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ON TRANSVERSALLY SIMPLE KNOTS

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Abstract

This paper studies knots that are transversal to the standard contact structure in $\mathbb{R}^3$, bringing techniques from topological knot theory to bear on their transversal classification. We say that a transversal knot type $\mathcal{TK}$ is *transversally simple* if it is determined by its topological knot type $\mathcal{K}$ and its Bennequin number. The main theorem asserts that any $\mathcal{TK}$ whose associated $\mathcal{K}$ satisfies a condition that we call *exchange reducibility* is transversally simple.

As a first application, we prove that the unlink is transversally simple, extending the main theorem in [10]. As a second application we use a new theorem of Menasco [17] to extend a result of Etnyre [11] to prove that all iterated torus knots are transversally simple. We also give a formula for their maximum Bennequin number. We show that the concept of exchange reducibility is the simplest of the constraints that one can place on $\mathcal{K}$ in order to prove that any associated $\mathcal{TK}$ is transversally simple. We also give examples of pairs of transversal knots that we conjecture are *not* transversally simple.

1. Introduction

Let ξ be the *standard* contact structure in oriented 3-space $\mathbb{R}^3 = (\rho, \theta, z)$, that is the kernel of $\alpha = \rho^2 d\theta + dz$. An oriented knot K in contact $\mathbb{R}^3$ is said to be a *transversal* knot if it is transversal to the planes of this contact structure. In this paper, the term 'transversal' refers to this contact structure only. If the knot K is parametrized by $(\rho(t), \theta(t), z(t))$, then K is transversal if and only if $\frac{z'(t)}{\theta'(t)} \neq -(\rho(t))^2$ for

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every t. We will assume throughout that $\alpha > 0$ for all t, pointing out later how our arguments adapt to the case $\alpha < 0$.

For the benefit of the reader who may be unfamiliar with the standard contact structure, Figure 1(a) illustrates typical 2-planes in this structure in $\mathbb{R}^3$, when z is fixed, and ρ and θ vary. The structure is radially symmetric. It is also invariant under translation of $\mathbb{R}^3$ parallel to the z axis. Typical 2-planes are horizontal at points on the z axis and twist clockwise (if the point of view is out towards increasing ρ from the z axis,) as $\rho \to \infty$.

There has been some discussion about whether the planes tend to vertical as $\rho \to \infty$ or to horizontal. If one looks at the limit of α, it appears that the limit is a rotation of $\pi/2$. However, if one derives the standard contact structure on S^3 from the Hopf fibration, as described below, and wants to have this structure be consistent with the one defined on $\mathbb{R}^3$, it is necessary to take the limit to be a rotation up to (but not through, as a rotation of more than π results in an overtwisted structure) π. The resulting contact structures on $\mathbb{R}^3$ are equivalent, through a contactomorphism that untwists the planes from π to $\pi/2$. Thus we can work in the standard contact structure on S^3, which has horizontal planes in the limit, while using the contact form α, which induces vertical planes in the limit. The details are below.

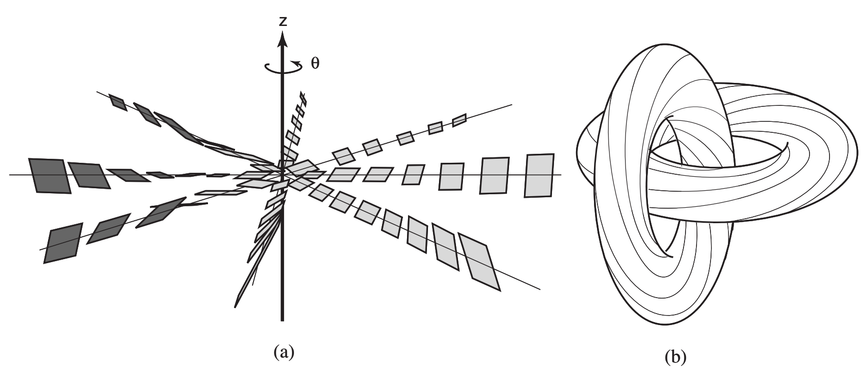

Figure 1: The standard contact structure on $\mathbb{R}^3$ and the Hopf fibration on S^3.

The standard contact structure extends to S^3 and has an interesting interpretation in terms of the geometry of S^3. Let

$$S^3 = \{(z_1, z_2) = (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) \in \mathbb{C}^2 \ / \ \rho_1^2 + \rho_2^2 = constant.\}$$

Then ξ is the kernel of $\rho_1^2 d\theta_1 + \rho_2^2 d\theta_2$. The field of 2-planes may be thought of as the field of hyperplanes which are orthogonal to the fibers of the Hopf fibration $\pi : S^3 \to S^2$. See Figure 1(b) for a picture of typical fibers. Identify the z axis in $\mathbb{R}^3$ with the core of one of the solid tori. There is a fiber through each point in S^3, and the 2-plane at a point is orthogonal to the fiber through the point.

The (topological) *type* $\mathcal{K}$ of a knot $K \subset \mathbb{R}^3$ is its equivalence class under isotopy of the pair $(K, \mathbb{R}^3)$. A sharper notion of equivalence is its *transversal knot type* $\mathcal{TK}$, which requires that $\frac{z'(t)}{\theta'(t)} + (\rho(t))^2$ be positive at every point of the deformed knot during every stage of the isotopy. The difference between these two concepts is the central problem studied in this paper.

A parametrized knot $K \subset \mathbb{R}^3$ is said to be represented as a *closed braid* if $\rho(t) > 0$ and $\theta'(t) > 0$ for all t. See Figure 2(a). It was proved by Bennequin in §23 of [1] that every transversal knot is transversally isotopic to a transversal closed braid. This result allows us to apply results obtained in the study of closed braid representatives of topological knots to the problem of understanding transversal isotopy. We carry Bennequin's approach one step further, initiating a comparative study of the two equivalence relations: topological equivalence of two closed braid representatives of the same transversal knot type, via closed braids, and transversal equivalence of the same two closed braids. Transversal equivalence is of course more restrictive than topological equivalence.

Topological equivalence of closed braid representatives of the same knot has been the subject of extensive investigations by the first author and W. Menasco, who wrote a series of six papers with the common title *Studying links via closed braids*. See, for example, [6] and [5]. See also the related papers [3] and [2]. In this paper we will begin to apply what was learned in the topological setting to the transversal problem. See also Vassiliev's paper [21], where we first learned that closed braid representations of knots were very natural in analysis; also our own contributions in [9], where we began to understand that there were deep connections between the analytic and the topological-algebraic approaches to knot theory.

A well-known invariant of a transversal knot type $\mathcal{TK}$ is its *Bennequin* number $\beta(\mathcal{TK})$. It is not an invariant of $\mathcal{K}$. We now define it in a way that will allow us to compute it from a closed braid representative K of $\mathcal{TK}$. The *braid index* $n = n(K)$ of a closed braid K is the linking number of K with the oriented z axis. A generic projection of K onto the (ρ, θ) plane will be referred to as a *closed braid projection*. An

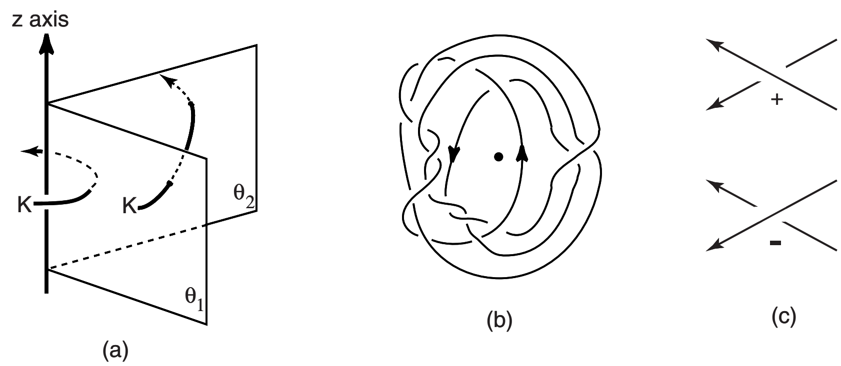

Figure 2: (a) Closed braid; (b) Example of a closed braid projection; (c) Positive and negative crossings

example is given in Figure 2(b). The origin in the (ρ, θ) plane is indicated as a black dot; our closed braid rotates about the z axis in the direction of increasing θ. The *algebraic crossing number* $e = e(K)$ of the closed braid is the sum of the signed crossings in a closed braid projection, using the sign conventions given in Figure 2(c). If the transversal knot type $\mathcal{TK}$ is represented by a closed braid K, then its Bennequin number $\beta(\mathcal{TK})$ is:

$$\beta(\mathcal{TK}) = e(K) - n(K).$$

Since $e(K) - n(K)$ can take on infinitely many different values as K ranges over the representatives of $\mathcal{K}$, it follows that there exist infinitely many transversal knot types for each topological knot type. It was proved by Bennequin in [1] that $e(K) - n(K)$ is bounded above by $-\chi(\mathcal{F})$, where $\mathcal{F}$ is a spanning surface of minimal genus for $\mathcal{K}$. Fuchs and Tabachnikov gave a different upper bound in [13]. However, sharp upper bounds are elusive and are only known in a few very special cases.

