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Research Article

On Tri-α-Open Sets in Fuzzifying Tritopological Spaces

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In this paper, we introduced and studied (1,2,3)-α-open set, (1,2,3)-α-neighborhood system, (1,2,3)-α-derived, (1,2,3)-α-closure, (1,2,3)-α-interior, (1,2,3)-α-exterior, (1,2,3)-α-boundary, (1,2,3)-α-convergence of nets, and (1,2,3)-α-convergence of filters in fuzzifying tritopological spaces.

1. Introduction

The fuzzy set is an important concept, which was introduced for the first time in 1965 by Zadeh [1]; it was then used in many studies in various fields. Here, we are interested in fuzzy with topology. The fuzzy and fuzzifying topologies are two branches of fuzzy mathematics. The basic concepts and properties of fuzzy topologies were subedited and investigated by Chang in 1968 [2] and Wong in 1974 [3]. After that, so many works of literature have appeared for different kinds of fuzzy topological spaces for, e.g., [4–8]. In 1991-1993, Ying introduced a new approach for fuzzy topology with fuzzy logic and established some properties in fuzzifying topology [9–11]. Also, we are interested in the concept of α-open set which was introduced by Njåstad in 1965 [12], and the tritopological space which was first initiated by Kovar in 2000 [13]. In 2017, Tapi and Sharma introduced α-open sets in tritopological spaces [14]. In 1999, Khedr et al. presented semiopen sets and semicontinuity in fuzzifying topology [15]. In 2016, Allam and et al. studied semiopen sets in fuzzifying bitopological spaces [16]. We will use in this paper Ying's basic fuzzy logic formulas with appropriate set theoretical notations from [9, 10].

The following are some useful definitions and results that will be used in the rest of the present work.

If X is the universe of discourse, and if $\tau \in \Im(P(X))$ satisfy the following three conditions:

(1) $\tau(X) = 1$ *and* $\tau(\emptyset) = 1$;

(2) for any $G, H, \tau(G \cap H) \geq \tau(G) \wedge \tau(H)$;

(3) for any $\{G_\lambda : \lambda \in \Lambda\}, \tau(\bigcup_{\lambda\in\Lambda} G_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \tau(G_\lambda)$;

then τ is a fuzzifying topology and (X, τ) a fuzzifying topological space [9].

The family of fuzzifying closed sets is denoted by $\mathscr{F}$ and defined as $G \in \mathscr{F} \coloneqq X \sim G \in \tau$, where $X \sim G$ is the complement of G [9].

The neighborhood system of x is denoted by $N_x \in \Im(P(X))$ and defined as $N_x(G) = \sup_{x\in H\subseteq G} \tau(H)$ [9].

The closure set of a set $G \subseteq X$ is denoted by $cl(G) \in \Im(X)$ and defined as $cl(G)(x) = 1 - N_x(X \sim G)$ [9].

The fuzzifying interior set of a set $G \subseteq X$ is denoted by $\operatorname{int}(G) \in \Im(X)$ and defined as $\operatorname{int}(G)(x) = N_x(G)$ [10].

The family of all fuzzifying α-open sets is denoted by $\alpha\tau$ and defined as $G \in \alpha\tau \coloneqq \forall x\ (x \in G \longrightarrow x \in \operatorname{int}(cl(\operatorname{int}(G)))$, i.e., $\alpha\tau(G) = \inf_{x\in G}(\operatorname{int}(cl(\operatorname{int}(G))))(x)$ [17].

The family of all fuzzifying α-closed sets is denoted by $\alpha\mathscr{F}$ and defined as $G \in \alpha\mathscr{F} \coloneqq X \sim G \in \alpha\tau$ [17].

The fuzzifying α-interior set of a set $G \subseteq X$ is denoted by $\alpha\operatorname{int}(G) \in \Im(X)$ and defined as follows: $\alpha\operatorname{int}(G)(x) = \alpha N_x(G)$, where αN_x is α–neighborhood system of x defined as $\alpha N_x(G) = \sup_{x\in H\subseteq G} \alpha\tau(H)$ [17].

The fuzzifying α-derived set of a set $G \subseteq X$ is denoted by $\alpha d(G) \in \Im(X)$ and defined as $x \in \alpha d(G) \coloneqq \forall H\ (H \in \alpha N_x \longrightarrow H \bigcap (G \sim \{x\}) \neq \emptyset))$, i.e., $\alpha d(G)(x) = \inf_{H\bigcap(G\sim\{x\})=\emptyset}(1 - \alpha N_x(H))$ [18].

The α-closure set of a set $G \subseteq X$ is denoted by $\alpha cl(G) \in \Im(X)$ and defined as $\alpha cl(G)(x) = \inf_{x \notin H, G \subseteq H}(1 - \alpha\mathscr{F}(H))$ [17].

2. $(1,2,3)$-α-Open Sets in Fuzzifying Tritopological Spaces

Definition 1. If $(X, \tau_1, \tau_2, \tau_3)$ is a fuzzifying tritopological space (FTTS), then we have the following:

(i) The family of all fuzzifying (1,2,3)-α-open sets is denoted by $\alpha\tau_{(1,2,3)} \in \Im(P(X))$ and defined as $G \in \alpha\tau_{(1,2,3)} := \forall x\ (x \in G \longrightarrow x \in \text{int}_1(cl_2(\text{int}_3(G))))$, i.e., $\alpha\tau_{(1,2,3)}(G) = \inf_{x \in G}(\text{int}_1(cl_2(\text{int}_3(G))))(x)$.

(ii) The family of all fuzzifying (1,2,3)-α-closed sets is denoted by $\alpha\mathscr{F}_{(1,2,3)} \in \Im(P(X))$ and defined as $G \in \alpha\mathscr{F}_{(1,2,3)} := X \sim G \in \alpha\tau_{(1,2,3)}$.

Lemma 2. *Let $(X, \tau_1, \tau_2, \tau_3)$ be a FTTS.*
If $[G \subseteq H] = 1$, then $\models \text{int}_1(cl_2(\text{int}_3(G))) \subseteq \text{int}_1(cl_2(\text{int}_3(H)))$.

Proof. If $[G \subseteq H] = 1$, then $\text{int}_3(G) \subseteq \text{int}_3(H) \Longrightarrow cl_2(\text{int}_3(G)) \subseteq cl_2(\text{int}_3(H))$ then $\text{int}_1(cl_2(\text{int}_3(G))) \subseteq \text{int}_1(cl_2(\text{int}_3(H)))$. □

Lemma 3. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS and $G \subseteq X$. Then*

(i) $\models X \sim \text{int}_1(cl_2(\text{int}_3(G))) \equiv cl_1(\text{int}_2(cl_3(X \sim G)))$;

(ii) $\models X \sim cl_1(\text{int}_2(cl_3(G))) \equiv \text{int}_1(cl_2(\text{int}_3(X \sim G)))$.

Proof. From Theorem 2.2-(5) in [10], we have

(i) $X \sim \text{int}_1(cl_2(\text{int}_3(G)))(x) = cl_1(X \sim cl_2(\text{int}_3(G)))(x) = cl_1(\text{int}_2(X \sim \text{int}_3(G)))(x) = cl_1(\text{int}_2(cl_3(X \sim G)))(x)$.

(ii) $X \sim cl_1(\text{int}_2(cl_3(G)))(x) = \text{int}_1(X \sim \text{int}_2(cl_3(G)))(x) = \text{int}_1(cl_2(X \sim cl_3(G)))(x) = \text{int}_1(cl_2(\text{int}_3(X \sim G)))(x)$.

□

Theorem 4. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then*

(i) $\alpha\tau_{(1,2,3)}(X) = 1$, $\alpha\tau_{(1,2,3)}(\emptyset) = 1$;

(ii) *for any* $\{G_\lambda : \lambda \in \Lambda\}$, $\alpha\tau_{(1,2,3)}(\bigcup_{\lambda \in \Lambda} G_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \alpha\tau_{(1,2,3)}(G_\lambda)$;

(iii) $\alpha\mathscr{F}_{(1,2,3)}(X) = 1$, $\alpha\mathscr{F}_{(1,2,3)}(\emptyset) = 1$;

(iv) *for any* $\{G_\lambda : \lambda \in \Lambda\}$, $\alpha\mathscr{F}_{(1,2,3)}(\bigcap_{\lambda \in \Lambda} G_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \alpha\mathscr{F}_{(1,2,3)}(G_\lambda)$.

Proof.

(i) $\alpha\tau_{(1,2,3)}(X) = \inf_{x \in X}(\text{int}_1(cl_2(\text{int}_3(X))))(x) = \inf_{x \in X}(\text{int}_1(cl_2(X)))(x) = \inf_{x \in X}(\text{int}_1(X))(x) = \inf_{x \in X}(X)(x) = 1$.
$\alpha\tau_{(1,2,3)}(\emptyset) = \inf_{x \in \emptyset}(\text{int}_1(cl_2(\text{int}_3(\emptyset))))(x) = \inf_{x \in \emptyset}(\text{int}_1(cl_2(\emptyset)))(x) = \inf_{x \in \emptyset}(\text{int}_1(\emptyset))(x) = \inf_{x \in \emptyset}(\emptyset)(x) = 1$.

