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Hairong Yuan

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# ON TRANSONIC SHOCKS IN TWO-DIMENSIONAL VARIABLE-AREA DUCTS FOR STEADY EULER SYSTEM*

HAIRONG YUAN†

**Abstract.** This paper concerns transonic shocks in compressible inviscid flow passing a two-dimensional variable-area duct for the complete steady Euler system. The flow is supersonic at the entrance of the duct, whose boundaries are slightly curved. The condition of impenetrability is posed on the boundaries. After crossing a nearly flat shock front, which passes through a fixed point on the boundary of the duct, the flow becomes subsonic. We show that to ensure the stability of such shocks, pressure should not be completely given at the exit: it only should be given with freedom one, that is, containing an unknown constant to be determined by the upstream flow and the profile of the duct. Careful analysis shows that this is due to the requirement of conservation of mass in the duct. We used Lagrangian transformation and characteristic decomposition to write the Euler system as a $2 \times 2$ system, which is valid for general smooth flows. Due to such a simplification, we can employ the theory of boundary value problems for elliptic equations to discuss well-posedness or ill-posedness of transonic shock problems in variable-area duct for various conditions giving at the exit.

**Key words.** Euler system, transonic shocks, free boundary problem, hyperbolic-elliptic composite system, ill-posed problem

**AMS subject classifications.** 35L65, 35L67, 76N10

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**1. Introduction.** Using nozzles and pipes to transport and control fluid flows has numerous applications. For instance, one of the important phenomena in gas dynamics is that by appropriate design of a nozzle there may generate one or several shock waves to adjust the supersonic gas flow at the entrance of the nozzle to a certain subsonic state at the exit, which is required, for example, for the jet engines of some types of supersonic airplanes to work. (See, for instance, [31, section 7.13] for detailed discussions.) Other examples are various wind tunnels [25]. Since compressible flows in nozzles exhibit abundant phenomena; such as chocking, local supersonic bubble, formation of shock waves and their interactions with the boundaries of nozzles, etc. (see [12, Chapter 5] and [24, section 6.2.3]), rigorous and thorough mathematical analysis of flows in nozzles is a formidable task.

Nevertheless, progress has been made for several model problems. For unsteady quasi–one-dimensional gas flow in a duct of variable area (see [27, section 8.1]), Liu showed in a series of papers [22], [23], [17] that supersonic and subsonic flows are stable, and for transonic flows, the shock waves tend to decelerate along an expanding duct and accelerate along a contracting duct. See also [18]. For inviscid isentropic irrotational gas flows, using the full velocity potential equation, Chen and Feldman studied the existence and stability of multidimensional transonic shocks through an infinite nozzle and determined the state of the flows at infinity by the data of the coming flows and the geometry of the nozzle [4], [5], [6]. In [28] Xin and Yin considered a similar problem for finite nozzle with a class of conditions involving potential giving at the exit. It is also remarkable that Kuz'min [19] studied subsonic-supersonic smooth

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†Department of Mathematics, East China Normal University, Shanghai 200062, China (hairongyuan0110@gmail.com, hryuan@math.ecnu.edu.cn).

flows in nozzles by using the Chaplygin equation (which is equivalent to the full velocity potential equation) and the simplified von Kármán equation. See also, for example, [9], [10], [13] for other interesting and important works concerning systems of conservation laws, multidimensional shock waves, and flow patterns in nature.

In this paper we study a class of transonic flows with shocks in a two-dimensional variable-area duct for the steady full Euler system, which is a more precise description of compressible inviscid flows. The flow is supersonic at the entrance of the duct, whose boundaries are suitable perturbations of straight lines, while the flow becomes subsonic across a nearly flat shock front. The condition of impenetrability is posed on the boundaries. We will show that for given pressure at the exit, in general such flow patterns may not exist except the pressure at the exit satisfies an additional restriction: something about the pressure has already been determined by the state of the flow at the entrance and the geometry of the duct. Precisely, to ensure the stability of such transonic shocks, the pressure at the exit can be given only apart from a constant difference, that is, it should contain an unknown constant to be solved simultaneously with the flow fields in the duct. The proof reveals that the requirement of conservation of mass in the duct is closely connected to this phenomenon (see Remark 8.1 in section 8.1).

On the other hand, given pressure at the exit is a physically well accepted condition for flows in nozzles [12]. So our result indicates that the transonic shock we investigated here is unstable and not likely to be observed in practice. However, since the flow fields of the transonic shock we studied here are relatively simple, it may help us gain some insight into understanding those more complicated transonic shocks appearing, for example, in de Laval nozzles. Note that all the works cited above on transonic shocks and [8], [7], [29] are devoted to the study of the class of transonic shocks we investigate here.

In this paper we also discuss, from the mathematical point of view, the well-posedness or ill-posedness of a transonic shock problem in variable-area ducts if other conditions are given at the exit. It is shown that for given density, entropy, Mach number, or the velocity component parallel to the axis of the duct, the problem is in general ill-posed; however, it is well-posed for the given velocity component, which is perpendicular to the axis of the duct. For a list of such results, see section 11.

We remark that in [8] Chen has discussed the special case when the boundaries of the duct are straight lines and the flow has certain symmetric properties, while the upstream supersonic flow is perturbed. The author [29] has also investigated the case for flows in a cylinder with cylindrical symmetry by a different method from [8], and the radial velocity vanishing condition was posed at the exit. In [7] Chen and Yuan developed the methods initiated in [29] and solved the transonic shock problem for a three-dimensional steady full Euler system under the periodic conditions on the lateral boundary of the duct, and hence obtained the solution of the transonic shock problem in a three-dimensional duct with a constant square section under certain assumptions on the symmetry of the coming flow. The ill-posedness for given pressure at the exit is also demonstrated in detail there. However, due to the special structure of the two-dimensional stationary Euler system, we developed a different and more powerful approach here and obtained more results.

Now we comment on several difficulties which lie in the transonic shock problem we investigate presently. First is the treatment of the shock front, which is a free boundary and should be determined with the solutions (the subsonic states of the gas flows) simultaneously. Fruitful techniques have been developed in [1], [2], [3],

[8], [20] to deal with the type of free boundaries we meet here. In a rough way, those techniques allow us, by suitable reformulation of the Rankine–Hugoniot jump conditions, to construct a boundary modifying mapping, whose fixed point is the desired free boundary. The definition of the boundary modifying mapping involves solving a series of nonlinear fixed boundary problems.

Second, the steady full Euler system is a hyperbolic-elliptic composite system for subsonic flow. For such a system, classical techniques such as energy estimates, maximum principle, and estimates of fundamental solutions, are not valid in a straightforward way. One needs to separate the "elliptic part" and "hyperbolic part" appropriately to use the classical theory of elliptic and hyperbolic differential equations. To cope with the curved boundary, we also write the system in Lagrangian coordinates by virtue of the law of conservation of mass, and then decompose it into a $2 \times 2$ system (which is elliptic for subsonic flow, hyperbolic for supersonic flow, and of mixed type for transonic flow) and two algebraic equations (one is Bernoulli's law and the other is the invariance of entropy along streamlines for $C^1$ flows). This simplifies greatly the Euler system and enables us to study the transonic shock problem comprehensively. For example, one of the merits of this approach is that it avoids loss of derivatives. We remark that this formulation may be used to study other smooth flow patterns in ducts. Such a technic has already been used by Chen to study a flat Mach configuration in [11] and by Fang to study the transonic shocks attached to a curved wedge [14].

Third, later on we will find out that we need to solve an elliptic system in a rectangular domain. It is well known that the corners may cause singularities in the solutions (even the well-posedness; see an example in [30]), which in turn affect the regularity of the shock front, and then the smoothness of the boundary itself, and then may cause new trouble in the regularity of solutions. Another feature is that the "hyperbolic part" may transport the singularity at the corners produced by the "elliptic part" to other points in the domain. We are lucky that we can use weighted Hölder spaces and the results established in [16] by Gilbarg, Hörmander and in [21] by Lieberman to overcome this difficulty.

Fourth, as mentioned above, it turns out we are in fact dealing with an ill-posed problem if we give directly the pressure at the exit. We will show its relation to the Neumann boundary problems for Poisson equations and determine appropriate boundary conditions to make such a problem well-posed.

We will use the following well-known Banach contraction mapping principle twice to solve the transonic shock problem:

> *Any contractive mapping on a complete metric space has one and only one fixed point.*

To find the transonic shock front, we will show that the boundary modifying mapping is contractive (see section 10). However, to define the boundary modifying mapping, we need again the Banach contraction mapping principle to show that a series of nonlinear fixed boundary problems are uniquely solvable under some hypothesis (see section 9). Due to our great efforts contributed to simplify the original problem (section 3–7), obtaining the necessary estimates is straightforward and not hard work.

The paper is organized as follows. In section 2 we rigorously formulate the problem of transonic shocks in variable-area ducts (denoted as problem (**A**)) and state our main results, i.e., Theorem 2.8. In section 3 we write the Euler system in Lagrangian coordinates, which also transform the curved boundaries of ducts into straight lines. In section 4 we decompose the resulted system into the "elliptic part" and "hyperbolic part" and in section 5 we show the existence of supersonic flow in the ducts. In section 6 we formulate a free boundary problem (denoted as problem (**B**)) and then reduce it to a set of fixed boundary problems (denoted as problem (**C**)) and a boundary modifying problem. In section 7, by rewriting the Rankine–Hugoniot jump conditions we express problem (**C**) in an equivalent but more transparent form (denoted as problem (**D**)). Section 8 is devoted to the linearized version of problem (**D**). In section 9 we solve problem (**C**) by the Banach contraction mapping principle. In section 10 we construct the boundary modifying mapping and show that it has a fix point by using Banach contraction mapping principle once again, thus finishing the proof of our main results, Theorem 2.8. The last section of this paper, section 11, is devoted to well-posedness or ill-posedness of the transonic shock problem (**A**) for various conditions given at the exits of the ducts. The detailed proofs of these results are omitted since they can be done in the same spirit as the proof of Theorem 2.8, but we have sketched out the main points.

## 2. Formulation of the transonic shock problem and main results.

**2.1. Problem (A) and background solution.** The Euler system, which models two-dimensional inviscid steady gas flow, is of the form

$$\begin{cases} \nabla\cdot\mathbf{m}=0, \\ \nabla\cdot\left(\dfrac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right)+\nabla p=0, \end{cases} \tag{2.1}$$

with Bernoulli's law

$$\frac{1}{2}\mathbf{u}^2+i=\text{const}, \tag{2.2}$$

where $\rho, p, i$ are the density, pressure, and enthalpy of the fluid, while $\mathbf{u}=(u,v)$ and $\mathbf{m}=\rho\mathbf{u}$ are the velocity and the momentum density vector, respectively. The first equation in (2.1) is the conservation of mass, the second is the conservation of momentum, and the Bernoulli's law corresponds to the conservation of energy. Note that the const in (2.2) depends on streamlines but is invariant on the same streamline even across a shock [12].