We now explain the geometric meaning of $\beta(\mathcal{TK})$. Choose a point $(z_1, z_2) = (x_1 + ix_2, x_3 + ix_4) \in TK \subset S^3$. Thinking of (z_1, z_2) as a point in $\mathbb{R}^4$, let $\vec{p} = (x_1, x_2, x_3, x_4)$. Let $\vec{q} = (-x_2, x_1, -x_4, x_3)$ and let $\vec{r} = (-x_3, x_4, x_1, -x_2)$. Then $\vec{r} \cdot \vec{p} = \vec{r} \cdot \vec{q} = \vec{p} \cdot \vec{q} = 0$. Then $\vec{q}$ may be interpreted as the outward-drawn normal to the contact plane at $\vec{p}$, so that $\vec{r}$ lies in the unique contact plane at the point $\vec{p} \in S^3$. Noting that a transversal knot is nowhere tangent to the contact plane, it follows that for each point $\vec{p}$ on a transversal knot $TK \subset S^3$ the vector $\vec{r}$ gives a well-defined direction for pushing TK off itself to a related simple closed curve TK'. The Bennequin number $\beta(\mathcal{TK})$ is the linking number $\mathcal{L}k(TK, TK')$. See §16 of [1] for a proof that $\beta(TK)$ is invariant under transverse isotopy and that $\beta(TK) = e(K) - n(K)$.

We say that a transversal knot is *transversally simple* if it is characterized up to transversal isotopy by its topological knot type and its Bennequin number. In [10] Eliashberg proved that a transversal unknot is transversally simple. More recently Etnyre [11] used Eliashberg's techniques to prove that transversal positive torus knots are transversally simple.

Our first main result, Theorem 1, asserts that if a knot type $\mathcal{K}$ is exchange reducible (a condition we define in Section 2), then its maximum Bennequin number is realized by any closed braid representative of minimum braid index. As an application, we are able to compute the maximum Bennequin number for all iterated torus knots. See Corollary 3. Our second main result, Theorem 2, asserts that if $\mathcal{TK}$ is a transversal knot type with associated topological knot type $\mathcal{K}$, and if $\mathcal{K}$ is exchange reducible, then $\mathcal{TK}$ is transversally simple. As an application, we prove in Corollary 2 that transversal iterated torus knots are transversally simple. The two corollaries use new results of Menasco [17], who proved (after an early version of this paper was circulated) that iterated torus knots are exchange reducible. In Theorem 3 we establish the existence of knot types that are *not* exchange reducible.

Here is an outline of the paper. Section 2 contains our main results. In it we will define the concept of an exchange reducible knot and prove Theorems 1 and 2. In Section 3 we discuss examples, applications and possible generalizations.

Acknowlegements. We thank Oliver Dasbach, William Menasco and John Etnyre for conversations and helpful suggestions relating to the work in this paper. We are especially grateful to Menasco for the manuscript [17]. In an early version of this paper, we conjectured that iterated torus knots might be exchange reducible. We explained our conjecture to him, and a few days later he had a proof! We also thank Wlodek Kuperberg for sharing his beautiful sketch of the Hopf fibration (Figure 1(b)) with us. Finally, we thank William Gibson, who noticed our formula for the Bennequin number of iterated torus knots in an earlier version of this paper and pointed out to us in a private conversation that it could be related to the upper bound which was given by Bennequin in [1], by the formula in Corollary 3, part (2).

1.1 Remarks on techniques

This subsection contains a discussion of the techniques used in the manuscripts [4], [5], [7] and [17], tools which form the foundation on which the results of this paper are based. We compare these techniques to those used in the manuscripts [1], [10] and [11], although Bennequin's paper rightfully belongs in both sets. This description and comparison is of great interest, but is not essential for the reading of this paper or the digestion of its arguments.

We concern ourselves with two known foliations of an orientable surface $\mathcal{F}$ associated to K: the *characteristic* foliation ξ_F from contact geometry and the *topological* foliation from braid theory. The characteristic foliation of $\mathcal{F}$ is the line field $\xi \cap T\mathcal{F}$, given by the intersection of the planes of the contact structure with the planes of the tangent space of the surface, which is then integrated to a singular foliation of $\mathcal{F}$. The *topological foliation* is the foliation of $\mathcal{F}$ which is induced by intersecting the foliation of $\mathbb{R}^3$ minus the z axis (see Figure 3), with the surface $\mathcal{F}$.

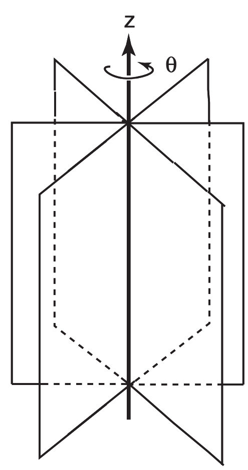

Figure 3: Half-planes in the braid structure on $\mathbb{R}^3$.

The foliation of $\mathbb{R}^3$ minus the z-axis by half-planes is called the *standard braid structure* on $\mathbb{R}^3$ − the z-axis. The surface used in [10] and [5] was a spanning surface for K; in [4] it was a 2-sphere which intersects K twice; in [7] it was a torus in the complement of K; in [11] and [17] it is a torus $T \subset S^3$ on which K is embedded. Menasco also considers the foliation of a meridian disc in the solid torus which T bounds. The characteristic foliation of a surface (associated to a transversal knot or to another transversal or Legendrian curve,) is a tool of study in contact geometry. It was the main tool in the manuscripts [1], [10] and [11].

In topological knot theory, one studies the topological foliation of $\mathcal{F}$ defined above. The review article [2] may be useful to the reader who is unfamiliar with this area. The study of the topological foliations has produced many results, for example the classification of knots that are closed 3-braids [6] and a recognition algorithm for the unknot [3]. Braid theory was also a major tool in the work in [1], but it appears that Bennequin's detailed study of the foliation is based entirely on the characteristic foliation, as it occurs for knots in $\mathbb{R}^3$ and S^3. To the best of our knowledge this paper contains the first application of braid foliations to the study of transversal knots.

We note some similarities between the two foliations: The characteristic foliation is oriented and the braid foliation is orientable. (The orientation is ignored, but a dual orientation, determined by an associated flow, plays an equivalent role.) The foliations can be made to agree in the limiting case, as $\rho \to \infty$ (see the comments above Figure 1).

After an appropriate isotopy of $\mathcal{F}$ both foliations have no leaves that are simple closed curves. Also, their singularities are finite in number, each being either an elliptic point or a hyperbolic point (the hyperbolic point corresponding to a saddle-point tangency of $\mathcal{F}$ with the 2-planes of the structure). The signs of the singularities of each foliation are determined by identical considerations: the surface is naturally oriented by the assigned orientation on the knot. If at a singularity the orientation of the surface agrees (resp. disagrees) with the orientation of the foliation, then the singularity is positive (resp. negative). See Figure 4.

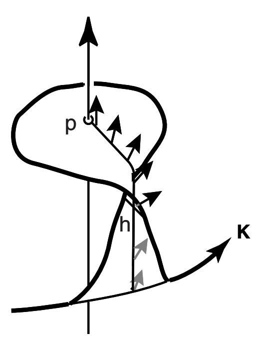

Figure 4: A positive elliptic singularity (p) and a positive hyperbolic singularity (h).

In both foliations the hyperbolic singularities are 4-pronged singularities. If s is a hyperbolic singular point, then the four branches of the singular leaf through s end at either elliptic points or at a point on K. (The condition that no singular leaves of the characteristic foliation connect hyperbolic points is a genericness assumption appearing in the literature on Legendrian and transversal knots). The three possible cases are illustrated in Figure 5. In that figure the elliptic points are depicted as circles surrounding $\pm$ signs (the sign of the elliptic singularity) and the hyperbolic singularities are depicted as black dots. Two of the four branches of the singular leaf end at positive elliptic points. The other two end at either two negative elliptic points, or one negative elliptic point and one point on K, or two points on K.

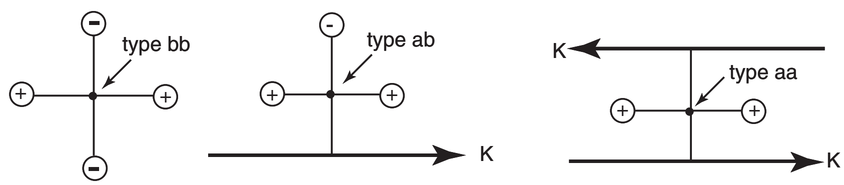

Figure 5: The three types of hyperbolic singularities.

There are also differences between the two foliations. In the braid foliation, elliptic points always correspond to punctures of the surface by the z axis. In the characteristic foliation, elliptic points on the surface may or may not correspond to punctures by the z-axis. That is, there may be elliptic points not corresponding to punctures, and there might be punctures not corresponding to elliptic points. Here is an example. In the braid foliation, if there is a piece of the surface along the boundary, foliated by a single positive pair of elliptic and hyperbolic singularities, then the only possible embedding for that piece is shown in Figure 4. On the other hand, in the characteristic foliation, if there is a piece of the surface, also along the boundary, also foliated by a positive elliptic-hyperbolic pair, then the corresponding embedding may or may not be the one shown in Figure 4. The embeddings will coincide if the tangent to the surface at the z axis is horizontal.