(ii) From Lemma 2 , we have $[G_\lambda \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda] = 1$, then $\models \text{int}_1(cl_2(\text{int}_3(G_\lambda))) \subseteq \text{int}_1(cl_2(\text{int}_3(\bigcup_{\lambda \in \Lambda} G_\lambda)))$,
$\alpha\tau_{(1,2,3)}(\bigcup_{\lambda \in \Lambda} G_\lambda) = \inf_{x \in \bigcup_{\lambda \in \Lambda} G_\lambda} \text{int}_1(cl_2(\text{int}_3(\bigcup_{\lambda \in \Lambda} G_\lambda)))(x) = \inf_{\lambda \in \Lambda} \inf_{x \in G_\lambda} \text{int}_1(cl_2(\text{int}_3(\bigcup_{\lambda \in \Lambda} G_\lambda)))(x) \geq \inf_{\lambda \in \Lambda} \inf_{x \in G_\lambda} \text{int}_1(cl_2(\text{int}_3(G_\lambda)))(x) = \bigwedge_{\lambda \in \Lambda} \alpha\tau_{(1,2,3)}(G_\lambda)(x)$.

(iii) and (iv) are obvious.

□

Lemma 5. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then $\models \tau_1 \equiv \tau_3 \longrightarrow \tau_1 \subseteq \alpha\tau_{(1,2,3)}$.*

Proof. From Theorem (2.2)-(3) in [10] and Lemma (2.1) in [15] , we have $[(G \in \tau_1) \wedge (G \in \tau_3)] = [(G \equiv \text{int}_1(G)) \wedge (G \equiv \text{int}_3(G))] \leq [G \equiv \text{int}_3(G)] = [(G \subseteq \text{int}_3(G)) \wedge (\text{int}_3(G) \subseteq G)] \leq [G \subseteq \text{int}_3(G)] \leq [cl_2(G) \subseteq cl_2(\text{int}_3(G))] \leq [G \subseteq cl_2(\text{int}_3(G))] \leq [\text{int}_1(G) \subseteq \text{int}_1(cl_2(\text{int}_3(G)))] = [G \subseteq \text{int}_1(cl_2(\text{int}_3(G)))] = [G \in \alpha\tau_{(1,2,3)}]$. □

Theorem 6. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS and $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ are the families of closed sets with respect to τ_1, τ_2, and τ_3, respectively, then*

(i) $\models E \in \tau_1 \wedge E \in \mathscr{F}_2 \wedge E \in \tau_3 \longrightarrow E \equiv \text{int}_1(cl_2(\text{int}_3(E)))$;

(ii) $\models E \in \mathscr{F}_1 \wedge E \in \tau_2 \wedge E \in \mathscr{F}_3 \longrightarrow E \equiv cl_1(\text{int}_2(cl_3(E)))$.

Proof.

(i) From Theorem (2.2)-(3) in [10] and Theorem (5.2)-(3) in [9], we have
$[(E \in \tau_1 \wedge E \in \mathscr{F}_2 \wedge E \in \tau_3)] = [(E \equiv \text{int}_1(E)) \wedge (E \equiv cl_2(E)) \wedge (E \equiv \text{int}_3(E))] = [((E \subseteq \text{int}_1(E)) \wedge (nt_1(E) \subseteq E)) \wedge ((E \subseteq cl_2(E)) \wedge (cl_2(E) \subseteq E)) \wedge ((E \subseteq \text{int}_3(E)) \wedge (\text{int}_3(E) \subseteq E))] = [(E \subseteq \text{int}_1(E)) \wedge (E \subseteq cl_2(E)) \wedge (E \subseteq \text{int}_3(E)) \wedge (\text{int}_1(E) \subseteq E) \wedge (cl_2(E) \subseteq E) \wedge (\text{int}_3(E) \subseteq E)] \leq [(E \subseteq \text{int}_1(E)) \wedge (E \subseteq cl_2(E)) \wedge (cl_2(E) \subseteq cl_2(\text{int}_3(E))) \wedge (\text{int}_1(E) \subseteq E) \wedge (cl_2(E) \subseteq E) \wedge (cl_2(\text{int}_3(E)) \subseteq cl_2(E))] \leq [(E \subseteq \text{int}_1(E)) \wedge (E \subseteq cl_2(\text{int}_3(E))) \wedge (\text{int}_1(E) \subseteq E) \wedge (cl_2(\text{int}_3(E)) \subseteq E))] \leq [(E \subseteq \text{int}_1(E)) \wedge (\text{int}_1(E) \subseteq \text{int}_1(cl_2(\text{int}_3(E)))) \wedge (\text{int}_1(E) \subseteq E) \wedge (\text{int}_1(cl_2(\text{int}_3(E))) \subseteq \text{int}_1(E))] \leq [E \subseteq \text{int}_1(cl_2(\text{int}_3(E)))] \wedge [\text{int}_1(cl_2(\text{int}_3(E))) \subseteq E] = [E \equiv \text{int}_1(cl_2(\text{int}_3(E)))]$.

(ii) It follows directly from (i).

□

Remark 7. The following example shows that
(i) $\alpha\tau_1 \subseteq \alpha\tau_{(1,2,3)}$;
(ii) $\alpha\tau_2 \subseteq \alpha\tau_{(1,2,3)}$;
(iii) $\alpha\tau_3 \subseteq \alpha\tau_{(1,2,3)}$;
(iv) $\alpha\tau_{(1,2,3)} = \alpha\tau_{(3,2,1)}$.
It may not be true for all FTTS $(X, \tau_1, \tau_2, \tau_3)$.

Example 8. For $X = \{a, b, c\}$ and $B = \{a, b\}$. Let τ_1, τ_2, τ_3 be a fuzzifying tritopological space X defined by

$$\tau_1(A) = \begin{cases} 1 & if\ A \in \{\emptyset, X, \{a\}\}, \\ \dfrac{3}{4} & if\ A \in \{\{c\}, \{a, c\}\}, \\ 0 & Otherwise. \end{cases}$$

$$\tau_2(A) = \begin{cases} 1 & if\ A \in \{\emptyset, X\}, \\ \dfrac{1}{4} & if\ A = \{c\}, \\ 0 & Otherwise. \end{cases} \quad (1)$$

$$\tau_3(A) = \begin{cases} 1 & if\ A \in \{\emptyset, X, \{b\}, \{a, c\}\}, \\ \dfrac{3}{4} & if\ A \in \{\{c\}, \{b, c\}\}, \\ 0 & if\ A \in \{\{a\}, \{a, b\}\}. \end{cases}$$

Now, we have $\text{int}_1(B)(a) = 1$, $\text{int}_1(B)(b) = 0$, $\text{int}_1(B)(c) = 0$, $cl_1(\text{int}_1(B))(a) = 1$, $cl_1(\text{int}_1(B))(b) = 1$, $cl_1(\text{int}_1(B))(c) = 1/4$, $\text{int}_1(cl_1(\text{int}_1(B)))(a) = 1$, $\text{int}_1(cl_1(\text{int}_1(B)))(b) = 1/4$, $\text{int}_1(cl_1(\text{int}_1(B)))(c) = 1/4$,

$\Longrightarrow \alpha\tau_1(B) = \inf_{x\in B}(\text{int}_1(cl_1(\text{int}_1(B))))(x) = 1/4.$

and $\text{int}_2(B)(a) = 0$, $\text{int}_2(B)(b) = 0$, $\text{int}_2(B)(c) = 0$, $cl_2(\text{int}_2(B))(a) = 0$, $cl_2(\text{int}_2(B))(b) = 0$, $cl_2(\text{int}_2(B))(c) = 0$,

$\text{int}_2(cl_2(\text{int}_2(B)))(a) = 0$, $\text{int}_2(cl_2(\text{int}_2(B)))(b) = 0$, $\text{int}_2(cl_2(\text{int}_2(B)))(c) = 0$,

$\Longrightarrow \alpha\tau_2(B) = \inf_{x\in B}(\text{int}_2(cl_2(\text{int}_2(B))))(x) = 0.$

and $\text{int}_3(B)(a) = 0$, $\text{int}_3(B)(b) = 1$, $\text{int}_3(B)(c) = 0$, $cl_3(\text{int}_3(B))(a) = 0$, $cl_3(\text{int}_3(B))(b) = 1$, $cl_3(\text{int}_3(B))(c) = 0$,

$\text{int}_3(cl_3(\text{int}_3(B)))(a) = 0$, $\text{int}_3(cl_2(\text{int}_1(B)))(b) = 1$, $\text{int}_3(cl_3(\text{int}_3(B)))(c) = 0$,

$\Longrightarrow \alpha\tau_3(B) = \inf_{x\in B}(\text{int}_3(cl_3(\text{int}_3(B))))(x) = 0.$

and $\text{int}_3(B)(a) = 0$, $\text{int}_3(B)(b) = 1$, $\text{int}_3(B)(c) = 0$, $cl_2(\text{int}_3(B))(a) = 1$, $cl_2(\text{int}_3(B))(b) = 1$, $cl_2(\text{int}_3(B))(c) = 3/4$,

$\text{int}_1(cl_2(\text{int}_3(B)))(a) = 0$, $\text{int}_1(cl_2(\text{int}_3(B)))(b) = 3/4$, $\text{int}_1(cl_2(\text{int}_3(B)))(c) = 3/4$,

$\Longrightarrow \alpha\tau_{(1,2,3)}(B) = \inf_{x\in B}(\text{int}_1(cl_2(\text{int}_3(B))))(x) = 3/4.$

and $\text{int}_1(B)(a) = 1$, $\text{int}_1(B)(b) = 0$, $\text{int}_1(B)(c) = 0$, $cl_2(\text{int}_1(B))(a) = 1$, $cl_2(\text{int}_1(B))(b) = 1$, $cl_2(\text{int}_1(B))(c) = 3/4$,

$\text{int}_3(cl_2(\text{int}_1(B)))(a) = 3/4$, $\text{int}_3(cl_2(\text{int}_1(B)))(b) = 1$, $\text{int}_3(cl_2(\text{int}_1(B)))(c) = 3/4$,

$\Longrightarrow \alpha\tau_{(3,2,1)}(B) = \inf_{x\in B}(\text{int}_3(cl_2(\text{int}_1(B))))(x) = 3/4.$

$\therefore\ \alpha\tau_{(1,2,3)}(B) = \alpha\tau_{(3,2,1)}(B)$. Therefore $\alpha\tau_{(1,2,3)} = \alpha\tau_{(3,2,1)}$, $\alpha\tau_1 \subseteq \alpha\tau_{(1,2,3)}$, $\alpha\tau_2 \subseteq \alpha\tau_{(1,2,3)}$, and $\alpha\tau_3 \subseteq \alpha\tau_{(1,2,3)}$.