In the case of polytropic gas $p=A(S)\rho^\gamma, \gamma\in(1,\infty)$, with $S$ the entropy, (2.2) takes the form

$$\frac{1}{2}(u^2+v^2)+\frac{a^2}{\gamma-1}=\text{const}, \tag{2.3}$$

where $a=\sqrt{\gamma A(S)\rho^{\gamma-1}}$ is the local speed of sound. For $C^1$ flow, (2.1) can also be written as a symmetric system:

$$\begin{pmatrix} \rho u & 0 & 1 \\ 0 & \rho u & 0 \\ 1 & 0 & \frac{u}{\rho a^2} \end{pmatrix}\partial_x U+\begin{pmatrix} \rho v & 0 & 0 \\ 0 & \rho v & 1 \\ 0 & 1 & \frac{v}{\rho a^2} \end{pmatrix}\partial_y U=0. \tag{2.4}$$

In this form, the first two equations are the conservation of momentum, and the last is the conservation of mass.

*Remark* 2.1. In (2.4) we have set

$$U=\begin{pmatrix} u & v & p \end{pmatrix}^t \tag{2.5}$$

as the state of the gas. Since $a^2 = \gamma p/\rho$, we may obtain $\rho$ from (2.3) once the const in it is known. Sometimes we will also set

$$U = \begin{pmatrix} u & v & p & \rho \end{pmatrix}^t. \tag{2.6}$$

In the latter we will introduce

$$w = \frac{v}{u}. \tag{2.7}$$

To simplify the notation, for $u$ nonzero we will also set $U$ as

$$\begin{pmatrix} u & w & p \end{pmatrix}^t. \tag{2.8}$$

There will be no confusion in using $U$ to express these vectors later.

Without loss of generality, we set $\mathbb{P} := \big\{(x,y) \in \mathbb{R}^2 : x \in [-1,1], 0 \le y \le \Gamma(x)\big\}$ to be a two-dimensional duct with variable sections, and denote the upper wall $\big\{y = \Gamma(x) : x \in [-1,1]\big\}$ as $\Gamma_+$ with $\Gamma(x)$ a positive function, and $\Gamma_- = \big\{y = 0 : x \in [-1,1]\big\}$ the lower wall. We also set $\Gamma_s = \big\{x = s : y \in [0, \Gamma(s)]\big\}$ for $s \in [-1,1]$.

We are interested in the following boundary value problem (**A**):

$$(\mathbf{A}) : \begin{cases} (2.1)(2.2) & \text{in} \quad \mathbb{P}, \\ v = 0 & \text{on} \quad \Gamma_-, \\ v - u\Gamma'(x) = 0 & \text{on} \quad \Gamma_+, \\ U = U_b^- & \text{on} \quad \Gamma_{-1}, \\ p = p_1 & \text{on} \quad \Gamma_1. \end{cases} \tag{2.9}$$

The conditions on $\Gamma_\pm$ mean that the boundary is impermeable and it is natural for ducts without holes on its boundaries. We suppose $U_b^- := \begin{pmatrix} u_b^- & 0 & p_b^- & \rho_b^- \end{pmatrix}^t$ is a constant supersonic state with $\rho_b^- > 0$ on $\Gamma_{-1}$, which represents supersonic flow entering the duct when $u_b^- > 0$. Hence the const in Bernoulli's law (2.3) is

$$c_0 = (u_b^-)^2/2 + (a_b^-)^2/(\gamma - 1)$$

and independent of streamlines. It is necessary to control the pressure at $\Gamma_1$ to obtain transonic flows in $\mathbb{P}$; otherwise the flow may be purely supersonic in $\mathbb{P}$, which is why we need the last condition in (2.9). However, the following simple but fundamental result indicates we may have trouble if we give $p_1$ in an arbitrary way. (Giving other conditions on $\Gamma_1$ instead of $p$ will be discussed in section 11. In the following sections 2–10 we concentrate only on the typical case, i.e., problem (**A**).)

PROPOSITION 2.1. *For the special case $\Gamma(x) \equiv 1$, suppose the solution $U$ to* (2.9) *depends only on $x$; then for given supersonic state $U_b^-$, there exists a unique constant $p_1 = p_b^+$ determined by $U_b^-$ such that*

$$S_b : \{x = 0\} \tag{2.10}$$

*is a shock front with uniform supersonic state $U_b^-$ ahead of it (i.e., in $\{x < 0\}$) and uniform subsonic state $U_b^+ = \begin{pmatrix} u_b^+ & 0 & p_b^+ & \rho_b^+ \end{pmatrix}^t$ behind it (i.e., in $\{x > 0\}$). $S_b$ and $(U_b^-, U_b^+)$ make up a piecewise smooth weak entropy solution to* (2.9) *containing shocks. Here "uniform" means that the states $U_b^\pm$ are constant vectors.*

*Proof.* From the one-dimensional steady Euler system it is obvious that any solution without a jump must be uniform. So $U \equiv U_b^-$ for $x \in [-1, 0]$. The Rankine–Hugoniot jump conditions (see [4, 13]) now takes the form

$$\rho u = \rho_b^- u_b^-, \quad \rho u^2 + p = \rho_b^- (u_b^-)^2 + p_b^-, \quad Eu = E_b^- u_b^-,$$

where $E = \frac{\gamma}{\gamma-1} p + \frac{1}{2}\rho |\mathbf{u}|^2$. By supersonic condition $u_b^- > a_b^-$ it is easy to get

$$\begin{aligned} u &= \frac{c_*}{u_b^-}, \qquad c_* := 2c_0 \frac{\gamma - 1}{\gamma + 1}, \\ \rho &= \frac{\rho_b^- (u_b^-)^2}{c_*}, \\ p &= \rho_b^- (u_b^-)^2 + p_b^- - c_* \rho_b^-. \end{aligned}$$

Direct calculation demonstrates that $u < a$ (subsonic) and the Lax entropy condition (see [13])

$$p > p_b^-. \tag{2.11}$$

So $U_b^+ \equiv U = (u, 0, 0, p, S)^t$ for $x \in [0, 1]$, and thus $p_1 = p$ is uniquely determined by $U_b^-$ as well as $U_b^\pm$. □

*Remark* 2.2. In the rest of this paper we call the above obtained $(U_b^\pm)$ and $S_b : \{x_0 = 0\}$ a *background solution* and denote it as $U_b = (U_b^-, U_b^+; S_b)$. Equation (2.10) is used to fix the position of the shock front since we may set $x = c$ for any $c \in (-1, 1)$ as the shock front and obtain the same $U_b^\pm$. One may also observe from the above result that the pressure $p = p_1$ at $\Gamma_1$ is necessary, though it cannot be given arbitrarily. The main result of this paper, Theorem 2.6 below, shows that for two-dimensional flow this observation is also valid.

**2.2. Function spaces.** Although the background solution provides us with some useful information, however, when $\Gamma(x)$ is slightly curved, rigorously solving problem (**A**) still involves several difficulties, as mentioned in the introduction. For subsonic flow, the steady Euler system is of hyperbolic-elliptic composite type: it has a real (generalized) eigenvalue of multiplicity 1 and a pair of conjugate complex eigenvalues. Due to conservation of mass we can introduce the Lagrangian transformation to reduce the original equations to two algebraic equations (Bernoulli's law and constancy of entropy along streamlines for $C^1$ flows) and a $2 \times 2$ system of partial differential equations, which is hyperbolic for supersonic flow, elliptic for subsonic flow, and of mixed type for transonic flow. Thus to obtain the subsonic flow behind the shock front $S$, we have to confront elliptic boundary value problems on rectangular domains

$$\Omega = \Big\{(x, y) : 0 \le y \le \Gamma(x), f(y) \le x \le 1\Big\}, \tag{2.12}$$

where $x = f(y)$ is the equation of the shock front $S$ which satisfies $f(0) = 0$. Suppose $S$ and $\Gamma_+$ intersect at the point $\Sigma_4 = (x_*, y_*)$. It is well known that the corners

$$\Sigma_1 = (0, 0), \quad \Sigma_2 = (1, 0), \quad \Sigma_3 = (1, \Gamma(1)), \quad \Sigma_4 = (x_*, y_*) \tag{2.13}$$

in general will cause singularities to the solutions, and the popular Schauder theory for $C^{2,\alpha}(\alpha \in (0, 1))$ domains (see [15]) may be invalid. The loss of regularity at the corners will also influence the regularity of the shock front $S$ itself, which in turn has an effect on the smoothness of the domain $\Omega$, and thus may cause new trouble when solving the elliptic problems. The hyperbolic part may also transport the singularities at corners to other points in the duct. Fortunately, Gilbarg and Hörmander [16] and Lieberman [21] have established the *intermediate Schauder estimates* to attack elliptic problems on nonsmooth domain (see also the Notes of Chapter 6 in [15]), and their theory is powerful enough to handle our dilemma. Following their ideas, we introduce the function spaces $H_a^{(b)}(\Omega)$, $H_a^{'b}[0, y_*]$ to describe precisely the regularity of our desired subsonic flow and the transonic shock front, respectively. Our definition is a little different from theirs, but due to the boundary regularity estimates (Lemmas 6.18 and 6.29 in [15]), there is no problem later in using their theorems. Note that in the following we always suppose that $0 \le k < a = k + \alpha \le k + 1, a + b > 0$, with $k$ an integer and $\alpha \in (0, 1]$ for such spaces.

The Banach spaces $H_a^{'(b)}[0, y_*]$ is defined as follows. A function $f$ on $[0, y_*]$ is in $H_a^{'(b)}[0, y_*]$ if and only if

$$\|f\|_{a;[0,y_*]}^{'(b)} := \sup_{\delta > 0} \delta^{a+b} \|f\|_{C^a[\delta, y_* - \delta]} < \infty, \tag{2.14}$$

with $C^a$ here the usual Hölder space $C^{k,\alpha}$. We define $\|\cdot\|_{a;[0,y_*]}^{'(b)}$ as the norm of $H_a^{'(b)}[0, y_*]$.

The Banach space $H_a^{(b)}(\Omega)$ is by definition the set of functions $\phi$ defined on $\Omega$ with the property that

$$\|\phi\|_{a;\Omega}^{(b)} := \sup_{\delta > 0} \delta^{a+b} \|\phi\|_{C^a(\Omega_\delta)} < \infty, \tag{2.15}$$

where

$$\Omega_\delta := \Big\{ P = (\bar\xi, \bar\eta) \in \Omega : \text{distance}(P, \Gamma^\pm) > \delta \Big\}. \tag{2.16}$$

Here $\Omega$ as in (2.12) with $f \in H_a^{'(b)}[0, y_*]$, while $\Gamma^+ := \{(x, \Gamma(x)) : x_* \le x \le 1\}, \Gamma^- := \{(x, 0) : 0 \le x \le 1\}$ are, respectively, the upper and lower boundary of $\Omega$.