In work on the braid foliation one uses certain properties that appear to have been ignored in work based upon the characteristic foliation. For example, the work on braid foliations makes much of the distinction between the three types of hyperbolic singularities which we just illustrated in Figure 5, calling them *types bb, ab* and *aa*. The resulting combinatorics play a major role in the study of braid foliations. It seems to us that the distinction between bb, ab and aa singularities can also be made in the situation of the characteristic foliation, but that this has not been done.

In the braid foliation the elliptic points have a natural cyclic order on the z axis, if we are considering the ambient space as S^3 and the braid axis as one of the core circles of the Hopf fibration, and the hyperbolic points have a natural cyclic order in $0 \leq \theta \leq 2\pi$. These orderings do not seem useful in the contact setting. On the other hand, the characteristic foliation is invariant under rotation by θ and translation by z, so the interesting parameter seems to be the coordinate ρ.

An essential tool in manipulating and simplifying the characteristic foliation is the Giroux Elimination Lemma ([14], [10]), which allows one to 'cancel' pairs of same sign singularities. In topological knot theory different modifications have been introduced that are the braid foliation analogue of isotopies of the Giroux Elimination Lemma, see [5] and also [2]. They are called *ab exchange moves* and *bb exchange moves*, and they use pairs of Giroux-like cancellations, but on a much larger scale.

2. Exchange reducibility and transversal simplicity

Our initial goal is to motivate and define the concept of exchange reducibility. Let $\mathcal{K}$ be a topological knot type and let K be a closed n-braid representative of $\mathcal{K}$. We consider the following three modifications of K

- Our first modification is *braid isotopy*, that is, an isotopy in the complement of the braid axis. In [18] it is proved that isotopy classes of closed n-braids are in one-to-one correspondence with conjugacy classes in the braid group B_n. Since the conjugacy problem in the braid group is a solved problem, each conjugacy class can then be replaced by a unique representative that can be assumed to be transversal. Braid isotopy preserves the Bennequin number since it preserves both braid index and algebraic crossing number.

- Our second move is *destabilization*. See Figure 6(a). The box labeled P contains an arbitrary $(n-1)$-braid, and the label $n-2$ on the braid strand denotes $n-2$ parallel braid strands. The destabilization move reduces braid index from n to $n-1$ by removing a 'trivial loop'. If the trivial loop contains a positive crossing, the move is called a *positive* or + destabilization. Positive destabilization reduces algebraic crossing number and preserves the Bennequin number. Negative ($-$) destabilization increases the Bennequin number by 2.

- Our third move is the *exchange move*. See Figure 6(b). In general the exchange move changes conjugacy class and so cannot be replaced by braid isotopy. The exchange move preserves both braid index and algebraic crossing number, hence preserves the Bennequin number.

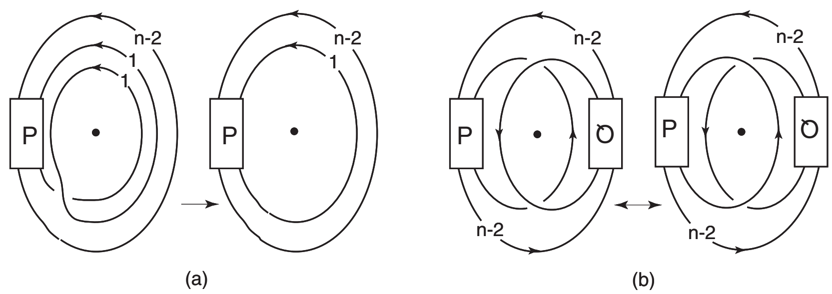

Figure 6: (a) positive destabilization and (b) The exchange move

To motivate our definition of exchange reducibility, we recall the following theorem, proved by the first author and W. Menasco:

Theorem A ([5], with a simplified proof in [2]). *Let K be a closed n-braid representative of the m-component unlink. Then K may be simplified to the trivial m-braid representative, i.e., a union of m disjoint round planar circles, by a finite sequence of the following three changes: braid isotopies, positive and negative destabilizations, and exchange moves.*

Motivated by Theorem A, we introduce the following definition:

Definition. A knot type $\mathcal{K}$ is said to be *exchange reducible* if an arbitrary closed braid representative K of arbitrary braid index n can be changed to an arbitrary closed braid representative of minimum braid index $n_{min}(\mathcal{K})$ by a finite sequence of braid isotopies, exchange moves and $\pm$-destabilizations. Note that this implies that any two minimal braid index representatives are either identical or are *exchange-equivalent*, i.e., are related by a finite sequence of braid isotopies and exchange moves.

Our first result is:

Theorem 1. *If $\mathcal{K}$ is an exchange reducible knot type, then the maximum Bennequin number of $\mathcal{K}$ is realized by any closed braid representative of braid index $n_{min}(\mathcal{K})$.*

The proof of Theorem 1 begins with a lemma. In what follows, we understand "transversal isotopy" to mean a topological isotopy that preserves the condition $\alpha = \rho^2 d\theta + dz > 0$ at every point of the knot and at every stage of the isotopy.

Lemma 1. *If a transversal closed braid is modified by one of the following isotopies, then the isotopy can be replaced by a transversal isotopy:*

(1) *Braid isotopy.*
(2) *Positive stabilization or positive destabilization.*
(3) *An exchange move.*

Proof of Lemma 1.

Proof of (1). Since the braid strands involved in the isotopy will be $\gg \epsilon$ away from the z-axis at each stage (so avoiding $-\rho^2 = 0$), any isotopy will be transversal if we keep the strands involved "relatively flat" ($dz/d\theta \sim 0$) at each stage. Since everything is happening locally there is space to flatten the strands involved without changing the braid.

*Proof o*f (2). See Figure 7(a). Consider a single trivial loop around the z axis, with a positive crossing. We have $d\theta > 0$ along the entire

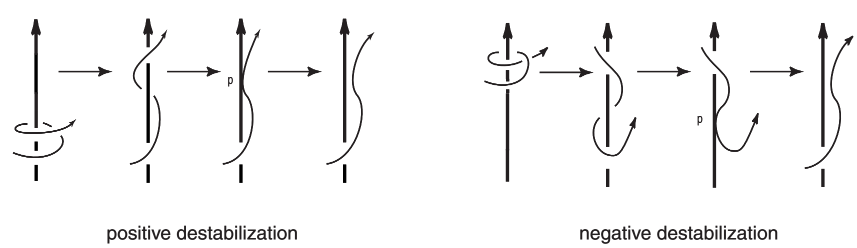

Figure 7: Destabilization, with a singularity at s, where $d\theta = 0$ and $\rho = 0$.

length of the loop since we are working with a closed braid. For a positive crossing we have $dz \geq 0$ throughout the loop as well. Therefore the inequality $dz/d\theta > -\rho^2$ is true for all non-zero real values of ρ. Crossing the z axis to destabilize the braid results in at least one singular point, where $d\theta = 0$, but if we continue to keep $dz \geq 0$ then in the limit, as $-\rho^2 \to 0$ from the negative real numbers, $dz/d\theta$ goes to ∞ through the positives. Therefore $dz/d\theta \neq -\rho^2$ at any stage in the isotopy.

Proof of (3). The sequence of pictures in Figure 8 shows that an exchange move can be replaced by a sequence of the following moves: isotopy in the complement of the z axis, positive stabilization, isotopy again, and finally positive destabilization. Claim 3 then follows from Claims 1 and 2. q.e.d.

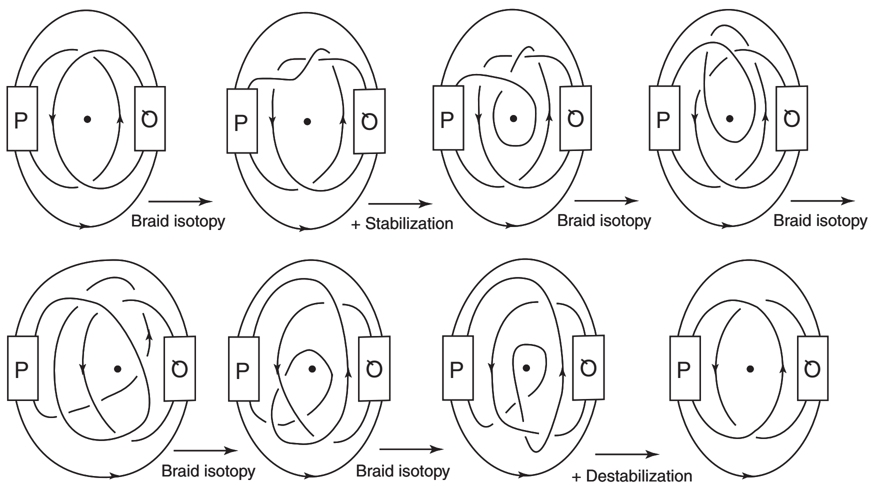

Figure 8: An exchange move corresponds to a sequence consisting of braid isotopies, a single positive stabilization and a single positive destabilization.

Remark 1. Observe that the argument given to prove (2) simply doesn't work for negative destabilization. See figure 7(b). The singularity in this destabilization is a point at which $dz/d\theta = -\rho^2$ in the limit. Indeed, a negative destabilization can't be modified to one which is transversal, because the Bennequin number (an invariant of transversal knot type) changes under negative destabilization.