Lemma 9. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then* $\models \tau_1 \equiv \tau_3 \longrightarrow \alpha\tau_{(1,2,3)} \equiv \alpha\tau_{(3,2,1)}$.

Proof. From Theorem (2.2)-(3) in [10], we have $[(G \in \tau_1) \wedge (G \in \tau_3)] = [(G \equiv \text{int}_1(G)) \wedge (G \equiv \text{int}_3(G))] = [(G \subseteq \text{int}_1(G)) \wedge (\text{int}_1(G) \subseteq G) \wedge (G \subseteq \text{int}_3(G)) \wedge (\text{int}_3(G) \subseteq G)] \leq [(G \subseteq \text{int}_3(G)) \wedge (G \subseteq \text{int}_1(G))] \leq [(cl_2(G) \subseteq cl_2(\text{int}_3(G))) \wedge (cl_2(G) \subseteq cl_2(\text{int}_1(G)))] \leq [(G \subseteq cl_2(\text{int}_3(G))) \wedge (G \subseteq cl_2(\text{int}_1(G)))] \leq [(\text{int}_1(G) \subseteq \text{int}_1(cl_2(\text{int}_3(G)))) \wedge (\text{int}_3(G) \subseteq \text{int}_3(cl_2(\text{int}_1(G))))] = [(G \subseteq \text{int}_1(cl_2(\text{int}_3(G)))) \wedge (G \subseteq \text{int}_3(cl_2(\text{int}_1(G))))] = [(G \in \alpha\tau_{(1,2,3)}) \wedge (G \in \alpha\tau_{(3,2,1)})]$.

Therefore $\alpha\tau_{(1,2,3)} \equiv \alpha\tau_{(3,2,1)}$. □

Theorem 10. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then* $\models G \in \alpha\mathscr{F}_{(1,2,3)} \longleftrightarrow \forall x\ (x \in cl_1(\text{int}_2(cl_3(G))) \longrightarrow x \in G)$.

Proof. From Lemma 3 -(ii), we have $[\forall x\ (x \in cl_1(\text{int}_2(cl_3(G))) \longrightarrow x \in G))] = [\forall x\ (x \in X \sim G \longrightarrow x \in X \sim cl_1(\text{int}_2(cl_3(G))))] = \inf_{x\in X\sim G}(X \sim cl_1(\text{int}_2(cl_3(G))))(x) = \inf_{x\in X\sim G}(\text{int}_1(cl_2(\text{int}_3(X \sim G))))(x) = [X \sim G \in \alpha\tau_{(1,2,3)}] = [G \in \alpha\mathscr{F}_{(1,2,3)}]$. □

Lemma 11. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then*

(i) $\models H \doteq cl_2(\text{int}_3(G)) \wedge G \in \alpha\tau_{(1,2,3)} \longrightarrow G \subseteq \text{int}_1(H)$;

(ii) $\models H \doteq \text{int}_2(cl_3(G)) \wedge G \in \alpha\mathscr{F}_{(1,2,3)} \longrightarrow cl_1(H) \subseteq G$.

Proof.

(i) $[(H \doteq cl_2(\text{int}_3(G))) \wedge (G \in \alpha\tau_{(1,2,3)})] = [(H \subseteq cl_2(\text{int}_3(G)) \wedge cl_2(\text{int}_3(G)) \subseteq H) \wedge (G \subseteq \text{int}_1(cl_2(\text{int}_3(G))))] \leq [(cl_2(\text{int}_3(G)) \subseteq H) \wedge (G \subseteq \text{int}_1(cl_2(\text{int}_3(G))))] \leq [(\text{int}_1(cl_2(\text{int}_3(G))) \subseteq \text{int}_1(H)) \wedge (G \subseteq \text{int}_1(cl_2(\text{int}_3(G))))] \leq [G \subseteq \text{int}_1(H)]$.

(ii) From Theorem 2.2-(5) in [10], we have

$[(H \doteq \text{int}_2(cl_3(G))) \wedge (G \in \alpha\mathscr{F}_{(1,2,3)})] = [(H \subseteq \text{int}_2(cl_3(G)) \wedge \text{int}_2(cl_3(G)) \subseteq H) \wedge (X \sim G \in \alpha\tau_{(1,2,3)})] \leq [(H \subseteq \text{int}_2(cl_3(G)) \wedge (X \sim G \subseteq \text{int}_1(cl_2(\text{int}_3(X \sim G))))] \leq [(cl_1(H) \subseteq cl_1(\text{int}_2(cl_3(G))) \wedge (X \sim G \subseteq X \sim cl_1(\text{int}_2(cl_3(G))))] \leq [(cl_1(H) \subseteq cl_1(\text{int}_2(cl_3(G))) \wedge (cl_1(\text{int}_2(cl_3(G))) \subseteq G)] \leq [cl_1(H) \subseteq G]$.

□

Theorem 12. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then*

(i) $\models \exists H\ (H \in \tau_3 \ \wedge\ H \subseteq G \subseteq \text{int}_1(cl_2(H))) \longrightarrow G \in \alpha\tau_{(1,2,3)}$;

(ii) $\models \exists K\ (K \in \mathscr{F}_3 \ \wedge\ cl_1(\text{int}_2(K)) \subseteq G \subseteq K) \longrightarrow G \in \alpha\mathscr{F}_{(1,2,3)}$.

Proof.

(i) From Theorem (2.2)-(3) in [10], we have

$[\exists H\ (H \in \tau_3 \bigwedge H \subseteq G \subseteq \text{int}_1(cl_2(H)))] = \sup_{H\in P(X)}([H \equiv \text{int}_3(H)] \bigwedge [H \subseteq G] \bigwedge [G \subseteq \text{int}_1(cl_2(H))]) \leq \sup_{H\in P(X)}([(H \subseteq \text{int}_3(H))$

$\wedge(\text{int}_3(H) \subseteq H)] \bigwedge [\text{int}_3(H) \subseteq \text{int}_3(G)] \bigwedge [G \subseteq \text{int}_1(cl_2(H))]) \leq \sup_{H\in P(X)}([H \subseteq \text{int}_3(H)] \bigwedge [\text{int}_3(H) \subseteq \text{int}_3(G)] \bigwedge [G \subseteq \text{int}_1(cl_2(H))]) \leq \sup_{H\subseteq G}([H \subseteq \text{int}_3(G)] \bigwedge [G \subseteq \text{int}_1(cl_2(H))]) \leq \sup_{H\subseteq G}([H \subseteq \text{int}_3(G)] \bigwedge [G \subseteq \text{int}_1(cl_2(H))]) \leq \sup_{H\subseteq G}([cl_2(H) \subseteq cl_2(\text{int}_3(G))] \bigwedge [G \subseteq \text{int}_1(cl_2(H))]) \leq \sup_{H\subseteq G}([\text{int}_1(cl_2(H)) \subseteq \text{int}_1(cl_2(\text{int}_3(G)))] \bigwedge [G \subseteq \text{int}_1(cl_2(H))]) \leq \sup_{H\subseteq G}[G \subseteq \text{int}_1(cl_2(\text{int}_3(G)))] = [G \in \alpha\tau_{(1,2,3)}]$.

(ii) From (i) above and Theorem (2.2)-(5) in [10]. We have $[G \in \alpha\mathcal{F}_{(1,2,3)}] = [X \sim G \in \alpha\tau_{(1,2,3)}] \geq [\exists H\ (H \in \tau_3 \bigwedge H \subseteq X \sim G \subseteq \text{int}_1(cl_2(H)))] = [\exists H\ (H \in \tau_3 \bigwedge X \sim \text{int}_1(cl_2(H)) \subseteq G \subseteq X \sim H)] = [\exists X \sim H\ (X \sim H \in \mathcal{F}_3 \bigwedge cl_1(\text{int}_2(X \sim H)) \subseteq G \subseteq X \sim H)] = [\exists K\ (K \in \mathcal{F}_3 \bigwedge cl_1(\text{int}_2(K)) \subseteq G \subseteq K)]$.

□

3. $(1,2,3)$-α–Neighborhood System in Fuzzifying Tritopological Spaces

Definition 13. If (X,τ_1,τ_2,τ_3) is a FTTS and $x \in X$. Then $\alpha N_x^{(1,2,3)} \in \mathfrak{F}(P(X))$ indicates the "(1,2,3)-α-neighborhood system of x" and defined as $G \in \alpha N_x^{(1,2,3)} := \exists H\ (H \in \alpha\tau_{(1,2,3)} \bigwedge x \in H \subseteq G)$, i.e., $\alpha N_x^{(1,2,3)}(G) = \sup_{x\in H\subseteq G}\alpha\tau_{(1,2,3)}(H)$.