Notice that the Lagrangian transformation also has the advantage that it straightens the curved boundary since it straightens the streamlines. So later by introducing certain homeomorphisms $\Phi : \Omega \to [0, 1; 0, 1]$ which are of class $H_a^{(b)}(\Omega)$ with $b < -1$, we will actually solve elliptic boundary problems on the square $[0, 1; 0, 1]$. For simplicity, we write $H_a^{(b)}([0, 1; 0, 1])$ as $H_a^{(b)}$, and $H_a^{'(b)}[0, 1]$ as $H_a^{'(b)}$. The corresponding norms are simply denoted as $\|\cdot\|_a^{(b)}, \|\cdot\|_a^{'(b)}$, respectively. By direct calculations one can verify the following.

Proposition 2.2. *Suppose $b < -1$, $\Omega$ as before, $u \in H_a^{(b)}$, and $\Phi : \Omega \to [0, 1; 0, 1]$ satisfy $\Phi \in H_a^{(b)}(\bar\Omega; \mathbb{R}^2)$. Then $u \circ \Phi \in H_a^{(b)}(\bar\Omega)$ and*

$$\| u \circ \Phi \|_{H_a^{(b)}(\bar\Omega)} \le C \| u \|_{H_a^{(b)}}, \tag{2.17}$$

*where $C = C\big(n, \| \Phi \|_{H_a^{(b)}(\bar\Omega; \mathbb{R}^n)}\big)$.*

A similar result also holds for $H_a^{'(b)}$. This means that the homeomorphisms introduced later (including the Lagrangian transformation) will not influence our resultant estimates.

We list here several useful properties of spaces $H_a^{(b)}$ (they also hold for $H_a^{'(b)}$) which are used later to obtain estimates of certain nonlinear terms. The proof and more information about this class of weighted Hölder spaces can be found in [16].

PROPOSITION 2.3.

$$\|\nabla\phi\|_{a-1}^{(1+b)} \le C\,\|\phi\|_a^{(b)} \qquad \textit{if} \quad a > 1, \tag{2.18}$$

$$\|\phi\|_a^{(b)} \le C\,\|\phi\|_a^{(b')} \qquad \textit{if} \quad b' \le b. \tag{2.19}$$

PROPOSITION 2.4. *If $0 \le a' \le a, a' + b \geqslant 0$, and $b$ is not an integer $\le 0$, then*

$$\|\phi\|_{a'}^{(b)} \le C\,\|\phi\|_a^{(b)}. \tag{2.20}$$

PROPOSITION 2.5. *If $0 \le c_j \le a + b, a \geqslant 0$, then*

$$\|\phi\psi\|_a^{(b)} \le C(\|\phi\|_a^{(b-c_1)}\,\|\psi\|_0^{(c_1)} + \|\phi\|_0^{(c_2)}\,\|\psi\|_a^{(b-c_2)}). \tag{2.21}$$

**2.3. Main results.** There are two main results in this paper. Now we can state precisely the first one as the following theorem. Another is indicated in Remark 2.12. For details, see section 11.

THEOREM 2.6. *There exists a $\varepsilon_0 > 0$ such that if*

$$\|\Gamma(x) - 1\|_{C^5[-1,1]} \le \varepsilon < \varepsilon_0, \tag{2.22}$$

$$\frac{\mathrm{d}^k(\Gamma(x)-1)}{\mathrm{d}x^k}\Big|_{x=-1} = 0, \quad k = 0,1,2,3,4,5, \tag{2.23}$$

*then there is a unique $e \in \mathbb{R}$ with*

$$|e| < C_0\varepsilon \tag{2.24}$$

*such that* (2.9) *with*

$$p_1 = p_b^+ + e \tag{2.25}$$

*has a unique weak entropy solution $(U_-, U_+; S)$ with the following properties:*

(i) *$U_-$ is supersonic, $U_+$ is subsonic, and $S$ is the shock front separating $U_-$ and $U_+$ with entropy condition.*

(ii) *$S : x = f(y)$, $y \in [0, y_*]$, with $y_*$ satisfying $y_* = \Gamma(f(y_*))$ and*

$$f(0) = 0. \tag{2.26}$$

(iii) *For some $\alpha \in (0,1)$, the following estimates hold:*

$$\left\|U_- - U_b^-\right\|_{C^{3,\alpha}(\mathbb{P}_-)} < C_0\varepsilon, \tag{2.27}$$

$$\left\|U_+ - U_b^+\right\|_{2+\alpha;\Omega}^{(-\alpha)} < C_0\varepsilon, \tag{2.28}$$

$$\|f\|_{3+\alpha;[0,y_*]}^{'(-1-\alpha)} < C_0\varepsilon. \tag{2.29}$$

*Here $\varepsilon_0, C_0$ are constants depending only on $U_b$, and we have*

$$\mathbb{P}_- := \{(x,y) \in \mathbb{P} : x < f(y)\},$$

*where $\Omega$ is the same as in* (2.12).

*Remark* 2.3. We suppose (2.26) holds to fix the position of the shock front. This is necessary, as indicated by the translation invariance along the $x$ axis for the background solution: for the same $p_1$ in Proposition 2.2 at the exit, the position of the shock cannot be uniquely determined (see also Remark 2.3 and [4], [28]). We note that this phenomena is different from those transonic shocks observed in de Laval nozzles.

*Remark* 2.4. Equation (2.25) may be replaced by

$$p_1 = p_b^+ + g(y) + e \tag{2.30}$$

with $e$ a constant to be determined simultaneously with $U$ for any $g \in C^{2,\alpha}[0, \Gamma(1)]$ with small norm. There is no additional difficulty in the proof. Giving pressure at the exit in this way implies that the value of the pressure at the exit can be given only apart from a constant difference. We note that $e$ in most cases does not vanish, as was shown by Proposition 2.2 if $g$ is a nonzero number: it has already been determined by the coming upstream flow and the shape of the duct and $g$. This implies that for given pressure at the exit of the duct the transonic shock problem is ill-posed. As can be seen from the background solution, the ill-posedness is not related to the fact that we fixed the position of the shock.

*Remark* 2.5. Our proof can be modified to treat the case when the upstream supersonic flow at the entrance of the duct is also perturbed slightly if certain orders of compatibility conditions hold at the entrance, and a similar result can be proved. The major difference is that the constant in Bernoulli's law may be different on different streamlines. In [7] we have studied this case for the three-dimensional Euler system.

*Remark* 2.6. We note that the method developed in this paper provides us with more information than just presented in Theorem 2.6. It indicates clearly the well-posedness of giving $v, w$ at $\Gamma_1$; ill-posedness of giving $u$, $\rho$, $S$, or the Mach number $M = |\mathbf{u}|/a$ as well as $p$ there will be discussed in detail in section 11.

*Remark* 2.7. We emphasize here that the uniqueness of transonic shock in Theorem 2.6 is proved only in the class of functions satisfying properties (i)–(iii) listed there. The "global uniqueness" is an interesting open problem. Noting the nonuniqueness of transonic shocks claimed by Smith in [26] and symmetry breaking phenomena discussed by Kuz'min [19] and references therein, rigorous analysis of uniqueness or nonuniqueness of certain problems in aerodynamics is very important to understand some widely used models in practice and numerical simulations.

**3. Euler equations in Lagrangian coordinates.** The Euler equations (2.1), (2.2) are difficult to handle directly. In this section we use conservation of mass to write them in Lagrangian coordinates, which simplifies the geometry of the domain, as well as the "hyperbolic" part of the Euler system, as will be shown in the next section.

Set

$$w = \frac{v}{u}, \tag{3.1}$$

and denote the integral curves of

$$\begin{cases} \dfrac{\mathrm{d}\hat{y}(x,h)}{\mathrm{d}x} = w(x, \hat{y}(x,h)), \\ \hat{y}(0,h) = h \end{cases} \tag{3.2}$$

by

$$\begin{cases} x = \xi, \\ \hat{y} = \hat{y}(\xi, h). \end{cases} \tag{3.3}$$

(This is exactly streamlines.) Let

$$\eta = \eta(x, h) = \int_{\hat{y}(x,0)}^{\hat{y}(x,h)} \rho u(x, y) \mathrm{d}y \tag{3.4}$$

be the flux of mass between two such curves. Then by using the first equation in (2.1) we have

$$\begin{aligned} \partial_x \eta = {} & \rho u(x, \hat{y}(x, h)) \frac{\partial \hat{y}(x, h)}{\partial x} \\ & - \rho u(x, \hat{y}(x, 0)) \frac{\partial \hat{y}(x, 0)}{\partial x} - \int_{\hat{y}(x,0)}^{\hat{y}(x,h)} \partial_y (\rho v(x, y)) \mathrm{d}y \\ = {} & 0. \end{aligned} \tag{3.5}$$

Hence $\eta = \eta(-1, h)$ and $\eta(-1, 0) = 0$. Thus if

$$\frac{\partial \eta(0, h)}{\partial h} = \rho u(0, \hat{y}(0, h)) \frac{\partial \hat{y}(0, h)}{\partial h} = \rho u(0, \hat{y}(0, h)) \neq 0, \tag{3.6}$$

we may obtain the inverse function $h = h(\eta)$ of $\eta = \eta(-1, h)$ and $h(0) = 0$. Set

$$y(x, \eta) = \hat{y}(x, h(\eta)); \tag{3.7}$$

then (3.4) becomes

$$\eta = \int_{y(x,0)}^{y(x,\eta)} \rho u(x, s) \mathrm{d}s, \tag{3.8}$$

and by differentiating it with $\eta$ one has

$$\frac{\partial y}{\partial \eta} = \frac{1}{\rho u}. \tag{3.9}$$

Now we introduce the following Lagrangian transformation $(x, y) \mapsto (\xi, \eta)$:

$$\begin{cases} x = \xi, \\ y = y(\xi, \eta). \end{cases} \tag{3.10}$$

Then by

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{pmatrix} 1 & 0 \\ w & \frac{1}{\rho u} \end{pmatrix}, \tag{3.11}$$

we have

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{pmatrix} 1 & 0 \\ -\rho v & \rho u \end{pmatrix}; \tag{3.12}$$

thus

$$\begin{cases} \partial_x = \partial_\xi - \rho v \partial_\eta, \\ \partial_y = \rho u \partial_\eta. \end{cases} \tag{3.13}$$

So a little computation shows that (2.1) or (2.4) may be written as conservation laws

$$\begin{cases} \partial_\xi(\frac{1}{\rho u}) - \partial_\eta w = 0, & \text{(conservation of mass)}, \\ \partial_\xi(u + \frac{p}{\rho u}) - \partial_\eta(pw) = 0, & \text{(conservation of momentum along } \xi), \\ \partial_\xi v + \partial_\eta p = 0, & \text{(conservation of momentum along } \eta), \end{cases} \tag{3.14}$$

or as symmetric system

$$A\partial_\xi U + B\partial_\eta U = 0, \tag{3.15}$$

with

$$A = \begin{pmatrix} u & 0 & \frac{1}{\rho} \\ 0 & u & 0 \\ \frac{1}{\rho} & 0 & \frac{u}{\rho^2 a^2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -v \\ 0 & 0 & u \\ -v & u & 0 \end{pmatrix} \tag{3.16}$$

and

$$U = \begin{pmatrix} u & v & p \end{pmatrix}^t. \tag{3.17}$$

The above transformation is valid if and only if

$$\rho u \neq 0. \tag{3.18}$$

Note that in (3.16) the last equation corresponds to conservation of mass.