Remark 2. If we had chosen $\alpha < 0$ along the knot, we would consider *negative* stabilizations and destabilizations as transversal isotopies and would use those instead of positive stabilizations and destabilizations in the exchange sequence. All the other moves translate to the negative setting without change.

Proof of Theorem 1. Let K be an arbitrary closed braid representative of the exchange reducible knot type $\mathcal{K}$. Let K_0 be a minimum braid index representative of $\mathcal{K}$, obtained from K by the sequence described in the definition of exchange reducibility. We must prove that the transversal knot TK_0 associated to K_0 has maximum Bennequin number for the knot type $\mathcal{K}$. Note that in general K_0 is not unique, however it will not matter, for if K_0' is a different closed braid representative of minimal braid index, then K_0 and K_0' are related by a sequence of braid isotopies and exchange moves, both of which preserve both braid index and algebraic crossing number, so $\beta(K_0') = \beta(K_0)$.

We obtain K_0 from K by a sequence of braid isotopies, exchange moves, and $\pm$-destabilizations. Braid isotopy, exchange moves and positive destabilization preserve $\beta(TK)$, but negative destabilization increases the Bennequin number by 2, so the sequence taking K to K_0 changes the Bennequin number from $\beta(TK) = c$ to $\beta(TK_0) = c + 2p$, where p is the number of negative destabilizations in the sequence. The question then is whether $c+2p$ is maximal for the knot type $\mathcal{K}$. If $c+2p$ is less than maximal, then there exists some other closed braid representative K' of the knot type $\mathcal{K}$ with maximum $\beta(TK') > \beta(TK_0)$. Since K_0 has minimum braid index for the knot type $\mathcal{K}$, it must be that $n(K') \geq n(K_0)$. If $n(K') = n(K_0)$, then the two braids are equivalent by a sequence of Bennequin number preserving exchange equivalences, so suppose instead that $n(K') > n(K_0)$. Then, since K' is a closed braid representative of the exchange reducible knot type $\mathcal{K}$, there must exist a sequence of braid isotopies, exchange moves, and $\pm$-destabilizations taking K' to a minimum braid index braid representative K_0'. Since $\beta(TK')$ is assumed to be maximum, and none of the moves in the sequence taking K' to K_0' reduce Bennequin number, it must be that $\beta(TK_0') = \beta(TK')$. But since K_0' and K_0 are both minimum braid index representatives of $\mathcal{K}$, they must be equivalent by a sequence of Bennequin number preserving exchange moves and isotopies. Thus $\beta(TK_0) = \beta(TK_0')$. q.e.d.

Our next result is:

Theorem 2. *If $\mathcal{TK}$ is a transversal knot type with associated topological knot type $\mathcal{K}$, where $\mathcal{K}$ is exchange reducible, then $\mathcal{TK}$ is transversally simple.*

The proof of Theorem 2 begins with two lemmas. Our first lemma had been noticed long ago by the first author and Menasco, who have had a long collaboration on the study of closed braid representatives of knots and links. However, it had never been used in any of their papers. It is therefore new to this paper, although we are indebted to Menasco for his part in its formulation. A contact-theory analogue of Lemma 3, below, appears as Lemma 3.8 of [11].

Lemma 2. *Using exchange moves and isotopy in the complement of the braid axis, one may slide a trivial loop on a closed braid from one location to another on the braid.*

Proof of Lemma 2. See Figure 9. It shows that, using braid isotopy and exchange moves, we can slide a trivial negative loop past any crossing to any place we wish on the braid. The argument for sliding a positive trivial loop around the braid is identical. q.e.d.

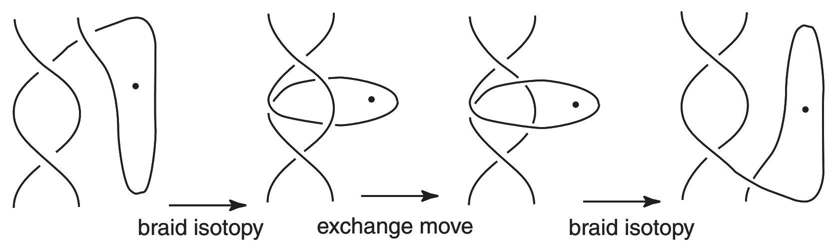

Figure 9: An exchange move that allows a negative trivial loop to slide along a braid.

Lemma 3. *Let K_1 and K_2 be closed n-braids that are exchange-equivalent. Let L_1 and L_2 be $(n+1)$-braids that are obtained from K_1 and K_2 by either negative stabilization on both or positive stabilization on both. Then L_1 and L_2 are exchange-equivalent.*

Proof of Lemma 3. We already know there is a way to deform K_1 to K_2, using exchange equivalence. Each braid isotopy may be broken up into a sequence of isotopies, each of which only involves local changes on some well-defined part of the braid. (For example, the defining relations in the braid group are appropriate local moves on cyclic braids). Similarly, exchange moves have local support. It may happen that the trivial loop which we added interferes with the support of one of the isotopies or exchange moves. If so, then by Lemma 2 we may use exchange equivalence to slide it out of the way. It follows that we may deform L_1 to L_2 by exchange equivalence. q.e.d.

Proof of Theorem 2. We are given an arbitrary representative of the transversal knot type $\mathcal{TK}$. Let $\mathcal{K}$ be the associated topological knot type. By the transversal Alexander's theorem [1] we may modify our representative transversally to a transversal closed n-braid $K = TK$

that represents the transversal knot type $\mathcal{TK}$ and the topological knot type $\mathcal{K}$. By the definition of exchange reducibility, we may then find a finite sequence of closed braids

$$K = K_1 \to K_2 \to \cdots \to K_{m-1} \to K_m,$$

all representing $\mathcal{K}$, such that each K_{i+1} is obtained from K_i by braid isotopy, a positive or negative destabilization or an exchange move, and such that K_m is a representative of minimum braid index $n_{min} = n_{min}(\mathcal{K})$ for the knot type $\mathcal{K}$. The knots $K_1, \ldots, K_m$ in the sequence will all have the topological knot type $\mathcal{K}$.

In general $\mathcal{K}$ will have more than one closed braid representative of minimum braid index. Let $\mathcal{M}_0(\mathcal{K}) = \{M_{0,1}, M_{0,2} \ldots\}$ be the set of all minimum braid index representatives of $\mathcal{K}$, up to braid isotopy. Clearly $K_m \in \mathcal{M}_0$. By [1], each $M_{0,i} \in \mathcal{M}_0$ may be assumed to be a transversal closed braid.

By Theorem 1 each $M_{0,i}$ has maximal Bennequin number for all knots that represent $\mathcal{K}$. In general this Bennequin number will not be the same as the Bennequin number of the original transversal knot type $\mathcal{TK}$. By Lemma 1 the moves that relate any two $M_{0,i}, M_{0,j} \in \mathcal{M}_0$ may be assumed to be transversal. After all these modifications the closed braids in the set $\mathcal{M}_0$ will be characterized, up to braid isotopy, by their topological knot type $\mathcal{K}$, their braid index $n_{min}(\mathcal{K})$ and their Bennequin number $\beta_{max}(\mathcal{K})$.

If the transversal knot type $\mathcal{TK}$ had Bennequin number $\beta_{max}(\mathcal{K})$, it would necessarily follow that $\mathcal{TK}$ is characterized up to transversal isotopy by its ordinary knot type and its Bennequin number. Thus we have proved the theorem in the special case of transversal knots that have maximum Bennequin number.

We next define new sets $\mathcal{M}_1, \mathcal{M}_2, \ldots$ of transversal knots, inductively. Each $\mathcal{M}_s$ is a collection of conjugacy classes of closed $(n_{min}(\mathcal{K}) + s)$-braids. We assume, inductively, that the braids in $\mathcal{M}_s$ all have topological knot type $\mathcal{K}$, braid index $n_{min}(\mathcal{K}) + s$ and Bennequin number $\beta_{max}(\mathcal{K}) - 2s$. Also, their conjugacy classes differ at most by exchange moves. Also, the collection of conjugacy classes of $(n_{min}(\mathcal{K}) + s)$-braids in the set $\mathcal{M}_s$ is completely determined by the collection of conjugacy classes of braids in the set $\mathcal{M}_0$. We now define the set $\mathcal{M}_{s+1}$ by choosing an arbitrary closed braid $M_{i,s}$ in $\mathcal{M}_s$ and adding a trivial negative loop. Of course, there is no unique way to do this, but by Lemma 2 we can choose one such trivial loop and use exchange moves to slide it completely around the closed braid $M_{i,s}$. Each time we use the exchange move of Lemma 2, we will obtain a new conjugacy class, which we then add to the collection $\mathcal{M}_{s+1}$. The set $\mathcal{M}_{s+1}$ is defined to be the collection of all conjugacy classes of closed braids obtained by adding trivial loops in every possible way to each $M_{i,s} \in \mathcal{M}_s$. The closed braids in $\mathcal{M}_{s+1}$ are equivalent under braid isotopy and exchange moves. They all have topological knot type $\mathcal{K}$, braid index $n_{min}(\mathcal{K}) + s + 1$, and Bennequin number $\beta_{max}(\mathcal{K}) - 2(s+1)$. The collection of closed braids in the set $\mathcal{M}_{s+1}$ is completely determined by the collection of closed braids in $\mathcal{M}_s$, and so by the closed braids in $\mathcal{M}_0$.