Theorem 14. *If (X,τ_1,τ_2,τ_3) is a FTTS, then $\models G \in \alpha\tau_{(1,2,3)} \longleftrightarrow \forall x\ (x \in G \longrightarrow \exists H\ (H \subseteq \text{int}_1(cl_2(\text{int}_3(G))) \wedge x \in H \subseteq G))$.*

Proof. $[\forall x\ (x \in G \longrightarrow \exists H\ (H \subseteq \text{int}_1(cl_2(\text{int}_3(G))) \wedge x \in H \subseteq G))] = \inf_{x\in G}\sup_{x\in H\subseteq G}\alpha\tau_{(1,2,3)}(H) = \inf_{x\in G}\alpha N_x^{(1,2,3)}(G) = \alpha\tau_{(1,2,3)}(G)$. □

Theorem 15. *If (X,τ_1,τ_2,τ_3) is a FTTS and $G \in P(X)$, then*

(i) $\models G \in \alpha\tau_{(1,2,3)} \longleftrightarrow \forall x\ (x \in G \longrightarrow \exists H\ (H \in \alpha N_x^{(1,2,3)} \bigwedge H \subseteq G))$,

(ii) $\models \tau_1 \equiv \tau_3 \longrightarrow N_x^{(1)}(G) \leq \alpha N_x^{(1,2,3)}(G)$.

Proof.

(i) From Theorem 14 we get

$[\forall x\ (x \in G \longrightarrow \exists H\ (H \in \alpha N_x^{(1,2,3)} \bigwedge H \subseteq G))] = \inf_{x\in G}\sup_{H\subseteq G}\alpha N_x^{(1,2,3)}(H) = \inf_{x\in G}\sup_{H\subseteq G}\sup_{x\in K\subseteq H}\alpha\tau_{(1,2,3)}(K) = \inf_{x\in G}\sup_{x\in K\subseteq G}\alpha\tau_{(1,2,3)}(K) = \alpha\tau_{(1,2,3)}(G)$.

(ii) From Lemma 5 we get

$\alpha N_x^{(1,2,3)}(G) = \sup_{x\in H\subseteq G}\alpha\tau_{(1,2,3)}(H) \geq \sup_{x\in H\subseteq G}\tau_1(H) = N_x^{(1)}(G)$.

□

Theorem 16. *If (X,τ_1,τ_2,τ_3) is a FTTS, the mapping $\alpha N^{(1,2,3)} : X \longrightarrow \mathfrak{F}^N(P(X))$, $x \longmapsto \alpha N_x^{(1,2)}$, where $\mathfrak{F}^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$, has the following properties:*

(i) $\models G \in \alpha N_x^{(1,2,3)} \longrightarrow x \in G$,

(ii) $\models G \subseteq H \longrightarrow (G \in \alpha N_x^{(1,2,3)} \longrightarrow H \in \alpha N_x^{(1,2,3)})$,

(iii) $\models G \in \alpha N_x^{(1,2,3)} \longrightarrow \exists K\ (K \in \alpha N_x^{(1,2,3)} \bigwedge K \subseteq G \bigwedge \forall y\ (y \in K \longrightarrow K \in \alpha N_x^{(1,2,3)})$.

Proof.

(i) If $[G \in \alpha N_x^{(1,2,3)}] = 0$, then (i) is obtained. If $[G \in \alpha N_x^{(1,2,3)}] = \sup_{x\in H\subseteq G}\alpha\tau_{(1,2,3)}(H) > 0$, then $\exists H_0$ such that $x \in H_0 \subseteq G$. Now we have $[x \in G] = 1$.

Therefore $[G \in \alpha N_x^{(1,2,3)}] \leq [x \in G]$.

(ii) $[G \in \alpha N_x^{(1,2,3)}] = \sup_{x\in E\subseteq G}\alpha\tau_{(1,2,3)}(E) \leq \sup_{x\in E\subseteq H}\alpha\tau_{(1,2,3)}(E) = [H \in \alpha N_x^{(1,2,3)}]$.

(iii) $[\exists K\ (K \in \alpha N_x^{(1,2,3)} \wedge K \subseteq G \bigwedge \forall y\ (y \in K \longrightarrow K \in \alpha N_y^{(1,2,3)})] = \sup_{K\subseteq G}(\alpha N_x^{(1,2,3)} \bigwedge \inf_{y\in K}\alpha N_y^{(1,2,3)}(K)) = \sup_{K\subseteq G}(\alpha N_x^{(1,2,3)} \bigwedge \alpha\tau_{(1,2,3)}(K) = \sup_{K\subseteq G}\alpha\tau_{(1,2,3)}(K) \geq \sup_{x\in K\subseteq G}\alpha\tau_{(1,2,3)}(K) = \alpha N_x^{(1,2,3)}(G) = [G \in \alpha N_x^{(1,2,3)}]$.

□

4. $(1,2,3)$-α-Derived Set and $(1,2,3)$-α-Closure Operator in Fuzzifying Tritopological Space

Definition 17. If (X,τ_1,τ_2,τ_3) is a FTTS, then $\alpha d_{(1,2,3)}(G)$ indicates the "(1,2,3)-α-derived set of G" and defined as follows: $x \in \alpha d_{(1,2,3)}(G) := \forall H\ (H \in \alpha N_x^{(1,2,3)} \longrightarrow H \cap (G \sim \{x\}) \neq \emptyset)$, i.e., $\alpha d_{(1,2,3)}(G)(x) = \inf_{H\cap(G\sim\{x\})=\emptyset}(1 - \alpha N_x^{(1,2,3)}(H))$.

Lemma 18. $\alpha d_{(1,2,3)}(G)(x) = 1 - \alpha N_x^{(1,2,3)}((X \sim G) \cup \{x\})$.

Proof. $\alpha d_{(1,2,3)}(G)(x) = 1 - \sup_{H\cap(G\sim\{x\})=\emptyset}\alpha N_x^{(1,2,3)}(H) = 1 - \sup_{H\cap(G\sim\{x\})=\emptyset}\sup_{x\in K\subseteq H}\alpha\tau_{(1,2,3)}(K) = 1 - \sup_{x\in K\subseteq(X\sim G)\cup\{x\}}\sup_{x\in K\subseteq H}\alpha\tau_{(1,2,3)}(K) = 1 - \sup_{x\in K\subseteq(X\sim G)\cup\{x\}}\alpha\tau_{(1,2,3)}(K) = 1 - \alpha N_x^{(1,2,3)}((X \sim G) \cup \{x\})$. □

Theorem 19. *If (X,τ_1,τ_2,τ_3) is a FTTS and $G, H \in P(X)$, then*

(i) $\models \alpha d_{(1,2,3)}(\emptyset) = 0$;

(ii) $\models G \subseteq H \longrightarrow \alpha d_{(1,2,3)}(G) \subseteq \alpha d_{(1,2,3)}(H)$;

(iii) $\models G \in \alpha\mathcal{F}_{(1,2,3)} \longleftrightarrow \alpha d_{(1,2,3)}(G) \subseteq G$;

(iv) $\models \tau_1 \equiv \tau_3 \longrightarrow \alpha d_{(1,2,3)}(G) \subseteq d_1(G)$, *where $d_1(G)$ is the fuzzifying derived set of G with respect to τ_1.*

Proof.

(i) From Lemma 18 we have

$\alpha d_{(1,2,3)}(\emptyset)(x) = 1 - \alpha N_x^{(1,2,3)}((X \sim \emptyset) \cup \{x\}) = 1 - \alpha N_x^{(1,2,3)}(X) = 1 - 1 = 0.$

(ii) Let $G \subseteq H$, then from Lemma 18 and Theorem 16 -(ii) we get

$\alpha d_{(1,2,3)}(G)(x) = 1 - \alpha N_x^{(1,2,3)}((X \sim G) \cup \{x\}) \leq 1 - \alpha N_x^{(1,2,3)}((X \sim H) \cup \{x\}) = \alpha d_{(1,2,3)}(H)(x)$

(iii) From Lemma 18 and Theorem 15 -(ii).We have

$[\alpha d_{(1,2,3)}(G) \subseteq G] = \inf_{x \in X \sim G}(1 - \alpha d_{(1,2,3)}(G)(x)) = \inf_{x \in X \sim G} \alpha N_x^{(1,2,3)}((X \sim G) \cup \{x\}) = \inf_{x \in X \sim G} \alpha N_x^{(1,2,3)}(X \sim G) = \inf_{x \in X \sim G} \sup_{x \in H \subseteq X \sim G} \alpha \tau_{(1,2,3)}(H) = \alpha \tau_{(1,2,3)}(X \sim G) = \alpha \mathscr{F}_{(1,2,3)}(G) = [G \in \alpha \mathscr{F}_{(1,2,3)}(x)].$

(iv) From Theorem 15 -(ii) and Lemma (5.1) in [9] we have

$\alpha d_{(1,2,3)}(G) = 1 - \alpha N_x^{(1,2,3)}((X \sim G) \cup \{x\}) \leq 1 - \alpha N_x^{(1)}((X \sim G) \cup \{x\}) = d_1(G)(x).$

□

Definition 20. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then $\alpha cl_{(1,2,3)}(G)$ indicates the "(1,2,3)-α-closure set of G" and defined as $x \in \alpha cl_{(1,2,3)}(G) := \forall H\ (H \supseteq G) \wedge (H \in \alpha \mathscr{F}_{(1,2,3)}) \longrightarrow x \in H)$, i.e., $\alpha cl_{(1,2,3)}(G)(x) = \inf_{x \notin H \supseteq G}(1 - \alpha \mathscr{F}_{(1,2,3)}(H))$.