Next we consider boundary conditions on $\Gamma_\pm$. By (2.9) and (3.2) (and uniqueness of solutions to Cauchy problems of ODE) $\Gamma_\pm$ are streamlines. Thus $\Gamma_-$ in the $(\xi, \eta)$ coordinates is

$$\tilde{\Gamma}_- : \ \eta = 0, \quad \xi \in [-1, 1], \tag{3.19}$$

while $\Gamma_+$ is

$$\tilde{\Gamma}_+ : \ \eta = \eta_0 = \int_0^1 \rho u(-1, s) \mathrm{d}s, \quad \xi \in [-1, 1]. \tag{3.20}$$

In the latter, without loss of generality we always suppose that $\eta_0 \equiv 1$ by suitable normalization of the unit of the coming flow. The corresponding boundary conditions are

$$\begin{cases} w = 0 & \text{on} \quad \tilde{\Gamma}_-, \\ w = \Gamma'(\xi) & \text{on} \quad \tilde{\Gamma}_+. \end{cases} \tag{3.21}$$

Since the shock front may be curved, with the equation

$$\xi = \psi(\eta), \quad \eta \in [0, 1], \tag{3.22}$$

we introduce further the following transformation $\mathbf{\Phi}_\psi : (\xi,\eta) \mapsto (\bar\xi,\bar\eta)$ to straighten the shock front:

$$\begin{cases} \bar\xi = \dfrac{\xi-\psi(\eta)}{1-\psi(\eta)}, \\ \bar\eta = \eta, \end{cases} \quad \text{or} \quad \begin{cases} \xi = (1-\psi(\bar\eta))\bar\xi + \psi(\bar\eta), \\ \eta = \bar\eta. \end{cases} \tag{3.23}$$

So

$$(\bar\xi,\bar\eta) \in [0,1;0,1]$$

and

$$\begin{cases} \dfrac{\partial}{\partial\xi} = \dfrac{1}{1-\psi(\bar\eta)}\dfrac{\partial}{\partial\bar\xi}, \\ \dfrac{\partial}{\partial\eta} = \dfrac{\partial}{\partial\bar\eta} + \dfrac{(\bar\xi-1)\psi'(\bar\eta)}{1-\psi(\bar\eta)}\dfrac{\partial}{\partial\bar\xi}. \end{cases} \tag{3.24}$$

Thus (3.15) becomes

$$\bar A \partial_{\bar\xi} U + \bar B \partial_{\bar\eta} U = 0, \tag{3.25}$$

with

$$\bar A = A + (\bar\xi - 1)\psi'(\bar\eta) B, \quad \bar B = (1-\psi(\bar\eta))B. \tag{3.26}$$

Finally, note that after passing the shock front, (3.21) in the $(\bar\xi,\bar\eta)$ coordinates is

$$\begin{cases} w = 0 & \text{on} \quad \bar\Gamma_- := \{(\bar\xi,0) : \bar\xi \in [0,1]\}, \\ w = \Gamma'((1-\psi(1))\bar\xi + \psi(1)) & \text{on} \quad \bar\Gamma_+ := \{(\bar\xi,1) : \bar\xi \in [0,1]\}. \end{cases} \tag{3.27}$$

**4. Decomposition of elliptic-hyperbolic composite system.** The idea involved in this section to write (3.25) as separate elliptic and hyperbolic equations is rudimentary. Let $\lambda$ be a generalized eigenvalue of $\bar B$ with respect to $\bar A$:

$$\det(\lambda \bar A - \bar B) = 0, \tag{4.1}$$

and let the corresponding generalized left (row) eigenvector be $l$, i.e.,

$$l\bar B = \lambda l \bar A, \tag{4.2}$$

then multiply (3.25) from the left by $l$ to get

$$l\bar A(\partial_{\bar\xi} + \lambda\partial_{\bar\eta})U = 0. \tag{4.3}$$

Now suppose $\lambda$ (and thus $l$) is complex:

$$\lambda = \lambda_R + i\lambda_I, \quad l = l_R + il_I, \quad i = \sqrt{-1}; \tag{4.4}$$

then (4.3) is equivalent to

$$\begin{cases} l_R\bar A\partial_I U + l_I\bar A\partial_R U = 0, \\ l_R\bar A\partial_R U - l_I\bar A\partial_I U = 0, \end{cases} \tag{4.5}$$

with

$$\partial_R = \partial_{\bar\xi} + \lambda_R \partial_{\bar\eta}, \qquad \partial_I = \lambda_I \partial_{\bar\eta}. \tag{4.6}$$

So roughly speaking, the real eigenvalue $\lambda$ corresponds to the hyperbolic equation (4.3) while the complex $\lambda$ and its conjugation $\bar\lambda$ correspond to the elliptic system (4.5).

System (3.25) has a real eigenvalue $\lambda_0$ and a pair of complex eigenvalues $\lambda = \lambda_R \pm i\lambda_I$ if the Mach number $M = |\mathbf{u}|/a < 1$ (the computation is straightforward and we omit it):

$$\lambda_0 = 0, \tag{4.7}$$

$$\lambda_R = -\frac{(1-\psi)\big((\bar\xi-1)\psi'(u^2+v^2) - \frac{v}{\rho}\big)}{\frac{1}{\rho^2}\left(\frac{u^2}{a^2}-1\right) - (\bar\xi-1)^2\psi'^2(u^2+v^2) + \frac{2}{\rho}(\bar\xi-1)\psi' v}, \tag{4.8}$$

$$\lambda_I = \frac{(1-\psi)\frac{u}{\rho}\sqrt{1-M^2}}{\frac{1}{\rho^2}\left(\frac{u^2}{a^2}-1\right) - (\bar\xi-1)^2\psi'^2(u^2+v^2) + \frac{2}{\rho}(\bar\xi-1)\psi' v}. \tag{4.9}$$

Here and in the following we write $\psi = \psi(\bar\eta)$, $\psi' = \psi'(\bar\eta)$. The corresponding left eigenvectors are

$$l_0 = \begin{pmatrix} u & v & 0 \end{pmatrix}, \tag{4.10}$$

$$l_R = \begin{pmatrix} \frac{\lambda_R}{\rho} - \lambda_R(\bar\xi-1)v\psi' + (1-\psi)v & \lambda_R(\bar\xi-1)\psi' u - (1-\psi)u & -\lambda_R u \end{pmatrix}, \tag{4.11}$$

$$l_I = \begin{pmatrix} \frac{\lambda_I}{\rho} - \lambda_I(\bar\xi-1)v\psi' & \lambda_I(\bar\xi-1)\psi' u & -\lambda_I u \end{pmatrix}. \tag{4.12}$$

Thus

$$l_0\bar A = \begin{pmatrix} u^2 & uv & \frac{u}{\rho} \end{pmatrix}, \tag{4.13}$$

$$l_R\bar A = \begin{pmatrix} (1-\psi)uv & -(1-\psi)u^2 & 0 \end{pmatrix}, \tag{4.14}$$

$$l_I\bar A = \begin{pmatrix} 0 & 0 & -(1-\psi)u\frac{\sqrt{1-M^2}}{\rho} \end{pmatrix}. \tag{4.15}$$

By (4.3), (4.7), and (4.13) we get the hyperbolic equation

$$\frac{1}{2}\partial_{\bar\xi}(u^2+v^2) + \frac{1}{\rho}\partial_{\bar\xi} p = 0 \tag{4.16}$$

if $u \neq 0$. On the other hand, from Bernoulli's law (2.3) one gets

$$\frac{1}{2}\partial_{\bar\xi}(u^2+v^2) + \frac{1}{\gamma-1}\partial_{\bar\xi} a^2 = 0;$$

hence

$$\frac{1}{\rho}\partial_{\bar\xi} p - \frac{1}{\gamma-1}\partial_{\bar\xi} a^2 = 0.$$

By $p = A(S)\rho^\gamma, a^2 = \gamma A(S)\rho^{\gamma-1}$ the above equation is actually the constancy of entropy along streamlines for $C^1$ solutions

$$\partial_{\bar\xi}\left(\frac{p}{\rho^\gamma}\right) = 0. \tag{4.17}$$

Similarly, we can write (4.5) as

$$\begin{cases} \partial_{\bar{\xi}} p + \lambda_R \partial_{\bar{\eta}} p - \beta_1 \partial_{\bar{\eta}} w = 0, \\ \partial_{\bar{\xi}} w + \beta_2 \partial_{\bar{\eta}} p + \lambda_R \partial_{\bar{\eta}} w = 0, \end{cases} \tag{4.18}$$

with

$$\begin{cases} \beta_1 = -\dfrac{(1-\psi)u^3}{\frac{1}{\rho^2}\left(\frac{u^2}{a^2}-1\right) - (\bar{\xi}-1)^2\psi'^2(u^2+v^2) + \frac{2}{\rho}(\bar{\xi}-1)\psi' v}, \\ \beta_2 = -\dfrac{\frac{1}{u\rho^2}(1-\psi)(1-M^2)}{\frac{1}{\rho^2}\left(\frac{u^2}{a^2}-1\right) - (\bar{\xi}-1)^2\psi'^2(u^2+v^2) + \frac{2}{\rho}(\bar{\xi}-1)\psi' v}. \end{cases} \tag{4.19}$$

When $M < 1$, (4.18) is an elliptic system. However, for $M > 1$, i.e., supersonic flow, we can also carry out similar calculations to obtain (4.18), which is a hyperbolic system. Thus if

$$\det \begin{pmatrix} l_0 \\ l_R \\ l_I \end{pmatrix} = (1-\psi)\lambda_I u(u^2+v^2) \tag{4.20}$$

is nonzero, (4.17) and (4.18) are equivalent to (3.25) for $C^1$ solutions. This is true if

$$\rho u \neq 0, \quad M \neq 1, \quad u \neq a, \qquad \|\psi\|_{C^1} \ll 1. \tag{4.21}$$

*Remark* 4.1. The first equation in (4.18) is in essence the conservation of mass. In fact, it is obtained by multiplying by $\lambda_I$ by (3.25). Notice that the third argument in $\lambda_I$ is nonzero and the last equation in (3.25) is the conservation of mass.

