^2+\alpha^2)^{1/2}]\sin\alpha x dx & \text{(A2.10)}\\ \Psi = \displaystyle\int_{-\infty}^{\infty} B\exp\left[-z\left(\frac{p^2}{k^2}+\alpha^2\right)^{1/2}\right]\cos\alpha x dx. & \text{(A2.11)}\end{cases}$$

Then, from equations (A2.8–9) one obtains for $z = 0$:

$$\begin{cases}\dfrac{S}{\rho c^2} = \dfrac{2k^2}{p^2}\displaystyle\int_{-\infty}^{\infty}\left[\alpha(p^2+\alpha^2)^{1/2}A - k^2\left(\frac{p^2}{2k^2}+\alpha^2\right)B\right]\cos\alpha|x|d\alpha & \text{(A2.12)}\\ 0 = \left(\dfrac{p^2}{2k^2}+\alpha^2\right)A - k^2\alpha\left(\dfrac{p^2}{k^2}+\alpha^2\right)^{1/2}B & \text{(A2.13)}\\ p^2 w_x = \displaystyle\int_{-\infty}^{\infty}\left[\alpha A - k^2\left(\frac{p^2}{k^2}+\alpha^2\right)^{1/2}B\right]\cos\alpha|x|d\alpha. & \text{(A2.14)}\end{cases}$$

Inversion of equation (A2.12) and use of equation (A2.13) gives

$$\begin{cases} A = -\dfrac{1}{2\pi G}\dfrac{\alpha p(p^2/k^2+\alpha^2)^{1/2}}{R(p,\alpha)}F\left(\dfrac{p}{\alpha}\right) \\ B = \dfrac{p^2/2k^2+\alpha^2}{k^2\alpha(p^2/k^2+\alpha^2)^{1/2}}A \end{cases}$$

where

$$F\left(\frac{p}{\alpha}\right) = p\int_0^\infty S(p\,|x|)\cos\alpha\,|x|\,d\,|x| \tag{A2.15}$$

and

$$R(p,\alpha) = (p^2/2k^2+\alpha^2)^2 - \alpha^2(p^2+\alpha^2)^{1/2}(p^2/k^2+\alpha^2)^{1/2}.$$

Being a one-sided Laplace-transform of a function that behaves algebraically as $\tau\to\infty$, $S(p\,|x|)$ is regular for $Re\,p>0$ and then also $F(p/\alpha)$ is regular for $Re\,p>0$. Since $F(-p/\alpha)=F(p/\alpha)$, due to the symmetry of the problem, the function $F(p/\alpha)$ is regular in the whole p plane, except for points on the imaginary axis.

Insertion of the expressions for A and B into equation (A2.14) and inversion of equation (A2.15) gives

$$p^2 w_x = \frac{1}{2\pi k^2 G}\int_0^\infty \frac{F(p/\alpha)p^3(p^2/k^2+\alpha^2)^{1/2}}{R(p,\alpha)}\cos\alpha\,|x|\,d\alpha \tag{A2.16}$$

$$pS = \frac{2}{\pi}\int_0^\infty F(p/\alpha)\cos\alpha\,|x|\,d\alpha. \tag{A2.17}$$

Restriction is now made to real, positive values of p. With the substitution

$$i\alpha\,|x| = -p\theta$$

one obtains

$$p^2 w_x = -\frac{1}{2\pi k^2 G\,|x|}\,\mathrm{Re}\,p\int_0^{-i\infty} H\left(\frac{\theta^2}{x^2}\right)L\left(\frac{\theta^2}{x^2}\right)\exp(-p\theta)\,d\theta \tag{A2.18}$$

$$pS = \frac{2}{\pi\,|x|}\,\mathrm{Re}\,p\int_0^{-i\infty} iH\left(\frac{\theta^2}{x^2}\right)\exp(-p\theta)\,d\theta \tag{A2.19}$$