Theorem 21. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, $G, H \in P(X)$ and $x \in X$, then*

(i) $\alpha cl_{(1,2,3)}(G)(x) = 1 - \alpha N_x^{(1,2,3)}(X \sim G)$;

(ii) $\models \alpha cl_{(1,2,3)}(\emptyset) = 0$;

(iii) $\models G \subseteq \alpha cl_{(1,2,3)}(G)$;

(iv) $\models \alpha cl_{(1,2,3)}(G) = \alpha d_{(1,2,3)}(G) \cup G$;

(v) $\models x \in \alpha cl_{(1,2,3)}(G) \longleftrightarrow \forall H\ (H \in \alpha N_x^{(1,2,3)} \longrightarrow G \cap H \neq \emptyset)$;

(vi) $\models G \equiv \alpha cl_{(1,2,3)}(G) \longleftrightarrow G \in \alpha \mathscr{F}_{(1,2,3)}(G)$;

(vii) $\models G \subseteq H \longrightarrow \alpha cl_{(1,2,3)}(G) \subseteq \alpha cl_{(1,2,3)}(H)$;

(viii) $\models H \doteq \alpha cl_{(1,2,3)}(G) \longrightarrow H \in \alpha \mathscr{F}_{(1,2,3)}$.

Proof.

(i) $\alpha cl_{(1,2,3)}(G)(x) = \inf_{x \notin H \supseteq G}(1 - \alpha \mathscr{F}_{(1,2,3)}(H) = \inf_{x \notin H \supseteq G}(1 - \alpha \tau_{(1,2,3)}(X \sim H)) = 1 - \sup_{x \in X \sim H \subseteq X \sim G} \alpha \tau_{(1,2,3)}(X \sim H) = 1 - N_x^{(1,2,3)}(X \sim G).$

(ii) $\alpha cl_{(1,2,3)}(\emptyset)(x) = 1 - \alpha N_x^{(1,2,3)}(X \sim \emptyset) = 1 - \alpha N_x^{(1,2,3)}(X) = 1 - \sup_{x \in G \subseteq X} \alpha \tau_{(1,2,3)}(G) = 1 - 1 = 0.$

(iii) If $G \in P(X)$ and for any $x \in X$ and if $x \notin G$, then $[x \in G] \leq [x \in \alpha cl_{(1,2,3)}(G)]$. If $x \in G$, then $\alpha cl_{(1,2,3)}(G)(x) = 1 - \alpha N_x^{(1,2,3)}(X \sim G) = 1 - 0 = 1$. Thus $[x \in G] \leq [x \in \alpha cl_{(1,2,3)}(G)] \Longrightarrow [G \subseteq \alpha cl_{(1,2,3)}(G)] = 1$.

(iv) From Lemma 18 and (iii) above, for any $x \in X$ we have

$[x \in (\alpha d_{(1,2,3)}(G) \cup G)] = \max((1 - \alpha N_x^{(1,2,3)}(X \sim G) \cup \{x\}), G(x))$. If $x \in G$, then $[x \in (\alpha d_{(1,2,3)}(G) \cup G)] = G(x) = 1 = [x \in \alpha cl_{(1,2,3)}(G)]$. If $x \notin G$, then $[x \in (\alpha d_{(1,2,3)}(G) \cup G)] = 1 - \alpha N_x^{(1,2,3)}(X \sim G) = [x \in \alpha cl_{(1,2,3)}(G)]$.

Thus $[\alpha cl_{(1,2,3)}(G)] = [\alpha d_{(1,2,3)}(G) \cup G]$.

(v) $[\forall H\ (H \in \alpha N_x^{(1,2,3)} \longrightarrow G \cap H \neq \emptyset)] = \inf_{H \subseteq X \sim G}(1 - \alpha N_x^{(1,2,3)}(H)) = 1 - \alpha N_x^{(1,2,3)}(X \sim G) = [x \in \alpha cl_{(1,2,3)}(G)]$.

(vi) From Theorem 19 -(iii), Lemma (8.2) in [15] and (iv) above, since

$[G \subseteq \alpha d_{(1,2,3)}(G) \cup G] = 1$, we get

$\alpha \mathscr{F}_{(1,2,3)}(G) = [\alpha d_{(1,2,3)}(G) \subseteq G] = [\alpha d_{(1,2,3)}(G) \cup G \subseteq G] = [\alpha d_{(1,2,3)}(G) \cup G \subseteq G] \bigwedge [G \subseteq \alpha d_{(1,2,3)}(G) \cup G] = [\alpha d_{(1,2,3)}(G) \cup G \equiv G] = [G \equiv \alpha cl_{(1,2,3)}(G)]$.

(vii) If $G \subseteq H$, then $X \sim H \subseteq X \sim G$. From (i) above and Theorem 16 -(ii) we get

$\alpha cl_{(1,2,3)}(G)(x) = 1 - \alpha N_x^{(1,2,3)}(X \sim H) \leq 1 - \alpha N_x^{(1,2,3)}(X \sim H) = \alpha cl_{(1,2,3)}(H)(x).$

Thus $\alpha cl_{(1,2,3)}(G) \subseteq \alpha cl_{(1,2,3)}(H)$.

(viii) If $[G \subseteq H] = 0$, then $[H \doteq \alpha cl_{(1,2,3)}(G)] = 0$. Assume that

$[G \subseteq H] = 1$, then $[H \subseteq \alpha cl_{(1,2,3)}(G)] = 1 - \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G)$ and

$[\alpha cl_{(1,2,3)}(G) \subseteq H] = \inf_{x \in X \sim H} \alpha N_x^{(1,2,3)}(X \sim G)$. Therefore $[H \doteq \alpha cl_{(1,2,3)}(G)] = \max(0, \inf_{x \in X \sim H} \alpha N_x^{(1,2,3)}(X \sim G) - \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G))$.

If $[H \doteq \alpha cl_{(1,2,3)}(G)] > c$, then $\inf_{x \in X \sim H} \alpha N_x^{(1,2,3)}(X \sim G) > c + \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G)$.

For any $x \in X \sim H$, we get $\alpha N_x^{(1,2,3)}(X \sim G) > c + \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G)$. Thus $\sup_{x \in E \subseteq X \sim G} \alpha \tau_{(1,2,3)}(E) > c + \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G)$, i.e., $\exists E_x$ such that $x \in E_x \subseteq X \sim G$ and $\alpha \tau_{(1,2,3)}(E_x) > c + \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G)$. To prove that $E_x \subseteq X \sim H$. If $E_x \nsubseteq X \sim H$, then $\exists x' \in E_x$ and $x' \in X \sim H$. Hence we get $\sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G) \geq \alpha N_{x'}^{(1,2,3)}(X \sim G) \geq \alpha \tau_{(1,2,3)}(E_x) > c + \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim H) \Longrightarrow$ Contradiction. Therefore $\alpha \mathscr{F}_{(1,2,3)}(H) = \alpha \tau_{(1,2,3)}(X \sim H) = \inf_{x \in X \sim H} \alpha N_x^{(1,2,3)}(X \sim H) \geq \inf_{x \in X \sim H} \alpha \tau_{(1,2,3)}(E_x) \geq \alpha \tau_{(1,2,3)}(E_x) > c + \sup_{x \in H \sim G} \alpha N_x^{(1,2,3)}(X \sim G) > c$, since c is arbitrary; thus $[H \doteq \alpha cl_{(1,2,3)}(G)] \leq [H \in \alpha \mathscr{F}_{(1,2,3)}]$.

□

5. $(1,2,3)$-α–Interior, $(1,2,3)$-α-Exterior, and $(1,2,3)$-α-Boundary Operators in Fuzzifying Tritopological Space

Definition 22. If (X,τ_1,τ_2,τ_3) is a FTTS and $G \in P(X)$, then $\alpha\text{int}_{(1,2,3)}(G)$ indicates the "(1,2,3)-α-interior set of G" defined as $\alpha\text{int}_{(1,2,3)}(G)(x) = \alpha N_x^{(1,2,3)}$

Theorem 23. *If (X,τ_1,τ_2,τ_3) is a FTTS, $G,H \in P(X)$ and $x \in X$, then*

(i) $\vDash \alpha\text{int}_{(1,2,3)}(X) \equiv X$;

(ii) $\vDash \alpha\text{int}_{(1,2,3)}(G) \subseteq G$;

(iii) $\vDash \tau_1 \equiv \tau_3 \longrightarrow \text{int}_1(G) \subseteq \alpha\text{int}_{(1,2,3)}(G)$;

(iv) $\vDash H \in \alpha\tau_{(1,2,3)} \bigwedge H \subseteq G \longrightarrow H \subseteq \alpha\text{int}_{(1,2,3)}(G)$;

(v) $\vDash G \equiv \alpha\text{int}_{(1,2,3)}(G) \longleftrightarrow G \in \alpha\tau_{(1,2,3)}$;

(vi) $\vDash G \subseteq H \longrightarrow \alpha\text{int}_{(1,2,3)}(G) \subseteq \alpha\text{int}_{(1,2,3)}(H)$;

(vii) $\vDash \alpha\text{int}_{(1,2,3)}(G) \equiv X \sim \alpha cl_{(1,2,3)}(X \sim G)$;

(viii) $\vDash \alpha\text{int}_{(1,2,3)}(G) \equiv G \cap (X \sim \alpha d_{(1,2,3)}(X \sim G))$;

(ix) $\vDash H \dot{\equiv} \alpha\text{int}_{(1,2,3)}(G) \longrightarrow H \in \alpha\tau_{(1,2,3)}$.