**5. Existence of supersonic flow.** In this section we always set $\psi \equiv 0$, and we will use (4.18) to show existence and uniqueness of supersonic flow in the duct $\mathbb{P}$ when its boundary is slightly curved, without considering the conditions at the exit. Now (4.18) is

$$\partial_\xi \begin{pmatrix} w \\ p \end{pmatrix} + \begin{pmatrix} \varpi & \beta \\ \kappa & \varpi \end{pmatrix} \partial_\eta \begin{pmatrix} w \\ p \end{pmatrix} = 0, \tag{5.1}$$

with

$$\varpi = \frac{\rho v a^2}{u^2-a^2}, \quad \beta = \frac{a^2(M^2-1)}{u(u^2-a^2)}, \quad \kappa = \frac{\rho^2 a^2 u^3}{u^2-a^2}.$$

Consider the following mixed initial-boundary value problem:

$$\begin{cases} (5.1) & \text{in} \quad \tilde{\mathbb{P}} = [-1,1]\times[0,1], \\ p = p_b^- & \text{on} \quad \tilde{\Gamma}_{-1} : \xi = -1, \\ w = 0 & \text{on} \quad \tilde{\Gamma}_{-1} : \xi = -1, \\ w = 0 & \text{on} \quad \tilde{\Gamma}_{-} : \eta = 0, \\ w = \Gamma'(\xi) & \text{on} \quad \tilde{\Gamma}_{+} : \eta = 1, \end{cases} \tag{5.2}$$

where

$$p = d_0 \rho^\gamma, \quad \text{or} \quad \rho = \left(\frac{p}{d_0}\right)^{\frac{1}{\gamma}}, \tag{5.3}$$

$$u = \left\{ \frac{2}{1+w^2} \left( c_0 - \frac{\gamma d_0}{\gamma - 1} \left( \frac{p}{d_0} \right)^{\frac{\gamma-1}{\gamma}} \right) \right\}^{\frac{1}{2}}, \tag{5.4}$$

with

$$d_0 = p_b^- / (\rho_b^-)^\gamma. \tag{5.5}$$

Equation (5.3) originates from (4.17), and (5.4) comes from Bernoulli's law (2.3) and (5.3).

*Remark* 5.1. The system (5.1) is genuinely nonlinear in a neighborhood of $U_b^-$. Direct computation shows that the eigenvalues are

$$\lambda_\pm = \varpi \pm \sqrt{\beta\kappa} = \frac{\rho a^2}{u^2 - a^2}(v \pm u\sqrt{M^2 - 1}), \tag{5.6}$$

and the corresponding left (resp., right) eigenvectors $l_\pm$ ($r_\pm$) are

$$l_\pm = \begin{pmatrix} \pm\sqrt{\kappa} & \sqrt{\beta} \end{pmatrix}, \tag{5.7}$$

$$r_\pm = \begin{pmatrix} \pm\sqrt{\beta} & \sqrt{\kappa} \end{pmatrix}^t. \tag{5.8}$$

Thus

$$\nabla\lambda_\pm \cdot r_\pm(U_b^-) = \frac{1}{2}(1+\gamma)\frac{\rho u^4 \sqrt{u}}{(u^2 - a^2)^2}\bigg|_{U = U_b^-} \neq 0. \tag{5.9}$$

Set $W = (w, p)^t$. The characteristic form of (5.1) is

$$(\sqrt{\kappa}, \sqrt{\beta})(\partial_\xi + (\varpi + \sqrt{\beta\kappa})\partial_\eta)W = 0, \tag{5.10}$$

$$(-\sqrt{\kappa}, \sqrt{\beta})(\partial_\xi + (\varpi - \sqrt{\beta\kappa})\partial_\eta)W = 0. \tag{5.11}$$

THEOREM 5.1. *There exists a positive $\varepsilon_0$ such that if* (2.22) *and* (2.23) *hold, then problem* (5.2) *has a unique solution $(w, p)$ satisfies*

$$\|w\|_{C^{3,\alpha}(\tilde{\mathbb{P}})} + \left\|p - p_b^-\right\|_{C^{3,\alpha}(\tilde{\mathbb{P}})} \leq C_0\varepsilon. \tag{5.12}$$

*The constants $\varepsilon_0, C_0$ depend solely on $U_b^-$, and $\alpha \in (0,1)$ may be arbitrary.*

COROLLARY 5.2. *Under the same assumptions of Theorem* 5.1*, by* (3.1)*,* (5.3)*,* (5.4)*, and* (5.12) *we also have*

$$\left\|u - u_b^-\right\|_{C^{3,\alpha}(\mathbb{P})} + \|v\|_{C^{3,\alpha}(\mathbb{P})} + \left\|\rho - \rho_b^-\right\|_{C^{3,\alpha}(\mathbb{P})} + \left\|p - p_b^-\right\|_{C^{3,\alpha}(\mathbb{P})} \leq C_0\varepsilon. \tag{5.13}$$

*Remark* 5.2. Hereafter we denote the $u, v, p, \rho, w, S$ obtained in Theorem 5.1 and Corollary 5.2 as $u_-, v_-, p_-, \rho_-, w_-, S_-$ in accordance with the notation in Theorem 2.6.

The proof of Theorem 5.1 is standard; it just needs a little modification of the proof of Theorem 3.3 in Chapter 4 of [20] (p. 180). (We may get local existence directly by this theorem.) In fact, if the boundary condition is a small perturbation of zero, then the existence can be semiglobal. It means that the life span of the smooth solution depends on the smallness of the perturbation of boundary data. In other words, the life span can be larger than any given number, provided the perturbation is small enough.

*Remark* 5.3. For $\Gamma'(\xi) < 0$ and the case when perturbation is not small, Chen has proved in [10] that the solution may blow up in finite distance from the entrance and shocks will appear.

## 6. Free boundary problem (B) and fixed boundary problem (C).

**6.1. Problem (B).** Knowing the supersonic flow, now we are in the position to determine the shock front and the subsonic state behind it simultaneously, satisfying the restrictions of pressure at the exit. We formulate it as the free boundary problem (**B**).

Let

$$S : \xi = \psi(\eta), \qquad \eta \in [0, 1], \tag{6.1}$$

be the shock front. By (3.14) the following Rankine–Hugoniot jump conditions [12] should hold across $S$:

$$-\left[\frac{1}{\rho u}\right] = [w]\ \psi'(\eta), \tag{6.2}$$

$$-\left[u + \frac{p}{\rho u}\right] = [pw]\ \psi'(\eta), \tag{6.3}$$

$$[v] = [p]\ \psi'(\eta). \tag{6.4}$$

By (6.4) we have

$$\psi'(\eta) = \frac{[v]}{[p]}. \tag{6.5}$$

Due to Remark 2.7 concerning (2.26), we set

$$\psi(0) = 0. \tag{6.6}$$

Substituting (6.5) in (6.2) and (6.3), we have

$$G_1(U, U_-) := [w][uw] + \left[\frac{1}{\rho u}\right] [p] = 0, \tag{6.7}$$

$$G_2(U, U_-) := [pw][uw] + \left[u + \frac{p}{\rho u}\right] [p] = 0. \tag{6.8}$$

Here $U := \begin{pmatrix} u & w & p \end{pmatrix}^t$, and $U_- := \begin{pmatrix} u_- & w_- & p_- \end{pmatrix}^t$ with $U_- = U_-(\psi(\eta), \eta)$. Note that (6.5), (6.7), and (6.8) are equivalent to (6.3)–(6.4) provided $[p] \neq 0$, which is guaranteed by (2.11) if the perturbations are small.

Problem (**B**) can be stated now as the following:

Find $U, \psi(\eta)$ and a real number $e$ such that
(i) $\psi(\eta)$ satisfies (6.1), (6.5), (6.6);
(ii) (6.7), (6.8) hold on $S$;
(iii) $w = 0$ on $\tilde{\Gamma}_-$;
(iv) $w = \Gamma'(\xi)$ on $\tilde{\Gamma}_+$;
(v) $p = p_b^+ + e$ on $\tilde{\Gamma}_1$.
(vi) Set $\Omega_\psi := \{(\xi, \eta) : \eta \in [0, 1], \psi(\eta) \leq \xi \leq 1\}$, then (4.17), (4.18) should hold in $\Omega_\psi$ as well as Bernoulli's law (2.3).

**6.2. Problem (C).** The idea of dealing with problem (**B**) is, roughly speaking, by iteration: first we fix the boundary and solve a fixed boundary problem, use (6.5), (6.6) to update the boundary, and then solve another fixed boundary problem, etc.

Set

$$\mathcal{S}_\sigma = \left\{\psi(\eta) \in H'^{(-1-\alpha)}_{3+\alpha}[0,1] : \|\psi\|'^{(-1-\alpha)}_{3+\alpha;[0,1]} \le \sigma, \psi(0) = 0\right\} \tag{6.9}$$

with

$$\sigma \le \sigma_0 < \frac{1}{2}. \tag{6.10}$$

For any $\psi \in \mathcal{S}_\sigma$, we may use transformation (3.23), which is of class $H^{(-1-\alpha)}_{3+\alpha;\Omega_\psi}$, to state the fixed boundary problem $(\mathbf{C}_\psi)$ as follows:

Find $U$ and $e \in \mathbb{R}$ such that
(i) (4.17), (4.18), and (2.3) hold in $\Omega := [0,1] \times [0,1]$;
(ii) (3.27) holds on $\bar{\Gamma}_\pm$;
(iii) (6.7), (6.8) hold on $\bar{\xi} = 0$;
(iv) $p = p_b^+ + e$ holds on $\bar{\xi} = 1$.

Now if problem $(\mathbf{C}_\psi)$ is uniquely solvable, with the solution $U_\psi$, then by the Cauchy problem of the ODE (note that $\eta = \bar{\eta}$)

$$\begin{cases} \tilde{\psi}'(\eta) = \frac{[v_\psi]}{[p_\psi]}, \\ \tilde{\psi}(0) = 0, \end{cases} \tag{6.11}$$

later (section 10) we will construct a mapping $\Psi : \mathcal{S}_\sigma \to \mathcal{S}_\sigma$ given by $\Psi(\psi) = \tilde{\psi}$ if $\varepsilon_0$ in Theorem 2.6 is small. Clearly the fixed point $\bar{\psi}$ of $\Psi$ corresponds to the desired shock front in problem $(\mathbf{B})$, and the solution $U_{\bar{\psi}}$ obtained by problem $(\mathbf{C}_{\bar{\psi}})$ is the subsonic state we are looking for. We call $\Psi$ the *boundary modifying mapping*.

**7. Problem (D): An equivalent form of problem (C).** This section is devoted to writing problem $(\mathbf{C}_\psi)$ in an equivalent, but more transparent and tractable, form called problem $(\mathbf{D}_\psi)$. This is a nonlinear boundary problem for nonlinear systems.