where

$$H\left(\frac{\theta^2}{x^2}\right) = F\left(\frac{|x|}{i\theta}\right)$$

$$L(u) = (u-1/k^2)^{1/2}\,[u(u-1)^{1/2}(u-1/k^2)^{1/2} - (1/2k^2-u)^2]^{-1}.$$

The path of integration can be deformed so that

$$\begin{cases} p^2 w_x = -\dfrac{1}{2\pi k^2 G\,|x|}\,\mathrm{Re}\,p\displaystyle\int_0^\infty H(\theta^2/x^2)L(\theta^2/x^2)\exp(-p\theta)\,d\theta & \text{(A2.20)} \\ pS = \dfrac{2}{\pi\,|x|}\,\mathrm{Re}\,p\displaystyle\int_0^\infty iH(\theta^2/x^2)\exp(-p\theta)\,d\theta & \text{(A2.21)} \end{cases}$$

with indentations below singular points on $\operatorname{Im}\theta = 0$.

The Laplace-transforms can now be inverted after inspection (Cagniard's method), giving

$$\begin{cases} \dfrac{\partial^2 u_x}{\partial\tau^2} = -\dfrac{1}{2\pi k^2 G\,|x|}\operatorname{Re}\{H(\tau^2/x^2)L(\tau^2/x^2)\} & \text{(A2.22)} \\[2ex] \dfrac{\partial\tau_{zx}}{\partial\tau} = \dfrac{2}{\pi|x|}\operatorname{Re}\{iH(\tau^2/x^2)\} = -\dfrac{2}{\pi|x|}\operatorname{Im}H(\tau^2/x^2). & \text{(A2.23)} \end{cases}$$

Now the boundary conditions

$$\begin{cases} u_x = 0 \quad \text{for } \tau < |x|/\beta & \text{(A2.24)} \\ \tau_{zx} = \tau_0(\tau^2/x^2) \quad \text{for } \tau > |x|/\beta & \text{(A2.25)} \end{cases}$$

shall be satisfied. After writing

$$\begin{cases} \partial^2 u_x/\partial\tau^2 = 0 \quad \text{for } \tau < |x|/\beta \\[1ex] \partial\tau_{xz}/\partial\tau = \dfrac{2\tau}{x^2}\tau_0'(\tau^2/x^2) \quad \text{for } \tau > |x|/\beta \end{cases}$$

one obtains:

$$\begin{cases} \operatorname{Re}\{HL\} = 0 \quad \text{for } \tau < |x|/\beta & \text{(A2.26)} \\[1ex] \operatorname{Im}H = -\dfrac{\pi\tau}{|x|}\cdot\tau_0'(\tau^2/x^2) \quad \text{for } \tau > |x|/\beta. \end{cases}$$

Since $L(\tau^2/x^2)$ is real for $\tau > |x|/\beta$ the last equation can be written:

$$\operatorname{Im}\{HL\} = -\frac{\pi\tau}{|x|}\tau_0'(\tau^2/x^2)L(\tau^2/x^2) \quad \text{for } \tau > |x|/\beta. \tag{A2.27}$$

Equations (A2.26–27) form a Hilbert problem (of the Keldysh–Sedov variety), the solution of which (*cf.*, e.g. Muskhelishvili 1953) is:

$$H(\tau^2/x^2)L(\tau^2/x^2) = -\pi i\frac{\tau}{|x|}\tau_0'\left(\frac{\tau^2}{x^2}\right)L\left(\frac{\tau^2}{x^2}\right)U\left(\frac{\tau}{|x|}-\frac{1}{\beta}\right)$$
$$-\left(\frac{\tau^2}{x^2}-\frac{1}{\beta^2}\right)^{1/2}I_1\left(\frac{\tau^2}{x^2}\right)-\left(\frac{\tau^2}{x^2}-\frac{1}{\beta^2}\right)^{-1/2}I_0 \tag{A2.28}$$