Proof.

(i) $\alpha\text{int}_{(1,2,3)}(X)(x) = \alpha N_x^{(1,2,3)}(X) = 1 \Longrightarrow \alpha\text{int}_{(1,2,3)}(X) \equiv X$

(ii) Let $G \in P(X)$, $x \in X$. If $x \notin G$, then $\alpha\text{int}_{(1,2,3)}(G)(x) = \alpha N_x^{(1,2,3)} = 0 \Longrightarrow \alpha\text{int}_{(1,2,3)}(G) \subseteq G$.

(iii) From Theorem 15 -(ii) we have

$\text{int}_1(G)(x) = N_x^{(1)}(G) \leq \alpha N_x^{(1,2,3)}(G) = \alpha\text{int}_{(1,2,3)}(G)(x)$. Therefore $\text{int}_1(G)(x) \subseteq \alpha\text{int}_{(1,2,3)}(G)$.

(iv) If $H \nsubseteq G$, then the result holds.

If $H \subseteq G$, then

$[H \subseteq \alpha\text{int}_{(1,2,3)}(G)] = \inf_{x\in H}\alpha\text{int}_{(1,2,3)}(G)(x) = \inf_{x\in H}\alpha N_x^{(1,2,3)}(G) \geq \inf_{x\in H}\alpha N_x^{(1,2,3)}(H) = \alpha\tau_{(1,2,3)}(H) = [(H \in \alpha\tau_{(1,2,3)}) \bigwedge (H \subseteq G)]$.

(v) $[G \equiv \alpha\text{int}_{(1,2,3)}(G)] = \min(\inf_{x\in G}\alpha\text{int}_{(1,2,3)}(G)(x), \inf_{x\in X\sim G}(1 - \alpha\text{int}_{(1,2,3)}(G)(x))) = \min(\inf_{x\in G}\alpha N_x^{(1,2,3)}(G), \inf_{x\in X\sim G}(1 - \alpha N_x^{(1,2,3)}(G))) = \inf_{x\in G}\alpha N_x^{(1,2,3)}(G) = \alpha\tau_{(1,2,3)}(G) = [G \in \alpha\tau_{(1,2,3)}]$

(vi) From Definition 22 and Theorem 16 -(ii) the proof follows.

(vii) From Theorem 21 -(i) we have $(X \sim \alpha cl_{(1,2,3)}(X \sim G))(x) = 1 - (1 - \alpha N_x^{(1,2,3)}(G)) = \alpha N_x^{(1,2,3)}(G) = \alpha\text{int}_{(1,2,3)}(G)(x)$. Therefore $\alpha\text{int}_{(1,2,3)}(G) = X \sim \alpha cl_{(1,2,3)}(X \sim G)$.

(viii) From Lemma 18 we get

$[G \cap (X \sim \alpha d_{(1,2,3)}(X \sim G))] = \min(G(x), \alpha N_x^{(1,2,3)}(G \cup \{x\}))$. If $x \notin G$, then

$[G \cap (X \sim \alpha d_{(1,2,3)}(X \sim G))] = 0 = \alpha N_x^{(1,2,3)}(G) = \alpha\text{int}_{(1,2,3)}(G)(x)$. If $x \in G$, then

$[G \cap (X \sim \alpha d_{(1,2,3)}(X \sim G))] = \alpha N_x^{(1,2,3)}(G) = \alpha\text{int}_{(1,2,3)}(G)(x)$. Therefore

$\alpha\text{int}_{(1,2,3)}(G) = G \cap (X \sim \alpha d_{(1,2,3)}(X \sim G))$.

(ix) From Theorem 21 -(ix) and (vii) above we get

$[H \dot{\equiv} \alpha\text{int}_{(1,2,3)}(G)] = [X \sim H \dot{\equiv} \alpha cl_{(1,2,3)}(X \sim G)] \leq [X \sim H \in \alpha\mathscr{F}_{(1,2,3)}] = [H \in \alpha\tau_{(1,2,3)}]$.

□

Definition 24. If (X,τ_1,τ_2,τ_3) is a FTTS and $G \subseteq X$. Then $\alpha\text{ext}_{(1,2,3)}(G)$ indicates the "(1,2,3)-α-exterior set of G" and defined as $x \in \alpha\text{ext}_{(1,2,3)}(G) \coloneqq x \in \alpha\text{int}_{(1,2,3)}(X \sim G)$, i.e., $\alpha\text{ext}_{(1,2,3)}(G)(x) = \alpha\text{int}_{(1,2,3)}(X \sim G)(x)$.

Theorem 25. *If (X,τ_1,τ_2,τ_3) is a FTTS and $G \subseteq X$. Then*

(i) $\vDash \alpha\text{ext}_{(1,2,3)}(\emptyset) \equiv X$;

(ii) $\vDash \alpha\text{ext}_{(1,2,3)}(G) \subseteq X \sim G$;

(iii) $\vDash \tau_1 \equiv \tau_3 \longrightarrow \alpha\text{ext}_1(G) \subseteq \alpha\text{ext}_{(1,2,3)}(G)$;

(iv) $\vDash G \in \alpha\mathscr{F}_{(1,2,3)} \longleftrightarrow \alpha\text{ext}_{(1,2,3)}(G) \equiv X \sim G$;

(v) $\vDash H \in \alpha\mathscr{F}_{(1,2,3)} \wedge G \subseteq H \longrightarrow X \sim H \subseteq \alpha\text{ext}_{(1,2,3)}(G)$;

(vi) $\vDash H \subseteq G \longrightarrow \alpha\text{ext}_{(1,2,3)}(H) \subseteq \alpha\text{ext}_{(1,2,3)}(G)$;

(vii) $\vDash \alpha\text{ext}_{(1,2,3)}(G) \equiv (X \sim G) \cap (X \sim \alpha d_{(1,2,3)}(G))$;

(viii) $\vDash \alpha\text{ext}_{(1,2,3)}(G) \equiv X \sim \alpha cl_{(1,2,3)}(G))$;

(ix) $\vDash x \in \alpha\text{ext}_{(1,2,3)}(G) \longleftrightarrow \exists H\ (x \in H \in \alpha\tau_{(1,2,3)} \bigwedge H \cap G = \emptyset)$.

Proof. The proofs of (i) - (vii) follow from Theorem 23 .

(ix) $[\exists H\ (x \in H \in \alpha\tau_{(1,2,3)} \bigwedge H \cap G = \emptyset)] = \sup_{x\in H\subseteq(X\sim G)}\alpha\tau_{(1,2,3)}(H) = \alpha N_x^{(1,2,3)}(X \sim G) = \alpha\text{int}_{(1,2,3)}(X \sim G)(x) = \alpha\text{ext}_{(1,2,3)}(G)(x)$. By Definition 24

□

Definition 26. If (X,τ_1,τ_2,τ_3) is a FTTS and $G \subseteq X$, then $\alpha b_{(1,2,3)}(G)$ indicates the "(1,2,3)-α-boundary of a set G" and defined as $x \in \alpha b_{(1,2,3)}(G) \coloneqq (x \notin \alpha\text{int}_{(1,2,3)}(G)) \bigwedge (x \notin \alpha\text{int}_{(1,2,3)}(X \sim G))$, i.e., $x \in \alpha b_{(1,2,3)}(G)(x) \coloneqq \min(1 - \alpha\text{int}_{(1,2,3)}(G)(x)) \bigwedge (1 - \alpha\text{int}_{(1,2,3)}(X \sim G)(x))$.

Lemma 27. *If (X,τ_1,τ_2,τ_3) is a FTTS, $G \in P(X)$ and $x \in X$, then $\vDash x \in \alpha b_{(1,2,3)}(A) \longleftrightarrow \forall H\ (H \in \alpha N_x^{(1,2,3)} \longrightarrow (H \cap G \neq \emptyset) \bigwedge (H \cap (X \sim G)) \neq \emptyset)$.*

Proof. $[\forall H\ (H \in \alpha N_x^{(1,2,3)} \longrightarrow (H \cap G \neq \emptyset) \bigwedge (H \cap (X \sim G)) \neq \emptyset)] = \min(\inf_{H\subseteq G}(1 - \alpha N_x^{(1,2,3)}(H)), \inf_{H\subseteq X\sim G}(1 - \alpha N_x^{(1,2,3)}(H))) = \min(1 - \alpha N_x^{(1,2,3)}(G), 1 - \alpha N_x^{(1,2,3)}(X \sim G)) = \min(1 - \alpha\text{int}_{(1,2,3)}(G)(x), 1 - \alpha\text{int}_{(1,2,3)}(X \sim G)(x)) = [x \in \alpha b_{(1,2,3)}(G)]$. □

Theorem 28. *If (X,τ_1,τ_2,τ_3) is a FTTS and $G \in P(X)$, then*

(i) $\vDash \alpha b_{(1,2,3)}(G) \equiv \alpha cl_{(1,2,3)}(G) \cap \alpha cl_{(1,2,3)}(X \sim G)$;

(ii) $\models \alpha b_{(1,2,3)}(G) \equiv \alpha b_{(1,2,3)}(X \sim G)$;

(iii) $\models x \sim \alpha b_{(1,2,3)}(G) \equiv \alpha \mathrm{int}_{(1,2,3)}(G) \cup \alpha \mathrm{int}_{(1,2,3)}(X \sim G)$;

(iv) $\models \alpha cl_{(1,2,3)}(G) \equiv G \cup \alpha b_{(1,2,3)}(G)$;

(v) $\models \alpha b_{(1,2,3)}(G) \subseteq G \longleftrightarrow G \in \alpha \mathscr{F}_{(1,2,3)}$;

(vi) $\models \alpha \mathrm{int}_{(1,2,3)}(G) \equiv G \cap (X \sim \alpha b_{(1,2,3)}(G))$;

(vii) $\models (\alpha b_{(1,2,3)}(G) \cap G \equiv \emptyset) \longleftrightarrow \mathrm{A} \in \alpha \tau_{(1,2,3)}$;

(viii) $\models \tau_1 \equiv \tau_3 \longrightarrow \alpha b_{(1,2,3)}(G) \subseteq b_1(G)$;

(ix) $\models X \sim \alpha b_{(1,2,3)}(G) \equiv \alpha \mathrm{int}_{(1,2,3)}(G) \cup \alpha \mathrm{ext}_{(1,2,3)}(X \sim G)$.