We first deal with the boundary conditions. Since $G_i(U_b^+, U_b^-) = 0$ for $i = 1, 2$ holds, we may write (6.7), (6.8) as

$$\begin{aligned} &\nabla_+ G_i(U_b^+, U_b^-) \cdot (U - U_b^+) \\ &\quad = \nabla_+ G_i(U_b^+, U_b^-) \cdot (U - U_b^+) - (G_i(U, U_b^-) - G_i(U_b^+, U_b^-)) \\ &+ (G_i(U, U_b^-) - G_i(U, U_-)) \\ &\quad := g_i(U, U_-), \end{aligned} \tag{7.1}$$

where $\nabla_+ G_i(U, U_-)$ is the gradient of $G_i(U, U_-)$ with respect to the variables $U$. By direct calculations (note that here by Bernoulli's law we consider $\rho$ as a function of $p, w, u$),

$$\begin{aligned} \nabla_+ G_i(U_b^+, U_b^-) &= \left.\frac{\partial(G_1, G_2)(U, U_-)}{\partial(u, w, p)}\right|_{(U,U_-)=(U_b^+,U_b^-)} \\ &= \begin{pmatrix} a_1 & 0 & b_1 \\ a_2 & 0 & b_2 \end{pmatrix} \end{aligned} \tag{7.2}$$

with

$$\begin{aligned} a_1 &= -\frac{2c_0+u^2}{2c_0-u^2}\cdot\frac{[p]}{\rho u^2}\Big|_{(U_b^+,U_b^-)}\\ &= -\frac{2c_0+{u_b^+}^2}{2c_0-{u_b^+}^2}\cdot\frac{p_b^+-p_b^-}{\rho_b^+{u_b^+}^2},\\ b_1 &= -\frac{[p]}{\rho up}\Big|_{(U_b^+,U_b^-)},\\ a_2 &= [p]\left(\frac{1}{\gamma}-\frac{p}{\rho u^2}\right)\Big|_{(U_b^+,U_b^-)},\\ b_2 &= \left[u+\frac{p}{\rho u}\right]\Big|_{(U_b^+,U_b^-)}\\ &= 0. \end{aligned}$$

Thus

$$d_1 := \det\begin{pmatrix} a_1 & b_1\\ a_2 & b_2\end{pmatrix} = \frac{[p]^2}{\rho up}\left(\frac{1}{\gamma}-\frac{p}{\rho u^2}\right)\Big|_{(U_b^+,U_b^-)} = -\frac{[p]^2}{\gamma\rho up}\left(\frac{a^2}{u^2}-1\right)\Big|_{(U_b^+,U_b^-)} \neq 0,$$

and we may rewrite (7.1) as

$$p-p_b^+ = \frac{1}{d_1}(a_1g_2-a_2g_1) := h_1(U,U_-), \tag{7.3}$$

$$u-u_b^+ = \frac{1}{d_1}(b_2g_1-b_1g_2) := h_2(U,U_-). \tag{7.4}$$

They should hold on $\bar{\xi}=0$.

Next we manipulate the equations. In the following we denote the value of $U$ on $\bar{\xi}=0$ as $U_0$. For example, by (7.3), (7.4), $p_0=h_1+p_b^+, u_0=h_2+u_b^+$.

Now by (4.17) and Bernoulli's law (2.3) we get

$$\rho_0 = \frac{\gamma}{\gamma-1}\cdot\frac{p_0}{c_0-\frac{1}{2}u_0^2(1+w_0^2)}, \tag{7.5}$$

$$\rho = \rho_0\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}, \tag{7.6}$$

$$u = \left\{\frac{2}{1+w^2}\cdot\left(c_0-\frac{\gamma}{\gamma-1}\cdot\frac{p}{\rho}\right)\right\}^{\frac{1}{2}}, \tag{7.7}$$

while $w,p$ may be solved from (4.18).

Let

$$\lambda_i := \beta_i|_{U=U_b^+,\psi=0} \qquad \text{for} \quad i=1,2, \tag{7.8}$$

with $\beta_i$ defined as in (4.19), and let

$$f_1(U,\psi) = -\lambda_R\partial_{\bar{\eta}}p+(\beta_1-\lambda_1)\partial_{\bar{\eta}}w, \tag{7.9}$$

$$f_2(U,\psi) = -\lambda_R\partial_{\bar{\eta}}w+(\lambda_2-\beta_2)\partial_{\bar{\eta}}p; \tag{7.10}$$

then $\lambda_i$ is positive and we may write (4.18) as

$$\begin{cases} \partial_{\bar{\xi}} p - \lambda_1 \partial_{\bar{\eta}} w = f_1(U, \psi), \\ \partial_{\bar{\xi}} w + \lambda_2 \partial_{\bar{\eta}} p = f_2(U, \psi). \end{cases} \tag{7.11}$$

For subsonic flow this is a first order nonlinear elliptic system.

So far problem (**C**) may be expressed in the following equivalent way if (4.21) holds.

Problem (**D1**)—boundary value problem for a first order elliptic system:

$$\begin{cases} \partial_{\bar{\xi}} p - \lambda_1 \partial_{\bar{\eta}} w = f_1(U, \psi) & \text{in} \quad \Omega, \\ \partial_{\bar{\xi}} w + \lambda_2 \partial_{\bar{\eta}} p = f_2(U, \psi) & \text{in} \quad \Omega, \\ p = p_b^+ + h_1(U, U_-) & \text{on} \quad \bar{\xi} = 0, \\ p = p_b^+ + e & \text{on} \quad \bar{\xi} = 1, \\ w = 0 & \text{on} \quad \bar{\eta} = 0, \\ w = \Gamma'((1 - \psi(1))\bar{\xi} + \psi(1)) & \text{on} \quad \bar{\eta} = 1. \end{cases} \tag{7.12}$$

Problem (**D2**)—algebraic equations (recall that $U_0 = U|_{\bar{\xi}=0}$):

$$\begin{cases} u_0 = u_b^+ + h_2(U, U_-), \\ p_0 = p_b^+ + h_1(U, U_-), \\ \rho_0 = \dfrac{\gamma}{\gamma - 1} \cdot \dfrac{p_0}{c_0 - \frac{1}{2} u_0^2 (1 + w_0^2)}, \\ \rho = \rho_0 \left( \frac{p}{p_0} \right)^{\frac{1}{\gamma}}, \\ u = \left\{ \frac{2}{1 + w^2} \cdot \left( c_0 - \frac{\gamma}{\gamma - 1} \cdot \frac{p}{\rho} \right) \right\}^{\frac{1}{2}}. \end{cases} \tag{7.13}$$

We call the above two coupled problems problem ($\mathbf{D}_\psi$) (or problem (**D**) for simplicity). The equivalence for smooth solutions is obvious from the deductions in the above sections.

**8. Solving linearized problem (D).** It is nature and standard to use iteration methods, such as the Banach contraction mapping principle, to solve problem (**D**). Thus in this section we concentrate on the related "linearized" problems.

**8.1. Linearized problem (D1).** This is to solve $\bar{p}, \bar{w}$, and $e \in \mathbb{R}$ satisfy

$$\begin{cases} \partial_{\bar{\xi}} \bar{p} - \lambda_1 \partial_{\bar{\eta}} \bar{w} = f_1 & \text{in} \quad \Omega, \\ \partial_{\bar{\xi}} \bar{w} + \lambda_2 \partial_{\bar{\eta}} \bar{p} = f_2 & \text{in} \quad \Omega, \\ \bar{p} = p_b^+ + h_1 & \text{on} \quad \bar{\xi} = 0, \\ \bar{p} = p_b^+ + e & \text{on} \quad \bar{\xi} = 1, \\ \bar{w} = 0 & \text{on} \quad \bar{\eta} = 0, \\ \bar{w} = g(\bar{\xi}) & \text{on} \quad \bar{\eta} = 1, \end{cases} \tag{8.1}$$

where $f_1, f_2, h_1, g$ are suitable nonhomogeneous terms. This is a boundary value problem on a domain with a piecewise smooth boundary, so we need a generalized version of the usual Schauder theory for elliptic equations.

We may separate problem (8.1) into the following two problems:

$$\begin{cases} \partial_{\bar\xi}\bar p_1 - \lambda_1\partial_{\bar\eta}\bar w_1 = f_1 & \text{in} \quad \Omega, \\ \partial_{\bar\xi}\bar w_1 + \lambda_2\partial_{\bar\eta}\bar p_1 = 0 & \text{in} \quad \Omega, \\ \bar p_1 = p_b^+ + h_1 & \text{on} \quad \bar\xi = 0, \\ \bar p_1 = p_b^+ + e & \text{on} \quad \bar\xi = 1, \\ \bar w_1 = 0 & \text{on} \quad \bar\eta = 0, \\ \bar w_1 = g(\bar\xi) & \text{on} \quad \bar\eta = 1; \end{cases} \tag{8.2}$$

$$\begin{cases} \partial_{\bar\xi}\bar p_2 - \lambda_1\partial_{\bar\eta}\bar w_2 = 0 & \text{in} \quad \Omega, \\ \partial_{\bar\xi}\bar w_2 + \lambda_2\partial_{\bar\eta}\bar p_2 = f_2 & \text{in} \quad \Omega, \\ \bar p_2 = 0 & \text{on} \quad \bar\xi = 0, \\ \bar p_2 = 0 & \text{on} \quad \bar\xi = 1, \\ \bar w_2 = 0 & \text{on} \quad \bar\eta = 0, \\ \bar w_2 = 0 & \text{on} \quad \bar\eta = 1. \end{cases} \tag{8.3}$$

Then

$$\bar p = \bar p_1 + \bar p_2, \qquad \bar w = \bar w_1 + \bar w_2 \tag{8.4}$$

is the solution of problem (8.1). Since $\Omega$ is simply connected, we may introduce potentials $\phi_1(\bar\xi,\bar\eta), \phi_2(\bar\xi,\bar\eta)$ such that

$$\partial_{\bar\xi}\phi_1 = -\lambda_2(\bar p_1 - p_b^+), \qquad \partial_{\bar\eta}\phi_1 = \bar w_1, \tag{8.5}$$

$$\partial_{\bar\xi}\phi_2 = \lambda_1\bar w_2, \qquad \partial_{\bar\eta}\phi_2 = \bar p_2, \tag{8.6}$$

and write (8.2), (8.3) as

$$\begin{cases} \frac{1}{\lambda_2}\partial^2_{\bar\xi}\phi_1 + \lambda_1\partial^2_{\bar\eta}\phi_1 = -f_1 & \text{in} \quad \Omega, \\ \partial_{\bar\xi}\phi_1 = -\lambda_2 h_1 & \text{on} \quad \bar\xi = 0, \\ \partial_{\bar\xi}\phi_1 = -\lambda_2 e & \text{on} \quad \bar\xi = 1, \\ \partial_{\bar\eta}\phi_1 = 0 & \text{on} \quad \bar\eta = 0, \\ \partial_{\bar\eta}\phi_1 = g(\bar\xi) & \text{on} \quad \bar\eta = 1; \end{cases} \tag{8.7}$$

$$\begin{cases} \frac{1}{\lambda_1}\partial^2_{\bar\xi}\phi_2 + \lambda_2\partial^2_{\bar\eta}\phi_2 = f_2 & \text{in} \quad \Omega, \\ \phi_2 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{8.8}$$

Equation (8.8) is the Dirichlet problem for Poisson equations. By Theorem 7.2 and Remark (2) in [16], we know there is a unique solution $\phi_2$ and

$$\|\phi_2\|^{(-1-\alpha)}_{3+\alpha} \le C\,\|f_2\|^{(1-\alpha)}_{1+\alpha}, \tag{8.9}$$

where $C$ depends only on $U_b$ and $\sigma_0$, and $\alpha \in (0,1)$ may be arbitrary.