In general negative destabilizations will occur in the chain $K_1 \to K_m$. Our plan is to change the order of the moves in the sequence $K_1 \to K_m$, pushing all the negative destabilizations to the right until we obtain a new sequence, made up of two subsequences:

$$K = K_1^\star \to K_2^\star \to \cdots \to K_r^\star = K_0' \to \cdots \to K_s' = K_p,$$

where K_p has minimum braid index $n_{min}(\mathcal{K})$. The first subsequence $\mathcal{S}_1$, will be $K = K_1^\star \to K_2^\star \to \cdots \to K_r^\star$, where every $K_i^\star$ is a transversal representative of $\mathcal{TK}$ and the connecting moves are braid isotopy, positive destabilizations and exchange moves. The second subsequence, $\mathcal{S}_2$, is $K_r^\star = K_0' \to \cdots \to K_q'$, where every K_{i+1}' is obtained from K_i' by braid isotopy and a single negative destabilization. Also, K_q' has minimum braid index $n_{min}(\mathcal{K})$.

To achieve the modified sequence, assume that $K_i \to K_{i+1}$ is the first negative destabilization. If the negative trivial loop does not interfere with the moves leading from K_{i+1} to K_m, just renumber terms so that the negative destabilization becomes K_m and every other $K_j, j > i$ becomes K_{j-1}. But if it does interfere, we need to slide it out of the way to remove the obstruction. We use Lemma 2 to do that, adding exchange moves as required.

So we may assume that we have our two subsequences $\mathcal{S}_1$ and $\mathcal{S}_2$. The braids in $\mathcal{S}_1$ are all transversally isotopic and so they all have the same Bennequin number and they all represent $\mathcal{TK}$. The braids $K_i' \in \mathcal{S}_2$ all have the same knot type, but $\beta(K_{i+1}') = \beta(K_i') + 2$ for each $i = 1, \ldots, s-1$. With each negative destabilization and braid isotopy, the Bennequin number increases by 2 and the braid index decreases by 1. Each braid represents the same knot type $\mathcal{K}$ but a different transversal knot type $\mathcal{TK}$.

Our concern now is with $\mathcal{S}_2$, i.e., $K_r^\star = K_0' \to \cdots \to K_s'$, where every K_{i+1}' is obtained from K_i' by a single negative destabilization and braid isotopy. The number of negative destabilizations in subsequence $\mathcal{S}_2$ is exactly one-half the difference between the Bennequin number $\beta(\mathcal{TK})$ of the original transversal knot $\mathcal{TK}$ and the Bennequin number $\beta_{max}(\mathcal{K})$.

Let us now fix on any particular minimum braid index representative of $\mathcal{K}$ as a minimum braid index closed braid representative of the transversal knot type that realizes $\beta_{max}(\mathcal{K})$. It will not matter which we choose, because all belong to the set $\mathcal{M}_0$ and so are exchange-equivalent. We may then take the final braid $K_r^\star$ in $\mathcal{S}_1$, which is the same as the initial braid K_0' in $\mathcal{S}_2$, as our representative of $\mathcal{TK}$, because it realizes the minimal braid index for $\mathcal{TK}$ and by our construction, any other such representative is related to the one we have chosen by transversal isotopy. We may also proceed back up the sequence $\mathcal{S}_2$ from K_s' to a new representative that is obtained from K_0' by adding s negative trivial loops, one at a time. By repeated application of Lemma 3 we know that choosing any other element of $\mathcal{M}_0$ will take us to an exchange-equivalent element of $\mathcal{M}_s$. In this way we arrive in the set $\mathcal{M}_s$, which also contains $K_r^\star$, and which is characterized by $\mathcal{K}$ and β. The proof of Theorem 2 is complete. q.e.d.

3. Examples, applications and possible generalizations

In this section we discuss examples which illustrate Theorems 1 and 2.

3.1 The unlink and the unknot

Theorem A, quoted earlier in this manuscript, asserts that the m-component unlink, for $m \geq 1$, is exchange reducible. In considering a link transversally, it should be mentioned that we are assuming each of the components of the link satisfy the same inequality $\alpha > 0$. We also need to define the Bennequin number properly for this transversal link. The natural way to do so, suggested by Oliver Dasbach, is by the following method. For a crossing involving two different components of the link, assign $\pm 1/2$ to each component depending on the sign of the crossing. Assign ± 1 to each crossing consisting of strands from the same component, as in the case of a knot. Then the Bennequin number of each component is the difference between the algebraic crossing number e (a sum of ± 1's and $\pm 1/2$'s) and n, the braid index of that component. Define the Bennequin number of the link to be the collection of the Bennequin numbers of the components of the link. The following corollary is an immediate consequence of Theorems 1 and A:

Corollary 1. *The m-component unlink, $m \geq 1$, is transversally simple. In particular, the unknot is transversally simple.*

Note that Corollary 1 gives a new proof of a theorem of Eliashberg [10].

3.2 Torus knots and iterated torus knots

In the manuscript [11] J. Etnyre proved that positive torus knots are transversally simple. His proof failed for negative torus knots, but he conjectured that the assertion was true for all torus knots and possibly also for all iterated torus knots. In an early draft of this manuscript we conjectured that torus knots and iterated torus knots ought to be exchange reducible, and sketched our reasons. Happily, the conjecture is now a fact, established by W. Menasco in [17]. Two corollaries follow. To state and prove our first corollary, we need to fix our conventions for the description of torus knots and iterated torus knots.

Definition. Let U be the unit circle in the plane $z = 0$, and let $N(U)$ be a solid torus of revolution with U as its core circle. Let λ_0 be a longitude for U, i.e., λ_0 is a circle in the plane $z = 0$ which lies on $\partial N(U)$, so that U and λ_0 are concentric circles in the plane $z = 0$. See Figure 10. A *torus knot* of *type* $e(p, q)$, where $e = \pm$, on $\partial N(U)$, denoted $K_{e(p,q)}$, is the closed p-braid $(\sigma_1\sigma_2 \cdots \sigma_{p-1})^{eq}$ on $\partial N(U)$, where $\sigma_1, \ldots, \sigma_{p-1}$ are elementary braid generators of the braid group B_p. Note that $K_{e(p,q)}$ intersects the curve λ_0 in q points, and note that the algebraic crossing number of its natural closed braid projection on the plane $z = 0$ is $e(p-1)q$. The knot $K_{e(p,q)}$ also has a second natural closed braid representation, with the unknotted circle U as braid axis and the closed q-braid $(\sigma_1\sigma_2 \cdots \sigma_{q-1})^{ep}$ as closed braid representative. Since p and q are coprime integers, one of these closed braids will have smaller braid index than the other, and without loss of generality we will assume in the pages which follow that we have chosen p to be smaller than q, so that $K_{e(p,q)}$ has braid index p.

Definition. We next define what we mean by an $e(s, t)$-*cable* on a knot X in 3-space. Let X be an arbitrary oriented knot in oriented S^3, and let $N(X)$ be a solid torus neighborhood of X in 3-space. A *longitude* λ for X is a simple closed curve on $\partial N(X)$ which is homologous to X in $N(X)$ and null-homologous in $S^3 \setminus X$. Let $f : N(U) \to N(X)$ be a

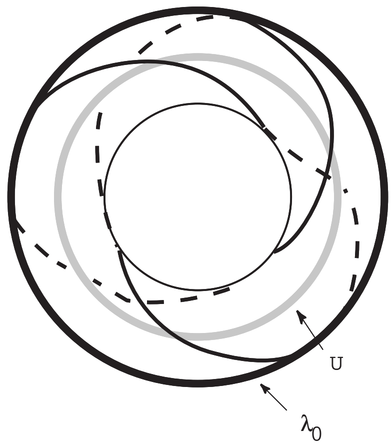

Figure 10: The standard solid torus $N(U)$, with $K_{+(2,3)} \subset \partial N(K)$

homeomorphism which maps λ_0 to λ. Then $f(K_{e(s,t)})$ is an $e(s,t)$-*cable* about X.

Definition. Let $\{e_i(p_i, q_i), i = 1, \dots, r\}$ be a choice of signs $e_i = \pm$ and coprime positive integers (p_i, q_i), ordered so that for each i we have $p_i, q_i > 0$. An *iterated torus knot* $K(r)$ of type $(e_1(p_1, q_1), \dots, e_r(p_r, q_r))$, is defined inductively by:

- $K(1)$ a torus knot of type $e_1(p_1, q_1)$, i.e., a type $e(p_1, q_1)$ cable on the unknot U. Note that, by our conventions, $p_1 < q_1$.
- $K(i)$ is an $e_i(p_i, q_i)$ cable about $K(i-1)$. We place no restrictions on the relative magnitudes of p_i and q_i when $i > 1$.

Here is one of the simplest non-trivial examples of an iterated torus knot. Let $K(1)$ be the positive trefoil, a torus knot of type $(2, 3)$. See Figure 11(a) and (b). Note that in the left sketch the core circle is our unit circle U, while in the right sketch the core circle is the knot $K(1)$. The iterated torus knot $K(2) = K_{(2,3),-(3,4))}$ is the $-(3, 4)$ cable about $K(1)$. See Figure 12.