where U is the unit step function, I_0 is a constant and

$$I_1(u) = \oint_{1/\beta^2}^{\infty}\frac{s^{1/2}\tau_0'(s)L(s)}{(s-1/\beta^2)^{1/2}(s-u)}\,ds$$

where C on the integral sign denotes the Cauchy principal value. Then from equation (A2.22) one obtains:

$$\frac{\partial^2 u_x}{\partial\tau^2} = \frac{1}{2\pi k^2 G\,|x|}\left\{\left(\frac{\tau^2}{x^2}-\frac{1}{\beta^2}\right)^{1/2}I_1\left(\frac{\tau^2}{x^2}\right)+\left(\frac{\tau^2}{x^2}-\frac{1}{\beta^2}\right)^{-1/2}I_0\right\}U\left(\frac{\tau}{|x|}-\frac{1}{\beta}\right). \tag{A2.29}$$

As $\tau/|x| \to \infty$, $\partial u_x/\partial\tau \to C_1 + C_2 x^2/\tau^2 + \ldots$, where C_1 and C_2 are constants. Thus

$$\frac{\partial^2 u_x}{\partial\tau^2} \to -2C_2 x^2/\tau^3 \quad \text{as } \tau/|x| \to \infty.$$

Hence,

$$I_0 = \int_{1/\beta^2}^{\infty} \frac{s^{1/2}\tau_0'(s)L(s)}{(s-1/\beta^2)^{1/2}}\, ds. \tag{A2.30}$$

Now one can write

$$I_1(u) = (1/\beta^2 - u)^{-1} I_0 - (1/\beta^2 - u)^{-1} I(u)$$

where

$$I(u) = \int_{1/\beta^2}^{\infty} \frac{s^{1/2}(s-1/\beta^2)^{1/2} L(s)\tau_0'(s)}{s-u}\, \mathrm{ds}.$$

Thus

$$\frac{\partial^2 u_x}{\partial\tau^2} = \frac{1}{2\pi k^2 G|x|} \cdot \left(\frac{\tau^2}{x^2} - \frac{1}{\beta^2}\right)^{-1/2} I\left(\frac{\tau^2}{x^2}\right) U\left(\frac{\tau}{|x|} - \frac{1}{\beta}\right) \tag{A2.31}$$

and, from equations (A2.23) and (A2.28), for $\tau > |x|$:

$$\frac{\partial\tau_{zx}}{\partial\tau} = \frac{2\tau}{x^2}\tau_0'\left(\frac{\tau^2}{x^2}\right) U\left(\frac{\tau}{|x|} - \frac{1}{\beta}\right) - \frac{2}{\pi|x|}\operatorname{Re} M\left(\frac{\tau^2}{x^2}\right) \tag{A2.32}$$

where

$$M(u) = [L(u)]^{-1}\left(\frac{1}{\beta^2} - u\right)^{-1/2} I(u). \tag{A2.33}$$

Integration of equation (A2.32) gives for $\tau > |x|$, assuming stress continuity:

$$\tau_{zx} = \tau_0(1/\beta^2) - |x| \int_{\tau/|x|}^{1/\beta} \frac{\partial\tau_{zx}}{\partial\tau}\, d\left(\frac{\tau}{|x|}\right) = \tau_0(1/\beta^2)$$

$$+ [\tau_0(\tau^2/x^2) - \tau_0(1/\beta^2)]\, U\left(\frac{\tau}{|x|} - \frac{1}{\beta}\right) + \frac{2}{\pi}\operatorname{Re} \int_{\tau/|x|}^{1/\beta} M(v^2)\, dv \tag{A2.34}$$

so that

$$\tau_\infty = \tau_0(1/\beta^2) + \frac{2}{\pi}\operatorname{Re}\int_1^{1/\beta} M(v^2)\, dv$$

since $\operatorname{Re} M(u) = 0$ for $u < 1$. A combination of this equation and equation (A2.34) gives for $|x| < \tau < |x|/\beta$:

$$\tau_{zx} = \tau_\infty - \frac{2}{\pi}\operatorname{Re}\int_1^{\tau/|x|} M(v^2)\, dv. \tag{A2.35}$$