Proof.

(i) From Theorem 23 -(vii), we have

$(\alpha cl_{(1,2,3)}(G) \cap \alpha cl_{(1,2,3)}(X \sim G)(x)) = \min(\alpha cl_{(1,2,3)}(G)(x), \alpha cl_{(1,2,3)}(X \sim G)(x)) = \min(1 - \alpha \mathrm{int}_{(1,2,3)}(G)(x), 1 - \alpha \mathrm{int}_{(1,2,3)}(X \sim G)(x)) = \alpha b_{(1,2,3)}(G)(x)$.

(ii) Since $\alpha b_{(1,2,3)}(G)(x) = \min(1 - \alpha N_x^{(1,2,3)}(G)(x), 1 - \alpha N_x^{(1,2,3)}(X \sim G)(x)) = \min(1 - \alpha N_x^{(1,2,3)}(X \sim G)(x), 1 - \alpha N_x^{(1,2,3)}(G)(x)) = \alpha b_{(1,2,3)}(X \sim G)(x)$.

(iii) From (i) above and Theorem 23 -(vii), we get

$X \sim \alpha b_{(1,2,3)}(G) \equiv X \sim (\alpha cl_{(1,2,3)}(G) \cap \alpha cl_{(1,2,3)}(X \sim G)) = (X \sim \alpha cl_{(1,2,3)}(G)) \cup (X \sim \alpha cl_{(1,2,3)}(X \sim G) = \alpha \mathrm{int}_{(1,2,3)}(X \sim G) \cup \alpha \mathrm{int}_{(1,2,3)}(G)$.

(iv) If $x \in G$, then $\alpha cl_{(1,2,3)}(G)(x) = 1 = (G \cup \alpha b_{(1,2,3)}(G))(x)$. If $x \notin G$, then $(G \cup \alpha b_{(1,2,3)}(G))(x) = \alpha b_{(1,2,3)}(G)(x) = \min(1 - \alpha \mathrm{int}_{(1,2,3)}(G)(x), 1 - \alpha \mathrm{int}_{(1,2,3)}(X \sim G)(x)) = 1 - \alpha \mathrm{int}_{(1,2,3)}(X \sim G)(x) = \alpha cl_{(1,2,3)}(G)(x)$.

(v) From Theorem 19 -(iii), Theorem 21 -(v), Lemma (8.2) in [15] and (iv) above, we get

$G \in \alpha \mathscr{F}_{(1,2,3)} \longleftrightarrow \alpha d_{(1,2,3)}(G) \subseteq G \longleftrightarrow G \cup \alpha d_{(1,2,3)}(G) \subseteq G \longleftrightarrow \alpha cl_{(1,2,3)}(G) \subseteq G \longleftrightarrow G \cup \alpha b_{(1,2,3)}(G) \subseteq G \longleftrightarrow \alpha b_{(1,2,3)}(G) \subseteq G$

(vi) From Theorem 23 -(vii) and (vi) above, we get

$\alpha \mathrm{int}_{(1,2,3)}(G) \equiv X \sim \alpha cl_{(1,2,3)}(X \sim G) \equiv X \sim ((X \sim G) \cup \alpha b_{(1,2,3)}(X \sim G)) \equiv G \cap (X \sim \alpha b_{(1,2,3)}(X \sim G)) \equiv G \cap (X \sim \alpha b_{(1,2,3)}(G))$.

(vii) From Theorem 23 -(v) and (vi) above, we have $(\alpha b_{(1,2,3)}(G) \cap G \equiv \emptyset) \longleftrightarrow (X \sim \alpha b_{(1,2,3)}(G)) \cup (X \sim G)) \equiv X \longleftrightarrow G \subseteq X \sim \alpha b_{(1,2,3)}(G) \longleftrightarrow G \cap (X \sim \alpha b_{(1,2,3)}(G)) \equiv G \longleftrightarrow \alpha \mathrm{int}_{(1,2,3)}(G) \equiv G \longleftrightarrow G \in \alpha \tau_{(1,2,3)}$.

(viii) From Theorem 23 -(iii), we get $\alpha b_{(1,2,3)}(G)(x) = \min(1 - \alpha \mathrm{int}_{(1,2,3)}(G)(x), 1 - \alpha \mathrm{int}_{(1,2,3)}(X \sim G)(x)) \leq \min(1 - \mathrm{int}_1(G)(x), 1 - \mathrm{int}_1(G)(X \sim G)(x) = \alpha b_1(G)$

$\alpha b_{(1,2,3)}(G) \subseteq b_1(G)(x)$.

(ix) From (iii) above, we have

$X \sim \alpha b_{(1,2,3)}(G) \equiv \alpha \mathrm{int}_{(1,2,3)}(G) \cup \alpha \mathrm{int}_{(1,2,3)}(X \sim G) \equiv \alpha \mathrm{int}_{(1,2,3)}(G) \cup \alpha \mathrm{ext}_{(1,2,3)}(G)$. □

6. $(1,2,3)$-α-Convergence of Nets in Fuzzifying Tritopological Spaces

Definition 29. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then the class of all nets in X is defined as $N(X) = \{S$ *such that* $S : D \longrightarrow X$, where $(D, \geq)$ is a directed set$\}$.

Definition 30. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then the binary fuzzy predicates $\triangleright^{\alpha}_{(1,2,3)(1,2,3)}, \propto^{\alpha}_{(1,2,3)} \in \mathfrak{I}(N(X) \times X)$, are defined as

$S \triangleright^{\alpha}_{(1,2,3)} x := \forall G\ (G \in \alpha N_x^{(1,2,3)} \longrightarrow S \underset{\sim}{\subseteq} G)$,

$S \propto^{\alpha}_{(1,2,3)} x := \forall G\ (G \in \alpha N_x^{(1,2,3)} \longrightarrow S \underset{\sim}{\sqsubseteq} G)$, $S \in N(X)$,

where $S \triangleright^{\alpha}_{(1,2,3)} x$ stand for "S is (*1,2,3*)-α-convergence to x" and $S \propto^{\alpha}_{(1,2,3)} x$ stand for "x is (*1,2,3*)-α-accumulation point of S". Also, the binary crisp predicate $\underset{\sim}{\subseteq}$ is "almost in" and $\underset{\sim}{\sqsubseteq}$ is "often in".

Definition 31. Let $T \in N(X)$. One has the following fuzzy sets:
$\lim^{\alpha}_{(1,2,3)} T(x) = [T \triangleright^{\alpha}_{(1,2,3)} x]$ is (1,2,3)-α-limit of T;
$adh^{\alpha}_{(1,2,3)} T(x) = [T \propto^{\alpha}_{(1,2,3)} x]$ is (1,2,3)-α-adherence of T.

Theorem 32. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS $x \in X$, $G \in P(X)$, and $S \in N(X)$, then*

(i) $\models \exists S\ ((S \subseteq G \sim \{x\}) \wedge (S \triangleright^{\alpha}_{(1,2,3)} x)) \longrightarrow x \in \alpha d_{(1,2,3)}(G)$;

(ii) $\models \exists S\ ((S \subseteq G) \bigwedge (S \triangleright^{\alpha}_{(1,2,3)} x)) \longrightarrow x \in \alpha cl_{(1,2,3)}(G)$;

(iii) $\models G \in \alpha \mathscr{F}_{(1,2,3)} \longrightarrow \forall S\ (S \subseteq G \longrightarrow \lim^{\alpha}_{(1,2,3)} S \subseteq G)$;

(iv) $\models \exists T\ ((T < S) \bigwedge (T \triangleright^{\alpha}_{(1,2,3)} x)) \longrightarrow S \propto^{\alpha}_{(1,2,3)} x$, *where $T < S$ standing for "T is a subnet of S".*

Proof.

(i) $[\exists S\ ((S \subseteq G \sim \{x\}) \bigwedge (S \triangleright^{\alpha}_{(1,2,3)} x))] = \sup_{S \subseteq G \sim \{x\}} \inf_{S \underset{\sim}{\not\subseteq} G} (1 - \alpha N_x^{(1,2,3)}(H))$. Now, since $S \subseteq G \sim \{x\}$, then $S \not\subseteq (X \sim G) \cup \{x\}$ and this implies $S \underset{\sim}{\not\subseteq} (X \sim G) \cup \{x\}$. Therefore

$\inf_{S \underset{\sim}{\not\subseteq} G} (1 - \alpha N_x^{(1,2,3)}(H)) \leq 1 - \alpha N_x^{(1,2,3)}((X \sim G) \cup \{x\}) = [x \in \alpha d_{(1,2,3)}(G)]$.