In [17], it was shown that the braid foliation machinery used for the torus in [7] could be adapted to the situation in which the knot is on the surface of the torus. The main result of that paper is the following theorem.

Theorem. ([17], Theorem 1) *Oriented iterated torus knots are exchange reducible.*

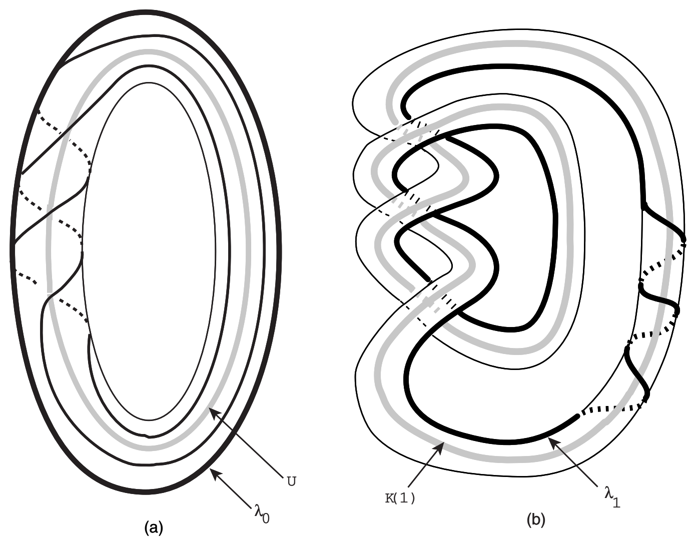

Figure 11: (a) the torus knot $K_{+(2,3)}$. (b)the solid torus neighborhood $N(K(1))$ of $K(1)$, with core circle $K(1)$ and longitude λ_1 marked.

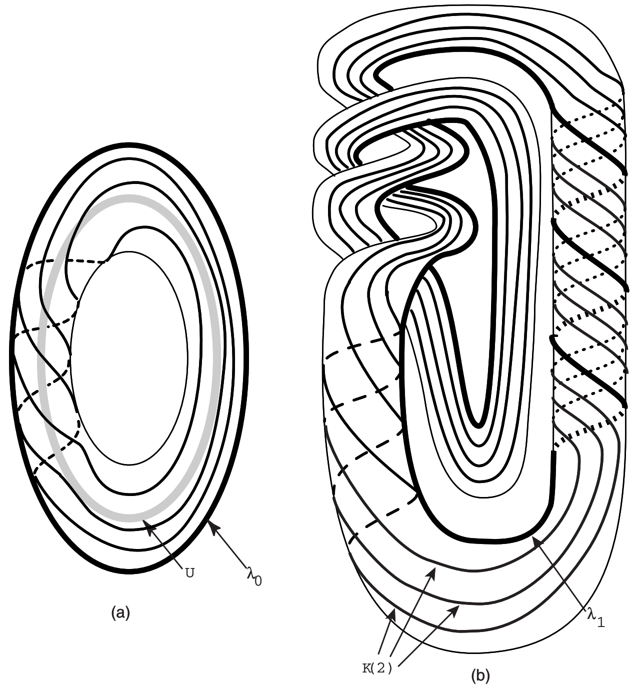

Figure 12: (a) the torus knot of type -(3,4) embedded in $\partial N(U)$, (b) the iterated torus knot $K(2) = K_{+(2,3),-(3,4)}$

Combining Menasco's Theorem with Theorem 2, we have the following immediate corollary:

Corollary 2. *Iterated torus knots are transversally simple.*

Our next contribution to the theory of iterated torus knots requires that we know the braid index of an iterated torus knot. The formula is implicit in the work of Schubert [20], but does not appear explicitly there.

Lemma 4. *Let $K(r) = K_{e_1(p_1,q_1),\ldots,e_r(p_r,q_r)}$ be an r-times iterated torus knot. Then the braid index of $K(r)$ is $p_1p_2\cdots p_r$.*

Proof. We begin with the case $r = 1$. By hypothesis $p_1 < q_1$, also the torus knot $K_{e_1(p_1,q_1)}$ is represented by a p_1-braid $(\sigma_1\sigma_2\cdots\sigma_{p_1-1})^{e_1q_1}$. By the formula given in [15] for the HOMFLY polynomial of torus knots, together with the Morton-Franks-Williams braid index inequality (discussed in detail in [15]), it follows that this knot cannot be represented as a closed m-braid for any $m < p_1$.

Passing to the general case, Theorem 21.5 of [20] tells us that the torus knot $K_{e_1(p_1,q_1)}$ and the array of integers $e_1(p_2,q_2),\ldots,e_r(p_r,q_r)$ form a complete system of invariants of the iterated torus knot $K(r) = K_{e_1(p_1,q_1),\ldots,e_r(p_r,q_r)}$. Lemma 23.4 of [20] tells us that, having chosen a p_1-braid representative for $K_{e_1(p_1,q_1)}$, there is a natural $p_1p_2\cdots p_r$-braid representative of $K(r)$. This representative is the only one on this number of strings, up to isotopy in the complement of the braid axis. Theorem 23.1 of [20] then asserts that $K(r)$ also cannot be represented as a closed braid with fewer strands. That is, its braid index is $p_1p_2\cdots p_r$.

q.e.d.

Remark. The iterated torus knot K_r has two natural closed braid representatives. The first is a $p_1p_2\cdots p_r$- braid which has the core circle U' of the unknotted solid torus $S^3 \setminus N(U)$ as braid axis. The second is a $q_1p_2\cdots p_r$-braid which has the core circle U of the unknotted solid torus $N(U)$ as braid axis. In the case $r = 1$, the second choice gives a closed braid which is reducible in braid index, i.e., it has $q_1 - p_1$ trivial loops. From this it follows that if $r > 1$ it will have $(q_1 - p_1)p_2\cdots p_r$ trivial loops, thus the second closed braid representation is reducible to the first.

We are now ready to state our second corollary about iterated torus knots. Let χ be the Euler characteristic of an oriented surface of minimum genus bounded by $K(r)$.

Corollary 3. *Let $K(r) = K_{e_1(p_1,q_1),\dots,e_r(p_r,q_r)}$ be an iterated torus knot, where $p_1 < q_1$. Then the maximum Bennequin number of $K(r) = K_{e_1(p_1,q_1),\dots,e_r(p_r,q_r)}$ is given by the following two equivalent formulas:*

(1) $\beta_{max}(K(r)) = a_r - p_1p_2\cdots p_r$, *where* $a_r = \sum_{i=1}^{r} e_iq_i(p_i-1)p_{i+1}p_{i+2}\dots p_r$.

(2) $\beta_{max}(K(r)) = -\chi - d$, *where* $d = \sum_{i=1}^{r}(1-e_i)(p_i-1)q_ip_{i+1}p_{i+2}\dots p_r$.

(3) *Moreover, the upper bound in the inequality $\beta_{max}(K(r)) \leq -\chi$ is achieved if and only if all of the e_i's are positive.*

Proof. We begin with the proof of (1). By Lemma 4 the braid index of $K(r)$ is $p_1p_2\cdots p_r$. Therefore $\beta_{max}(K(r)) = a_r - p_1p_2\cdots p_r$, where a_r is the algebraic crossing number of the unique $p_1p_2\cdots p_r$-braid representative of $K(r)$. To compute a_r we proceed inductively. If $r = 1$ then $K(1)$ is a type $e_1(p_1, q_1)$ torus knot, which is represented by the closed p_1-braid $(\sigma_1\sigma_2\dots\sigma_{p-1})^{e_1q_1}$. Its algebraic crossing number is $a_1 = e_1(p_1-1)q_1$.

The knot $K(i)$ is an $e_i(p_i, q_i)$ cable on $K(i-1)$. Note that $K_{e_i(p_i,q_i)} \subset \partial N(K_0)$, also $K_{e_i(p_i,q_i)}$ is a p_i-braid, also $K_{e_i(p_i,q_i)} \cap \lambda_0$ consists of q_i points. We shall think of the projection of the braided solid torus $N(K(i-1))$, which is a $p_1p_2\cdots p_{i-1}$-braid, as being divided into three parts. The reader may find it helpful to consult Figures 13 (a), (b), (c) as we examine the contributions to a_i from each part.

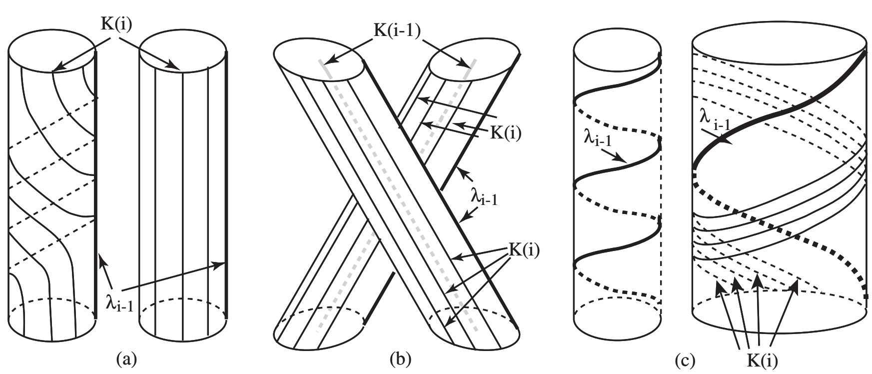

Figure 13: Iterated torus knots.