Appendix 3

Since $\tau_0'(s) = 0$ for $|x|/\tau < \delta$ and $\epsilon \ll 1$ (*cf.* equation (47)), equation (46) can be written:

$$\tau_\infty = \tau_0(1/\beta^2) + \frac{2}{\pi\beta}\int_1^{1/\beta} \frac{\text{Re}\,[L(1/\beta^2)/L(v^2) - 1]}{(1/\beta^2 - v^2)^{3/2}} \int_{1/\beta^2}^{1/\delta^2} (s - 1/\beta^2)^{1/2}\tau_0'(s)\,ds\,dv$$

$$+ \frac{2}{\pi}\int_1^{1/\beta} \frac{1}{(1/\beta^2 - v^2)^{1/2}} \int_{1/\beta^2}^{1/\delta^2} \frac{s^{1/2}(s - 1/\beta^2)^{1/2}\tau_0'(s)}{s - v^2}\,ds\,dv$$

$$= \tau_0(1/\beta^2) - \frac{1}{\pi\beta}\int_1^{1/\beta} \frac{\text{Re}\,[L(1/\beta^2)/L(v^2) - 1]}{(1/\beta^2 - v^2)^{3/2}}\,dv \int_{1/\beta^2}^{1/\delta^2} \frac{\tau_2(s)\,ds}{(s - 1/\beta^2)^{1/2}}$$

$$+ \frac{2}{\pi}\int_{1/\beta^2}^{1/\delta^2} s^{1/2}(s - 1/\beta^2)^{1/2}\tau_0'(s) \int_1^{1/\beta} \frac{dv}{(s - v^2)\,(1/\beta^2 - v^2)^{1/2}}\,ds$$

where $\tau_2(s) = \tau_0(s) - \tau_D$. Since

$$\int_0^{1/\beta} \frac{dv}{(s - v^2)\,(1/\beta^2 - v^2)^{1/2}} = \frac{\pi}{2s^{1/2}(s - 1/\beta^2)^{1/2}}$$

and $s \approx 1/\beta^2$ the last term can be written:

$$\int_{1/\beta^2}^{1/\delta^2} \tau_0'(s)\,ds - \frac{2}{\pi}\int_{1/\beta^2}^{1/\delta^2} \frac{1}{\beta}(s - 1/\beta^2)^{1/2}\tau_0'(s)\,ds \cdot \int_0^1 \frac{dv}{(1/\beta^2 - v^2)^{3/2}}.$$

Then

$$\tau_\infty = \tau_D + \frac{1}{\pi\beta}\int_0^{1/\beta} \frac{\text{Re}\,[1 - L(1/\beta^2)/L(v^2)]}{(1/\beta^2 - v^2)^{3/2}}\,dv \cdot \int_{1/\beta^2}^{1/\delta^2} \frac{\tau_2(s)\,ds}{(s - 1/\beta^2)^{1/2}}$$

$$= \tau_D + \frac{L(1/\beta^2)g_2(\beta)}{4\pi k^2(k^2 - \beta^2)} \cdot \frac{1}{\beta}\left(\frac{2}{a_*}\right)^{1/2} T$$

where

$$g_2(\beta) = 4k^2(k^2 - \beta^2)\beta^2 \int_0^{1/\beta} \frac{\text{Re}\,\{[L(1/\beta^2)]^{-1} - [L(v^2)]^{-1}\}}{(1/\beta^2 - v^2)^{3/2}}\,dv$$

$$T = \beta \cdot \left(\frac{a_*}{2}\right)^{1/2} \int_{1/\beta^2}^{1/\delta^2} \frac{\tau_2(s)\,ds}{(s - 1/\beta^2)^{1/2}}$$

$$= \int_{a_* - d_*}^{a_*} \frac{\tau_2(\tau_*^2/x^2)}{(a_* - |x|)^{1/2}}\,d\,|x|$$