(ii) If $x \in G$, then from Theorem 21 -(i) and (i) above we have

$[\exists S\ ((S \subseteq G) \bigwedge (S \triangleright^{\alpha}_{(1,2,3)} x))] = \sup_{S \subseteq G} \inf_{S \underset{\sim}{\not\subseteq} G} (1 - \alpha N_x^{(1,2,3)}(H)) \leq 1 - \alpha N_x^{(1,2,3)}(X \sim G) = [x \in \alpha cl_{(1,2,3)}(G)]$.

If $x \notin G$, then $G \sim \{x\} = G$. From Theorem 21 -(i) and (i) above we have

$[\exists S\ ((S \subseteq G) \bigwedge (S \triangleright^{\alpha}_{(1,2,3)} x))] = [\exists S\ ((S \subseteq G \sim \{x\}) \bigwedge (S \triangleright^{\alpha}_{(1,2,3)} x))] \leq 1 - \alpha N_x^{(1,2,3)}(X \sim G) = \alpha cl_{(1,2,3)}(G) = [x \in \alpha cl_{(1,2,3)}(G)]$.

(iii) From Theorem 21 -(vi) and (ii) above, we get

$[G \in \alpha\mathscr{F}_{(1,2,3)}] = [G \equiv \alpha cl_{(1,2,3)}(G)] = [G \subseteq \alpha cl_{(1,2,3)}(G)] \bigwedge [\alpha cl_{(1,2,3)}(G) \subseteq G] \leq [\alpha cl_{(1,2,3)}(G) \subseteq G] = [X \sim G \subseteq X \sim \alpha cl_{(1,2,3)}(G)] = \inf_{x\in X\sim G}(1 - \alpha cl_{(1,2,3)}(G)(x)) \leq \inf_{x\in X\sim G}(1 - \sup_{S\subseteq G}\inf_{S\underset{\sim}{\not\subseteq} H}(1 - \alpha N_x^{(1,2,3)}(H))) = \inf_{x\notin G}\inf_{S\subseteq G}(1 - \inf_{S\underset{\sim}{\not\subseteq} H}(1 - \alpha N_x^{(1,2,3)}(H))) = [\forall S\, (S \subseteq G \longrightarrow \lim^{\alpha}_{(1,2,3)} S \subseteq G)]$.

(iv) We have if $S \underset{\sim}{\not\subseteq} G$, then $S \underset{\sim}{\not\subseteq} G$, for any $S \in N(X)$ and any $G \subseteq X$. Therefore

$[\exists T\, ((T < S) \wedge (T \rhd^{\alpha}_{(1,2,3)} x))] = \sup_{T<S}\inf_{T\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = \inf_{T\underset{\sim}{\not\subseteq} G}(1 - \inf_{T<S}\alpha N_x^{(1,2,3)}(G)) \leq \inf_{T\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) \leq \inf_{S\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = \inf_{S\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = [S \propto^{\alpha}_{(1,2,3)} x]$.

□

Theorem 33. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS and T is a universal net, then* $\models \lim^{\alpha}_{(1,2,3)} T = adh^{\alpha}_{(1,2,3)} T$.

Proof. $\lim^{\alpha}_{(1,2,3)} T(x) = \inf_{T\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = \inf_{T\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = adh^{\alpha}_{(1,2,3)} T(x)$. □

Lemma 34. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then* $\models (T \rhd^{\alpha}_{(1,2,3)} x)) \longleftrightarrow \forall G\, (x \in G \in \alpha\tau_{(1,2,3)} \longrightarrow T \underset{\sim}{\subseteq} G)$.

Proof. If $H \subseteq G$ and $T \underset{\sim}{\not\subseteq} G$, then $T \underset{\sim}{\not\subseteq} H$.

$[T \rhd^{\alpha}_{(1,2,3)} x)] = \inf_{T\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = 1 - \sup_{T\underset{\sim}{\not\subseteq} G}\sup_{x\in H\subseteq G}\alpha\tau_{(1,2,3)}(H) \geq 1 - \sup_{T\underset{\sim}{\not\subseteq} H, x\in H}\alpha\tau_{(1,2,3)}(H) = \inf_{T\underset{\sim}{\not\subseteq} H, x\in H}(1 - \alpha\tau_{(1,2,3)}(H)) = [\forall G\, (x \in G \in \alpha\tau_{(1,2,3)} \longrightarrow T \underset{\sim}{\subseteq} G]$.

Conversely,

$[\forall G\, (x \in G \in \alpha\tau_{(1,2,3)} \longrightarrow T \underset{\sim}{\subseteq} G)] = \inf_{T\underset{\sim}{\not\subseteq} G, x\in G}(1 - \alpha\tau_{(1,2,3)}(G)) = \inf_{T\underset{\sim}{\not\subseteq} G, x\in G}(1 - \inf_{x\in G}\sup_{H\subseteq G}\alpha N_x^{(1,2,3)}(H)) \geq 1 - \sup_{T\underset{\sim}{\not\subseteq} G, x\in G}\alpha N_x^{(1,2,3)}(H) = \inf_{T\underset{\sim}{\not\subseteq} G, x\in G}(1 - \alpha N_x^{(1,2,3)}(H)) = [T \rhd^{\alpha}_{(1,2,3)} x]$. □

7. $(1,2,3)$-α-Convergence of Filters in Fuzzifying Tritopological Spaces

Definition 35. If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS and $F(X)$ is the set of all filters on X, then the binary fuzzy predicates $\rhd^{\alpha}_{(1,2,3)}$, $\propto^{\alpha}_{(1,2,3)} \in \mathfrak{I}(F(X) \times X)$ are defined as

$K \rhd^{\alpha}_{(1,2,3)} x := \forall G\, (G \in \alpha N_x^{(1,2,3)} \longrightarrow G \in K)$;

$K \propto^{\alpha}_{(1,2,3)} x := \forall G\, (G \in K \longrightarrow x \in \alpha cl_{(1,2,3)}(G))$, where $K \in F(X)$.

Definition 36. The fuzzy sets

$\lim^{\alpha}_{(1,2,3)} K(x) = [K \rhd^{\alpha}_{(1,2,3)} x]$ are (1,2,3)-α-limit of K;

$adh^{\alpha}_{((1,2,3)} K(x) = [K \propto^{\alpha}_{(1,2,3)} x]$ are (1,2,3)-α-adherence of K.

Theorem 37. *If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, then we have the following.*

(1) *If $T \in N(X)$ and K^T is the filter corresponding to T, i.e., $K^T = \{G : T \underset{\sim}{\subseteq} G\}$, then*

(i) $\models \lim^{\alpha}_{(1,2,3)} K^T = \lim^{\alpha}_{(1,2,3)} T$;

(ii) $\models adh^{\alpha}_{(1,2,3)} K^T = adh^{\alpha}_{(1,2,3)} T$.

(2) *If $K \in F(X)$ and T^K is the net corresponding to K, i.e., $T^K : D \longrightarrow X, (x, G) \longmapsto x, (x, G) \in D$, where $D = \{(x, G) : x \in G \in K\}, (x, G) \geq (y, H)$ iff $G \subseteq H$, then*

(i) $\models \lim^{\alpha}_{(1,2,3)} T^K = \lim^{\alpha}_{(1,2,3)} K$;

(ii) $\models adh^{\alpha}_{(1,2,3)} T^K = adh^{\alpha}_{(1,2,3)} K$.

Proof.

(1)

(i) $\lim^{\alpha}_{(1,2,3)} K^T(x) = \inf_{G\notin K^T}(1 - \alpha N_x^{(1,2,3)}(G)) = \inf_{T\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = \lim^{\alpha}_{(1,2,3)} T$.

(ii) $adh^{\alpha}_{(1,2,3)} K^T = \inf_{G\in K^T}\alpha cl_{(1,2,3)}(G)(x) = \inf_{T\underset{\sim}{\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(X \sim G)) = \inf_{T\underset{\sim}{\not\subseteq} X\sim G}(1 - \alpha N_x^{(1,2,3)}(X \sim G)) = \inf_{T\underset{\sim}{\not\subseteq} X\sim G}(1 - \alpha N_x^{(1,2,3)}(X \sim G)) = adh^{\alpha}_{(1,2,3)} T$.

(2) Similar to (i) above

(i) $\lim^{\alpha}_{(1,2,3)} T^K = [T^K \rhd^{\alpha}_{(1,2,3)} x] = \inf_{T^K\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = \inf_{G\notin K}(1 - \alpha N_x^{(1,2,3)}(G)) = \lim^{\alpha}_{(1,2,3)} K$.

(ii) $adh^{\alpha}_{(1,2,3)} T^K(x) = [T^K \propto^{\alpha}_{(1,2,3)} x] = \inf_{T^K\underset{\sim}{\not\subseteq} G}(1 - \alpha N_x^{(1,2,3)}(G)) = \inf_{X\sim G\in K}\alpha cl_{(1,2,3)}(X \sim G) = adh^{\alpha}_{(1,2,3)} K$.

□

8. Conclusion

The main contribution of the present paper is to give characterization of tri-α-open sets in fuzzifying tritopological space. We also define the concepts of tri-α-closed sets, tri-α-neighborhood system, tri-α-interior, tri-α-closure, tri-α-derived, tri-α-boundary, tri-α-exterior, and tri-α-convergence in fuzzifying tritopological spaces and some basics of such spaces. We present some problems for future study.

(1) Study the results of the present paper by considering the quad-α-open sets in fuzzifying quad-topological spaces.

(2) Investigate relations between fuzzifying quad-topology, tritopology, bitopology and fuzzifying topology.

(3) Study of quad-α-separation axioms in fuzzifying quad-topological spaces.

(4) Generalize the results in the present work to soft fuzzifying topology.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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