Next we consider (8.7), which is actually the Neumann problem for Poisson equations:

$$\begin{cases} \triangle\phi = f & \text{in} \quad \Omega, \\ \frac{\partial\phi}{\partial\nu} = g & \text{on} \quad \partial\Omega, \end{cases}$$

where $\nu$ is the unit outward normal of $\partial\Omega$. It is well known that such a problem is solvable if and only if

$$\int_{\Omega} f \ \mathrm{d}\bar{\xi}\mathrm{d}\bar{\eta} = \int_{\partial\Omega} g \ \mathrm{d}s. \tag{8.10}$$

This is in essence the reason why we have to introduce the number $e$ in the pressure giving at the exit of the duct.

Applying (8.10) to (8.7), we get

$$\int_{\Omega} f_1 \ \mathrm{d}\bar{\xi}\mathrm{d}\bar{\eta} = \int_0^1 (e - h_1) \ \mathrm{d}\bar{\eta} - \lambda_1 \int_0^1 g(\bar{\xi}) \ \mathrm{d}\bar{\xi};$$

thus if we take

$$e = \int_{\Omega} f_1 \ \mathrm{d}\bar{\xi}\mathrm{d}\bar{\eta} + \lambda_1 \int_0^1 g(\bar{\xi}) \ \mathrm{d}\bar{\xi} + \int_0^1 h_1 \ \mathrm{d}\bar{\eta}, \tag{8.11}$$

(8.7) is solvable, and any two solutions differ only from a constant.

Now if (8.11) holds, then by Theorem 1.4 in [21] there exists a unique solution $\phi_1$ to (8.7) with $\phi_1(0,0) = 0$, and furthermore the following estimate holds for some $\alpha \in (0,1)$:

$$\|\phi_1\|_{3+\alpha}^{(-1-\alpha)} \leq C\Big( \|f_1\|_{1+\alpha}^{(1-\alpha)} + \|g\|_{C^{2+\alpha}[0,1]} + \|h_1\|_{2+\alpha}^{'(-\alpha)} \Big). \tag{8.12}$$

So finally by (8.4)–(8.6), (8.9), and (8.12) we get

$$\begin{aligned} &\|\bar{p} - p_b^+\|_{2+\alpha}^{(-\alpha)} + \|\bar{w}\|_{2+\alpha}^{(-\alpha)} + |e| \\ &\leq C\Big( \|f_1\|_{1+\alpha}^{(1-\alpha)} + \|f_2\|_{1+\alpha}^{(1-\alpha)} + \|g\|_{C^{2+\alpha}[0,1]} + \|h_1\|_{2+\alpha}^{'(-\alpha)} \Big). \end{aligned} \tag{8.13}$$

*Remark* 8.1. An important observation is that the first equation in (8.1), which is responsible for the well-posedness or ill-posedness of the transonic shock problem under pressure giving on the exit, is in essence the equation of conservation of mass, as can be checked by tracing its origin. This fact can be seen more clearly by using another completely different method which was developed in [7].

**8.2. "Linearized" problem (D2).** This problem solves (recall that $U_0 = U|_{\bar{\xi}=0}$):

$$\begin{cases} \bar{u}_0 = u_b^+ + h_2(U, U_-), \\ \bar{p}_0 = p_b^+ + h_1(U, U_-), \\ \bar{\rho}_0 = \dfrac{\gamma}{\gamma - 1} \cdot \dfrac{\bar{p}_0}{c_0 - \frac{1}{2}\bar{u}_0^2(1 + \bar{w}_0^2)}, \\ \bar{\rho} = \bar{\rho}_0 \left(\frac{\bar{p}}{\bar{p}_0}\right)^{\frac{1}{\gamma}}, \\ \bar{u} = \left\{ \frac{2}{1+\bar{w}^2} \cdot \left(c_0 - \frac{\gamma}{\gamma-1} \cdot \frac{\bar{p}}{\bar{\rho}}\right)\right\}^{\frac{1}{2}}. \end{cases} \tag{8.14}$$

**9. Solution of problem (C).** With the above preparations, we solve in this section problem (**D**) (i.e., problem (**C**)) by the Banach contraction mapping principle.

Set

$$\mathcal{O}_\delta := \Big\{ U = (u, w, p, \rho)^t : \big\|u - u_b^+\big\|_{2+\alpha}^{(-\alpha)} + \|w\|_{2+\alpha}^{(-\alpha)} + \big\|p - p_b^+\big\|_{2+\alpha}^{(-\alpha)} + \big\|\rho - \rho_b^+\big\|_{2+\alpha}^{(-\alpha)} \le \delta \Big\}, \tag{9.1}$$

with

$$\delta < \delta_0 \tag{9.2}$$

and $\delta_0$ a constant depending only on $U_b$ such that our preparations in the preceding sections are valid. $\mathcal{O}_\delta$ is a closed subset of Banach space $(H_{2+\alpha}^{(-\alpha)})^4$. By Proposition 2.4 the norm here is in fact equivalent to that used in Theorem 2.6. We will construct a mapping $\mathcal{T}$ from $\mathcal{O}_\delta$ to $\mathcal{O}_\delta$ by problem (**D**) and show this mapping contracts when $\varepsilon_0$ (the perturbation of the wall of the duct) is small.

For any $U = (u, w, p, \rho)^t \in \mathcal{O}_\delta$, substitute

$$f_1 = f_1(U, \psi), \tag{9.3}$$

$$f_2 = f_2(U, \psi), \tag{9.4}$$

$$h_1 = h_1(U, U_-), \tag{9.5}$$

$$g(\bar\xi) = \Gamma'((1 - \psi(1))\bar\xi + \psi(1)) \tag{9.6}$$

in problem (**D1**), with $f_i(U, \psi)(i = 1, 2)$ as in (7.9), (7.10) and $h_1(U, U_-)$ defined by (7.3). By (2.18)–(2.21) and (7.1), (7.9), (7.10) we have the following estimates for $i = 1, 2$:

$$\|f_i(U, \psi)\|_{1+\alpha}^{(1-\alpha)} \le C(\delta^2 + \delta\sigma), \tag{9.7}$$

$$\|h_i(U, U_-)\|_{2+\alpha}^{\prime(-\alpha)} \le C(\delta^2 + \varepsilon). \tag{9.8}$$

$$\|g\|_{C^{2,\alpha}[0,1]} \le C\varepsilon. \tag{9.9}$$

We may get unique $\bar p, \bar w, e$ from problem (**D1**), and by (8.13) we have

$$\big\|\bar p - p_b^+\big\|_{2+\alpha}^{(-\alpha)} + \|\bar w\|_{2+\alpha}^{(-\alpha)} + |e| \le C(\delta^2 + \delta\sigma + \varepsilon). \tag{9.10}$$

Now consider problem (**D2**). By (8.14) and analyticity of each expression, we easily obtain that

$$\big\|\bar u - u_b^+\big\|_{2+\alpha}^{(-\alpha)} \le C(\delta^2 + \delta\sigma + \varepsilon), \tag{9.11}$$

$$\big\|\bar \rho - \rho_b^+\big\|_{2+\alpha}^{(-\alpha)} \le C(\delta^2 + \delta\sigma + \varepsilon). \tag{9.12}$$

So far we obtained the unique $\bar U := (\bar p, \bar u, \bar w, \bar \rho)^t$ from $U = (p, u, w, \rho)^t \in \mathcal{O}_\delta$ and have the estimate

$$\big\|\bar p - p_b^+\big\|_{2+\alpha}^{(-\alpha)} + \|\bar w\|_{2+\alpha}^{(-\alpha)} + \big\|\bar u - u_b^+\big\|_{2+\alpha}^{(-\alpha)} + \big\|\bar \rho - \rho_b^+\big\|_{2+\alpha}^{(-\alpha)} + |e| \le C(\delta^2 + \delta\sigma + \varepsilon). \tag{9.13}$$

Now choosing $\varepsilon_0, \sigma_0$ such that

$$C\delta_0 \le \frac{1}{4}, \tag{9.14}$$

$$C\sigma_0 \le \frac{1}{4}, \tag{9.15}$$

and

$$\delta = 2C\varepsilon, \tag{9.16}$$

we get

$$\left\|\bar{p} - p_b^+\right\|_{2+\alpha}^{(-\alpha)} + \|\bar{w}\|_{2+\alpha}^{(-\alpha)} + \left\|\bar{u} - u_b^+\right\|_{2+\alpha}^{(-\alpha)} + \left\|\bar{\rho} - \rho_b^+\right\|_{2+\alpha}^{(-\alpha)} + |e| \leq \delta. \tag{9.17}$$

Thus the mapping $\mathcal{T} : U = (u, w, p, \rho)^t \mapsto \bar{U} = (\bar{u}, \bar{w}, \bar{p}, \bar{\rho})^t$ is into $\mathcal{O}_\delta$.