$$a_* = \beta\tau_*$$

$$d_* = \epsilon\beta\tau_*$$

with a_* being an arbitrarily chosen half-length of the sliding region. The function $g_2(\beta)$ has been determined in terms of complete elliptic integrals, see equation (52). It turns out that

$$g_2(\beta) \to 2k^2(1 - k^2) \quad \text{as } \beta \to 0.$$

$g_2(\beta)$ is the mode 2 equivalent to the function $g(\beta)$ for mode 1, given by Broberg (1960).

Appendix 4

The energy dissipation per unit of area of the interface after the tearing region has passed by is

$$dW/dS = 2\int_0^t \tau_{zx} \cdot \frac{\partial u_x}{\partial t} dt.$$

With $\tau_{zx} = \tau_{zx}(\tau/|x|)$ and $u_x = u_x(\tau/|x|)$ one obtains

$$dW/dS = 2\int_{1/\beta}^{\tau/|x|} \tau_{zx}(v) \frac{\partial u_x(v)}{\partial v} dv$$

$$= 2\tau_D u_x(\tau/|x|) + 2\int_{1/\beta}^{1/\delta} (\tau_{zx} - \tau_D)(\partial u_x/\partial v) dv$$

$$= 2\tau_D u_x(\tau/|x|) + 2\int_{1/\beta}^{1/\delta} (\tau_{zx} - \tau_D)\int_{1/\beta}^{v} \frac{\partial^2 u_x}{\partial u^2} du\, dv$$

where $\partial^2 u_x/\partial u^2$, given by equation (41), can be written (since $v \approx w \approx 1/\beta$);

$$\partial^2 u_x/\partial u^2 = \frac{|x|L(1/\beta^2)}{4\pi k^2 G}(u - 1/\beta^2)^{-1/2}\int_{1/\beta}^{1/\delta} \frac{(w - 1/\beta)^{1/2}\tau'_{zx}(w)}{w - u} dw.$$

Insertion of this expression and change of the order of integration gives

$$\frac{dW}{dS} = 2\tau_D u_x - \frac{|x|L(1/\beta^2)}{2\pi k^2 G}\int_{1/\beta}^{1/\delta} [\tau_{zx}(v) - \tau_D]\int_{1/\beta}^{1/\delta} (w - 1/\beta)^{1/2}\tau'_{zx}(w)$$

$$\cdot \int_{1/\beta}^{v} \frac{du}{(u - 1/\beta)^{1/2}(u - w)} dw\, dv = 2\tau_D u_x$$

$$+ \frac{|x|L(1/\beta^2)}{2\pi k^2 G}\int_{1/\beta}^{1/\delta} [\tau_{zx}(v) - \tau_D]\int_{1/\beta}^{1/\delta} \tau'_{zx}(w)\ln|N|\, dw\, dv$$

where

$$N = [(v - 1/\beta)^{1/2} + (w - 1/\beta)^{1/2}] \cdot [(v - 1/\beta)^{1/2} - (w - 1/\beta)^{1/2}]^{-1}.$$

Partial integration gives

$$\frac{dW}{dS} = 2\tau_D u_x + \frac{|x|L(1/\beta^2)}{2\pi k^2 G}\int_{1/\beta}^{1/\delta} [\tau_{zx}(v) - \tau_D](v - 1/\beta)^{1/2}$$

$$\cdot \int_{1/\beta}^{1/\delta} \frac{\tau_{zx}(w) - \tau_D}{(w - 1/\beta)^{1/2}(w - v)} dw\, dv.$$

Then, by using the same technique as the one leading to equation (22) one obtains:

$$\frac{dW}{dS} = 2\tau_D u_x\left(\frac{\tau}{|x|}\right) + \frac{f(\beta)}{2\pi(1 - k^2)G} T^2$$

where $f(\beta)$ is the same function as used in equation (22).