(a) The first part of $N(K(i-1))$ is the trivial $p_1p_2\cdots p_{i-1}$-braid. The longitude λ_{i-1} is parallel to the core circle of $N(K(i-1))$ in this part. The surface $\partial N(K(i-1))$ contains on one of its $p_1p_2\cdots p_{i-1}$ cylindrical branches the image under f of the braided part of $K_{e_r(p_r,q_r)}$. See Figure 13(a), which shows the braid when $e_i(p_i,q_i)=-(3,4)$. This part of $K(i)$ contributes $e_i(p_i-1)q_i$ to a_i. Note that there are q_i points where $f(K_{e_i(p_i,q_i)})$ intersects λ_{i-1}.

(b) The second part of $N(K(i-1))$ contains all of the braiding in $K(i-1)$, and so also in $N(K(i-1))$. In Figure 13(b) we have illustrated a single crossing in $K(i-1)$ and the associated segments of $N(K(i-1))$. We show a single crossing of λ_{i-1} (as a thick line) over $K(i-1)$. The single signed crossing contributes p_i^2 crossings to $K(i)$, so the total contribution from all of the crossings in $K(i-1)$ will be $a_{i-1}p_i^2$. The illustration shows the case $p_i=3$.

(c) The third part of $N(K(i-1))$ is again the trivial $p_1p_2\cdots p_{i-1}$-braid. It contains corrections to the linking number of λ_{i-1} with $K(i-1)$ which result from the fact that a curve which is everywhere 'parallel' to the core circle will have linking number a_{i-1}, not 0, with $K(i-1)$. To correct for this, we must allow the projected image of λ_{i-1} to loop around $\partial N(K(i-1))$ exactly $-a_{i-1}$ times, so that its total linking number with $K(i-1)$ is zero. See the left sketch in Figure 13(c), which shows the 3 positive loops which occur if $a_{i-1}=-3$. We have already introduced q_i intersections between $f(K_{e_i(p_i,q_i)})$ and λ_{i-1}, and therefore we must avoid any additional intersections which might arise from the $-a_{i-1}$ loops. See the right sketch in Figure 13(c). When λ_{i-1} wraps around $N(K(i-1))$ the additional $-a_{i-1}$ times, the p_i-braid $f(K_{e_i(p_i,q_i)}$ must follow. Each loop in λ_{i-1} introduces $(p_i-1)+(p_i-2)+\cdots+2+1=\frac{p_i(p_i-1)}{2}$ crossings per half-twist in $f(K_{e_i(p_i,q_i)})$. Since there is a full twist to go around the positive loop this number is doubled to $p_i(p_i-1)$. We have shown the 12 crossings in $K(i)$ which come from a single loop in λ_{i-1} when $p_i=4$. The total contribution is $-a_{i-1}p_i(p_i-1)$.

Adding up all these contributions we obtain

$$a_i=e_i(p_i-1)q_i+a_{i-1}(p_i)^2-a_{i-1}(p_i^2-p_i)=e_i(p_i-1)q_i+a_{i-1}p_i.$$

Summing the various terms to compute a_r, we have proved part (1) of the Corollary.

The proof of (2) will follow from that of (1) if we can show that:

$$\chi = p_1 p_2 \cdots p_r - d - a_r,$$

where $d = \sum_{i=1}^{r} (1 - e_i)(p_i - 1)q_i p_{i+1} p_{i+2} \ldots p_r$. To see this, we must find a natural surface of minimum genus bounded by $K(r)$ and compute its Euler characteristic. By Theorem 12, Lemma 12.1 and Theorem 22 of [20], a surface of minimum genus bounded by $K(r)$ may be constructed by Seifert's algorithm, explained in Chapter 5 of [19], from a representative of K(r) which has minimal braid index. We constructed such a representative in our proof of Part (1). To compute its Euler characteristic, use the fact that χ is the number of Seifert circles minus the number of unsigned crossings (Exercises 2 and 10 on pages 119 and 121 of [19]). By a theorem of Yamada [22] the number of Seifert circles is the same as the braid index, i.e., $p_1 p_2 \cdots p_r$ in our situation. The number of unsigned crossings is b_r, where $b_i = (p_i - 1)q_i + b_{i-1} p_i$ and $b_1 = (p_1 - 1)q_1$ and $\chi = p_1 p_2 \cdots p_r - b_r$. Adding up the contributions from all the $b_i's$ we get $b_r = \sum_{i=1}^{r} (p_i - 1) q_i p_{i+1} p_{i+2} \ldots p_r$ which can be rewritten as $\sum_{i=1}^{r} (1 - e_i + e_i)(p_i - 1) q_i p_{i+1} p_{i+2} \ldots p_r$. Separating terms:

$$\begin{aligned} b_r = & \sum_{i=1}^{r} (1 - e_i)(p_i - 1) q_i p_{i+1} p_{i+2} \ldots p_r \\ & + \sum_{i=1}^{r} (e_i)(p_i - 1) q_i p_{i+1} p_{i+2} \ldots p_r = d + a_r. \end{aligned}$$

The claimed formula for χ follows.

To prove (3), observe that the only case when $\beta_{max} = -\chi$ exactly occurs when $d = 0$, i.e., the sum

$$\sum_{i=1}^{r} (1 - e_i)(p_i - 1) q_i p_{i+1} p_{i+2} \ldots p_r = 0.$$

That is, all the e_i's are $+1$. q.e.d.

3.3 Knots that are not exchange reducible

A very naive conjecture would be that all knots are exchange reducible, however that is far from the truth. We begin with a simple example. In the manuscript [6] Birman and Menasco studied knots that are represented by closed 3-braids, up to braid isotopy, and identified the proper subset of those knots whose minimum braid index is 3 (i.e., not 2 or 1). They prove that the knots that have minimum braid index representatives of braid index 3 fall into two groups: those that have a unique such representative (up to braid isotopy) and infinitely many examples that have exactly two distinct representatives, the two being related by 3-braid *flypes*. A flype is the knot type preserving isotopy shown in Figure 14. Notice that the flype is classified as *positive* or *negative* depending on the sign of the isolated crossing. After staring at the figure, it should become clear to the reader that the closed braids in a 'flype pair' have the same topological knot type. We say that a braid representative *admits a flype* if it is conjugate to a braid that has the special form illustrated in Figure 14.

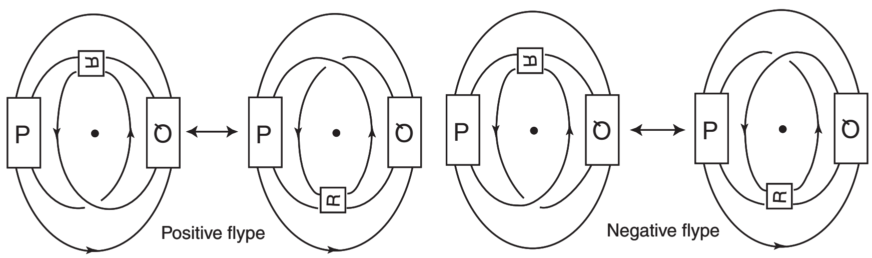

Figure 14: Closed 3-braids that are related by a flype.

Theorem 3 [6]. *The infinite sequence of knots of braid index 3 in* [6]*, each of which has two closed 3-braid representatives, related by flypes, are examples of knots that are* not *exchange reducible.*

Proof. It is proved in [6] that for all but an exceptional set of P, Q, R the closed braids in a flype pair are in distinct conjugacy classes. Assume from now on that a 'flype pair' means one of these non-exceptional pairs. Since conjugacy classes are in one-to-one correspondence with braid isotopy equivalence classes, it follows that the braids in a flype pair are not related by braid isotopy. On the other hand, it is proved in [6] that when the braid index is ≤ 3 the exchange move can always be replaced by braid isotopy, so the braids in a flype pair cannot be exchange equivalent. q.e.d.

On closer inspection, it turns out that a positive flype can be replaced by a sequence of braid isotopies and positive stabilizations and destabilizations, which shows that it is a transversal isotopy. See Figure 15.

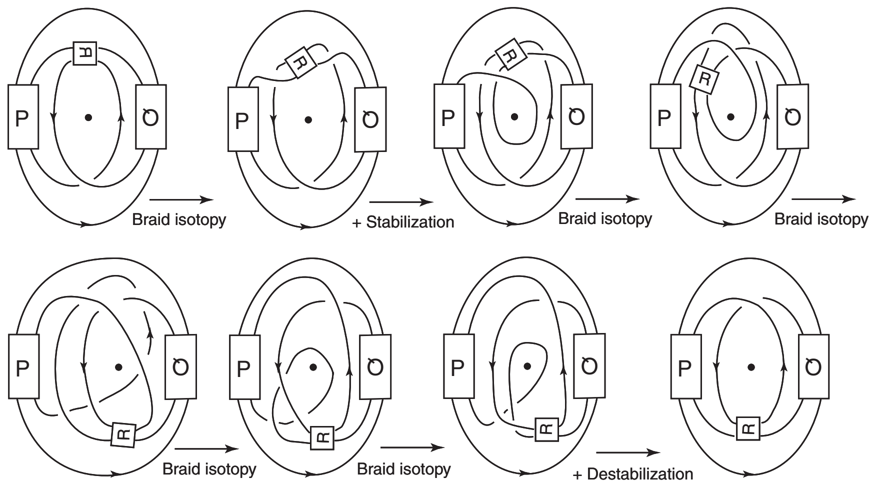

Figure 15: A positive flype can be replaced by a sequence of transversal isotopies.