What is left is to show the contraction of $\mathcal{T}$. For $i = 1, 2$, suppose

$$\psi^{(i)} \in \mathcal{S}_\sigma,$$

$$U^{(i)} = (u^{(i)}, w^{(i)}, p^{(i)}, \rho^{(i)})^t \in \mathcal{O}_\delta,$$

and denote

$$\bar{U}^{(i)} = (\bar{u}^{(i)}, \bar{w}^{(i)}, \bar{p}^{(i)}, \bar{\rho}^{(i)})^t = \mathcal{T}((u^{(i)}, w^{(i)}, p^{(i)}, \rho^{(i)})^t),$$

$$U_-^{(i)} = U_-(\psi^{(i)}(\bar{\eta}), \bar{\eta}), \tag{9.18}$$

$$g^{(i)} = \Gamma'((1 - \psi^{(i)}(1))\bar{\xi} + \psi^{(i)}(1)). \tag{9.19}$$

Recall that $\left\|U_- - U_b^-\right\|_{C^{3,\alpha}(\mathbb{P})} \leq C_0\varepsilon$; we get

$$\left\|U_-^{(1)} - U_-^{(2)}\right\|_{2+\alpha}^{'(-\alpha)} \leq C\varepsilon \left\|\psi^{(1)} - \psi^{(2)}\right\|_{2+\alpha}^{'(-\alpha)} \leq C\varepsilon \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{'(-1-\alpha)}.$$

Then direct calculation shows that for $j = 1, 2$,

$$\begin{aligned} &\left\|f_j(U^{(1)}, \psi^{(1)}) - f_j(U^{(2)}, \psi^{(2)})\right\|_{1+\alpha}^{(1-\alpha)} \\ &\leq C(\delta + \sigma)\Big( \left\|U^{(1)} - U^{(2)}\right\|_{2+\alpha}^{(-\alpha)} + \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{'(-1-\alpha)} \Big), \end{aligned} \tag{9.20}$$

$$\begin{aligned} &\left\|h_j(U^{(1)}, U_-^{(1)}) - h_j(U^{(2)}, U_-^{(2)})\right\|_{2+\alpha}^{'(-\alpha)} \\ &\leq C(\delta + \varepsilon)\Big( \left\|U^{(1)} - U^{(2)}\right\|_{2+\alpha}^{(-\alpha)} + \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{'(-1-\alpha)} \Big). \end{aligned} \tag{9.21}$$

$$\left\|g^{(1)} - g^{(2)}\right\|_{C^{2,\alpha}[0,1]} \leq C\varepsilon \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{'(-1-\alpha)}. \tag{9.22}$$

Thus by solving corresponding problem (**D**) we get

$$\begin{aligned} &\left\|\bar{U}^{(1)} - \bar{U}^{(2)}\right\|_{2+\alpha}^{(-\alpha)} \\ &\leq C(\delta + \sigma + \varepsilon)\Big( \left\|U^{(1)} - U^{(2)}\right\|_{2+\alpha}^{(-\alpha)} + \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{'(-1-\alpha)} \Big). \end{aligned} \tag{9.23}$$

For $\psi^{(1)} = \psi^{(2)} = \psi$, we obtain contraction of $\mathcal{T}$ by choosing $\delta_0.\varepsilon_0, \sigma_0$ such that

$$C(\varepsilon_0 + \delta_0 + \sigma_0) < \frac{1}{2}. \tag{9.24}$$

This solves problem ($\mathbf{C}_\psi$) for any $\psi \in \mathcal{S}_\sigma$ with $\sigma_0$ small.

**10. Solution of problem (B): Determination of shock front.** For any $\psi \in \mathcal{S}_\sigma$, we have solved problem $(\mathbf{C}_\psi)$ to obtain the unique solution $U_\psi$. Now by solving the Cauchy problem

$$\begin{cases} \tilde{\psi}'(\eta) = \frac{[v_\psi]}{[p_\psi]}, \\ \tilde{\psi}(0) = 0, \end{cases} \tag{10.1}$$

we constructed a mapping by $\Psi(\psi) = \tilde{\psi}$.

Obviously we have

$$\left\|\tilde{\psi}\right\|_{3+\alpha}^{\prime(-1-\alpha)} \leq C(\delta + \varepsilon) \leq C\varepsilon \tag{10.2}$$

by using (9.16). Taking

$$\sigma = C\varepsilon, \tag{10.3}$$

we then have $\Psi : \mathcal{S}_\sigma \mapsto \mathcal{S}_\sigma$ provided $\varepsilon_0$ is sufficiently small.

For $\psi^{(i)} \in \mathcal{S}_\sigma$, $i = 1, 2$, denote $U^{(i)}$ as $U_{\psi^{(i)}}$; then by (9.23) (recall now that $\bar{U}^{(i)} = U^{(i)}$) one gets

$$\left\|U^{(1)} - U^{(2)}\right\|_{2+\alpha}^{(-\alpha)} \leq C(\delta + \sigma + \varepsilon) \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{\prime(-1-\alpha)}; \tag{10.4}$$

thus

$$\begin{aligned} &\left\|\tilde{\psi}^{(1)} - \tilde{\psi}^{(2)}\right\|_{3+\alpha}^{\prime(-1-\alpha)} \\ &\leq C \left\|U^{(1)} - U^{(2)}\right\|_{2+\alpha}^{(-\alpha)} + C\varepsilon \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{\prime(-1-\alpha)} \\ &\leq C(\delta + \sigma + \varepsilon) \left\|\psi^{(1)} - \psi^{(2)}\right\|_{3+\alpha}^{\prime(-1-\alpha)}. \end{aligned} \tag{10.5}$$

If (9.24) holds, $\Psi$ contracts on $\mathcal{S}_\sigma$. By the Banach contraction mapping principle we know the free boundary problem $(\mathbf{B})$ is uniquely solvable. Combining this result and Theorem 5.1 we proved Theorem 2.6.

*Remark* 10.1. We explain here further why the uniqueness claimed in Theorem 2.6 holds. By (10.5) we see the fixed point, i.e., the transonic shock front $\bar{\psi}$ is unique. Uniqueness for nonlinear problem $(\mathbf{D}_{\bar{\psi}})$ in section 9 (thus problem $(\mathbf{C}_{\bar{\psi}})$) follows from the contraction argument of the mapping $\mathcal{T}$ defined on the line after estimate (9.17) (consult (9.23) with the case $\psi^{(1)} = \psi^{(2)} = \bar{\psi}$). Moreover, uniqueness of constant $e$ for nonlinear problem $(\mathbf{D}_{\bar{\psi}})$ follows by writing the nonlinear problem as a linear problem (8.1) with right-hand sides defined by (9.3)–(9.6), and then using uniqueness of $e$ for linear problem (8.1). Thus the subsonic flow is also unique.

**11. Discussion on well-posedness or ill-posedness for other conditions given at the exit of duct.** In this section we discuss the well-posedness or ill-posedness of transonic shock problem $(\mathbf{A})$ if other conditions are given at the exit of the duct. Our tool is the well-posedness or ill-posedness of the corresponding boundary value problem for elliptic systems (8.1). We remark that all the results listed below can be proved rigorously in the same fashion as we have done for giving pressure at the exit before. We just sketch out the key points that are needed for such proofs.

**1.** As have been shown above (in section 8), given $p$ at $\Gamma_1$ corresponds to the Neumann problem for Poisson equations. So in general for given pressure at the exit, problem (**A**) is ill-posed.

**2.** However, if we give $w$ at the exit, then we need to solve a mixed boundary value problem for Poisson equations with Dirichlet data nonempty. It is well known that such a problem is well-posed; i.e., we can give $w$ at the exit in an arbitrary way and also obtain a unique solution (although we require that $w|_{\Gamma_1}$ should be small since we are dealing with a small perturbation problem). So for $w$ given at the exit, problem (**A**) is well-posed.

**3.** For arbitrarily given $v = g$ at the exit, Problem (**A**) is still well-posed. In fact, this corresponds to given

$$w = \frac{g}{u}$$

for any $U = (u, v, p, \rho)^t \in \mathcal{O}_\delta$ in the linearized problem (**D1**). The linearized problem (**D1**) is then uniquely solvable, and the contraction by

$$w^{(1)} - w^{(2)} = g\frac{u^{(1)} - u^{(2)}}{u^{(1)}u^{(2)}}$$

is still available, since $g$ is small.

**4.** For given entropy $S = g$ at the exit, problem (**A**) is ill-posed. In fact, by constancy of entropy along streamlines behind the shock front, we get

$$\frac{p_0}{\rho_0^\gamma} = g.$$

Now by the third equation in (7.13), one has

$$w_0 = \left\{ 2\frac{c_0 - p_0\frac{\gamma}{\gamma-1}\left(\frac{g}{p_0}\right)^{\frac{1}{\gamma}}}{u_0^2} \right\}^{\frac{1}{2}}.$$

Since $p_0, u_0$ are known, we obtain both $p_0$ and $w_0$ on $\bar{\xi} = 0$. This means we encounter an initial-boundary value problem for the elliptic system (i.e., the first two equations in (8.1) or (7.12)). It is well known that such problems are ill-posed.

**5.** For giving Mach number $M = g$ at the exit, Problem (**A**) is ill-posed. Indeed, by Bernoulli's law (2.3) we have

$$\gamma\frac{p}{\rho} = a^2 = \frac{c_0}{\frac{1}{\gamma-1} + \frac{1}{2}g^2}.$$

From constancy of entropy we get

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma},$$

where $\rho_0$ can be expressed by the third equation in (7.13). Thus we can solve the above two equations to obtain

$$p = p_0 h,$$

where

$$h = \left\{ \frac{c_0}{(c_0 - \frac{1}{2}u_0^2(1+w_0^2))(1+\frac{1}{2}(\gamma-1)g^2)} \right\}^{\frac{\gamma}{\gamma-1}}.$$

Thus we get

$$p - p_b^+ = (p_0 - p_b^+)h + p_b^+(h-1).$$

Notice that

$$h(U_b^+) = 1$$

and

$$\|h-1\| \le C(\|u_0 - u_b^+\| + \|g - M_b^+\| + \|w_0\|^2).$$

(Note here that $w_0$ is itself small and that fortunately $w_0^2$ appears in the expression of $h$. This observation is of crucial importance!) With such an estimate in hand, the remaining work is the same as giving pressure at the exit. Thus problem (**A**) is not well-posed for the given arbitrary Mach number at the exit of the duct.

**6.** For given arbitrary density $\rho = g$ at the exit, problem (**A**) is not well-posed. From constancy of entropy we have

$$p = \frac{p_0 g^\gamma}{\rho_0^\gamma}.$$

Due to the third equation in (7.13), we get

$$p = \left(\frac{\gamma}{\gamma-1}\right)^\gamma p_0^{1-\gamma} g^\gamma \left(c_0 - \frac{1}{2}u_0^2(1+w_0^2)\right)^\gamma.$$

The left analysis is the same as in **5**.

**7.** For given $u = g$ at the exit, problem (**A**) is still ill-posed. By Bernoulli's law we get

$$\frac{p}{\rho} = \frac{\gamma-1}{\gamma}\Big(c_0 - \frac{1}{2}g^2(1+w_1^2)\Big);$$

here $w_1$ is the value of $w$ restricted on $\bar{\xi} = 1$. Using constancy of entropy we also have

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}.$$

By the expressions of $\rho_0$ (the third equation in (7.13)) we can solve from the above two equations that

$$p = p_0 \left( \frac{c_0 - \frac{1}{2}g^2(1+w_1^2)}{c_0 - \frac{1}{2}u_0^2(1+w_0^2)} \right)^{\frac{\gamma}{\gamma-1}}.$$

Note $w_0, w_1$ are small quantities and appeared as squares in the above expressions; $u_0, p_0$ are known and are second order terms as shown in section 9. Thus we can use similar methods as above to show the ill-posedness of problem (**A**).

*Remark* 11.1. All the above results may be surprising at first glance. However, a deep consideration of the background solution suggests these results are natural, since for the one-dimensional case, $S, p, \rho, M, u$ have already been completely determined.

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