The figure is a generalization of Figure 8 (proving that exchange moves are transversal), because flypes are generalizations of exchange moves. We replace one of the $\sigma_n^{\pm 1}$ with the braid word we label R. A negative flype also has a replacement sequence similar to the one pictured in Figure 15, but the stabilizations and destabilizations required are negative. Therefore the negative flype sequence cannot be replaced by a transversal isotopy using these methods. There may well be some other transversal isotopy that can replace a negative flype, but we did not find one. Thus we are lead to the following conjecture:

Conjecture. *Any transversal knot type whose associated topological knot type $\mathcal{K}$ has a minimum braid index representative that admits a negative flype is not transversally simple.*

A simple example which illustrates our conjecture is shown in Figure 16.

The essential difficulty we encountered in our attempts to prove or disprove this conjecture is that the only effective invariants of transversal knot type that are known to us at this writing are the topological knot type and the Bennequin number, but they do not distinguish these examples.

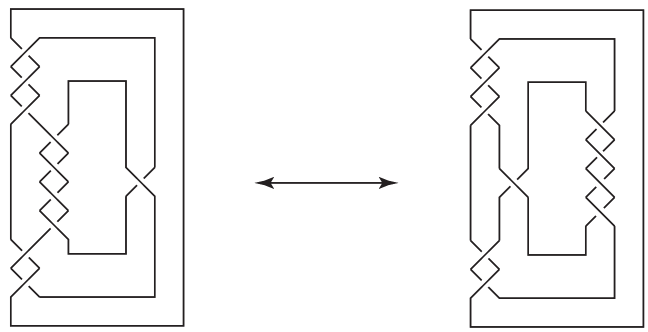

Figure 16: A simple example which illustrates our conjecture

3.4 Knots with infinitely many transversally equivalent closed braid representatives, all of minimal braid index

At this writing the only known examples of transversally simple knots are iterated torus knots. By Theorem 24.4 of [20] iterated torus knots have unique closed braid representatives of minimum braid index, and it follows from this and Theorem 1 that they have unique representatives of maximum Bennequin number. It seems unlikely to us that all transversally simple knots have unique closed braid representatives of minimum braid index, and we now explain our reasons.

The exchange move was defined in Figure 6(b) of §2. It seems quite harmless, being nothing more than a special example of a Reidemeister move of type II. It also seems unlikely to produce infinitely many examples of anything, however that is exactly what happens when we combine it with braid isotopy. See Figure 17 with $n = 4$. Proceeding from the right to the left and following the arrows, we see how braid isotopy and exchange moves can be used to produce infinitely many examples of closed braids which are transversally equivalent. It is not difficult to choose the braids R and S in Figure 17 so that the resulting closed braids are all knots, and also so that they actually have braid index 4, and also so that they are in infinite many distinct braid isotopy classes (using an invariant of Fiedler [12] to distinguish the braid isotopy classes). We omit details because, at this writing, we do not know whether the knots in question are exchange reducible, so we cannot say whether they all realize the maximum Bennequin number for their associated knot type.

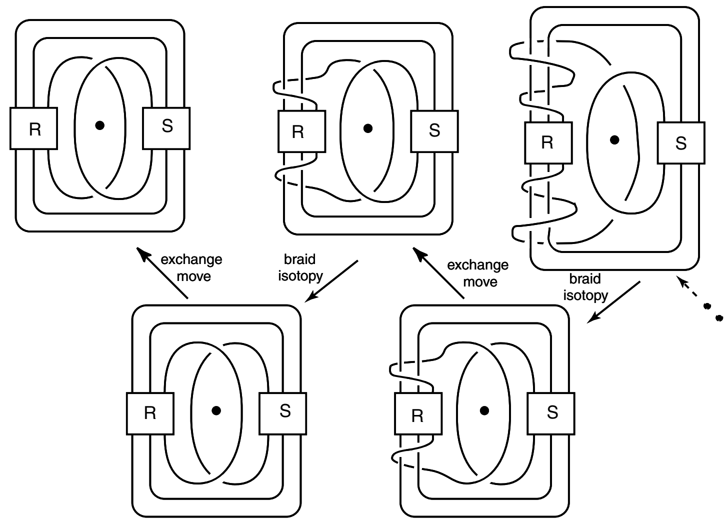

Figure 17: The exchange move and braid isotopy can lead to infinitely many distinct closed n-braid representatives of a single knot type.

3.5 Generalizing the concept of 'exchange reducibility'

Some remarks are in order on the concept of 'exchange reducibility'. Define two closed braids $A \in B_n$, $A' \in B_m$ to be *Markov-equivalent* if the knot types defined by the closed braids coincide. Markov's well-known theorem (see [16]) asserts that Markov-equivalence is generated by braid isotopy, $\pm$-stabilization, and $\pm$-destabilization. However, when studying this equivalence relation one encounters the very difficult matter that $\pm$-stabilization is sensitive to the exact spot on the closed braid at which one attaches the trivial loop. On the other hand, Lemma 2 shows that exchange moves are the obstruction to moving a trivial loop from one spot on a knot to another. Therefore if we allow exchange moves in addition to braid isotopy, $\pm$-stabilization, and $\pm$-destabilization, one might hope to avoid the need for stabilization. That is the idea behind the definition of exchange reducibility, and behind the proof of Theorem A. However, that hope is much too naive, as was shown by the examples in §3.3.

A way to approach the problem of transversal knots is to augment the definition of exchange reducible by allowing additional 'moves'. In their series of papers *Studying knots via closed braids I-VI*, the first author and Menasco have been working on generalizing the main result in Theorem A to all knots and links. In the forthcoming manuscript

[8], a general version of the 'Markov theorem without stabilization' is proved. The theorem states that for each braid index n a finite set of new moves suffices to reduce any closed braid representative of any knot or link to minimum braid index 'without stabilization'. These moves include not only exchange moves and positive and negative flypes, but more generally *handle moves* and *G-flypes*. Handle moves can always be realized transversally. The simplest example of a G-flype is the 3-braid flype that is pictured in Figure 14, with weights assigned to the strands. This will change it to an m-braid flype, for any m. But other examples exist, and they are much more complicated. We note, because it is relevant to the discussion at hand, that any *positive* G-flype can be realized by a transversal isotopy. The sequence in Figure 15 is a proof of the simplest case. Awaiting the completion of [8], we leave these matters for future investigations.

References

[1] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque **107-8** (1983) 87–161; English version: Russian Math. Surveys **44** (1989) 1–65. Section numbers in the two versions agree.

[2] J. S. Birman & E. Finkelstein, *Studying surfaces via closed braids*, J. Knot Theory Ramifications **7** (1998) 267–334.

[3] J. S. Birman & M. Hirsch, *A new algorithm for recognizing the unknot*, Geom. Topology **2** (1998) 175–220.

[4] J. S. Birman & W. Menasco, *Studying links via closed braids. IV: Composite and split links*, Invent. Math. **102** (1990) 115–139.

[5] ———, *Studying links via closed braids. V: The unlink*, Trans. Amer. Math. Soc. **329** (1992) 585–606.

[6] ———, *Studying links via closed braids. III: Classifying links which are closed 3-braids*, Pacific J. Math. **161** (1993) 25–113.

[7] ———, *Special positions for essential tori in link complements*, Topology **33** No. 3 (1994) 525–556. See also Erratum, Topology **37** (1998) 225.

[8] ———, *Stabilization in the braid groups*, manuscript in preparation.

[9] J. S. Birman & N. C. Wrinkle, *Holonomic and Legendrian parametrizations of knots*, J. Knot Theory Ramifications, **9** (2000) 293–309.

[10] Y. Eliashberg, *Legendrian and transversal knots in tight contact* 3*-manifolds*, Topological Methods in Modern Math. (1991) 171–193.

[11] J. Etnyre, *Transversal torus knots*, Geom. Topology **3** (1999) 253–268.

[12] T. Fiedler *A small state sum for knots*, Topology **32** (1993) 281–294.

[13] D. Fuchs & S. Tabachnikov, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology **36** (1997) 1025–1053.

[14] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. **66** (1991) 637–677.

[15] V. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. Math. **126** (1987) 335–388.

[16] A. A. Markov, *Über die freie Äquivalenz der geschlossenen Zöpfe*, Recueil Math. Moscou **1** (1935) 73–78.

[17] W. Menasco, *On iterated torus knots and transversal knots*, Preprint: http://www.math.buffalo.edu/ menasco/index.html

[18] H. Morton, *Infinitely many fibered knots having the same Alexander polynomial*, Topology **17** (1978) 101–104.

[19] D. Rolfsen, *Knots and links*, Publish or Perish 1976, reprinted 1990.

[20] H. Schubert, *Knoten und Vollringe*, Acta Math. **90** (1953) 131–226.

[21] V. Vassiliev, *Holonomic links and Smale principles for multisingularities*, J. Knot Theory Ramifications **6** (1997) 115–123.

[22] S. Yamada, *The minimum number of Seifert circles equals the braid index of a link*, Invent. Math. **89** (1987) 347–356.

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