On Trigonometric Series Associated with Weak Closed Subspaces of Continuous Functions*

HASKELL P. ROSENTHAL*

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If f is a complex Lebesgue integrable function on the unit circle, $\hat{f}(n)$, the n^{th} Fourier coefficient of f, is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}\, d\theta \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Let C be the Banach space of continuous functions on the unit circle, A those functions in C with absolutely convergent Fourier series, i.e., those f in C for which $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$, and L^{∞} (resp. L^1) the Banach spaces of equivalence classes of bounded measurable (resp. integrable) functions on the circle; L^{∞} is endowed with the essential supremum norm, denoted $||\cdot||_{\infty}$; and we regard C as being a closed subspace of L^{∞}.

If $\mathfrak{B}$ is one of the spaces, C, A, L^{∞} or L^1, and E is a subset of $\mathbf{Z}$, the integers, we define

$$\mathfrak{B}_E = \{f \in \mathfrak{B}: \hat{f}(n) = 0 \text{ for all } n \notin E\}.$$

We show here that there exists a set E such that $L_E^{\infty} \subset C_E$, but $C_E \not\subset A$. Indeed, examples of such sets may be given explicitly as follows:

Given S a subset of $\mathbf{Z}$, and $l \in \mathbf{Z}$, define $lS = \{ln: n \in S\}$. Let $E_n = \{1, 2, \cdots, n\}$. Then $E^1 = \bigcup_{n=0}^{\infty} (19)^n n! E_{n+1}$ and $E^2 = \bigcup_{n=1}^{\infty} (2n)! E_{2n}$ both have this property.

In Theorem 3, we give a general class of sets E for which $L_E^{\infty} \subset C_E$; the above examples follow as a corollary to this result.

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1. Motivation. A set E is called a Sidon set if $C_E \subset A$. It is known that if E is a Sidon set, then $L_E^\infty \subset C_E$ (cf. p. 207 of [2]). Our results show that the converse to this is false.

If $f \in C$, $f \notin A$, then it is fairly easy to show that there exists a sequence $\{\epsilon_n\}$, $\epsilon_n = 0$ or 1 for $n = 0, \pm 1, \pm 2, \cdots$ such that $\sum \epsilon_n \hat{f}(n) e^{in\theta}$ is not the Fourier series of a bounded measurable function; i.e., such that for no g in L^∞ is it true that $\hat{g}(n) = \epsilon_n \hat{f}(n)$ for all n. One might think that it would then be possible to find another sequence δ_n of zeros and ones, so that $\sum \delta_n \hat{f}(n) e^{in\theta}$ was the Fourier series of a bounded measurable function, but not of a continuous one. Our results show that this is not the case, in general.

Finally, we note that $L_E^\infty \subset C_E$ if and only if C_E is a weak* closed subspace of L^∞. Since L_E^∞ may be isometrically identified with the conjugate space of a certain Banach space (namely $L^1/L^1_{\sim E}$), our examples show that there are non-Sidon sets E such that C_E is isometric to a conjugate space.

2. We first have need of a lemma which shows that the sum of the sups may be dominated by the sup of the sum, under certain conditions. (See théorème IV, page 155 of [1] for a related result.)

Lemma. **1.** *Let* $p_1, p_2, \cdots$ *be a sequence of bounded real-valued periodic functions defined on the real line, and* $K_1, K_2, \cdots$ *a sequence of positive constants, satisfying the following conditions for all numbers* n:

(1) $|p_n(x) - p_n(y)| \leqq K_n \|p_n\|_\infty |x - y|$ *for all real numbers* x *and* y (*where* $\|p_n\|_\infty = \sup |p_n(x)|$).
(2) *Period* $p_{n+1} \leqq (1/3K_n)$. (*By which we mean, there exists a positive real number* τ, *so that* $p_{n+1}(x + \tau) = p_{n+1}(x)$ *for all real* x, *and* $\tau \leqq (1/3K_n)$.)
(3) p_n *assumes the value zero.*
(4) $2K_n \leqq K_{n+1}$

Conclusion. For all N,

$$\sum_{n=1}^{N} \|p_n\|_\infty \leqq 6 \left\| \sum_{n=1}^{N} p_n \right\|_\infty .$$

Proof. Fix N, and let $a_n = \|p_n\|_\infty$ for all n. Let G denote the set of those integers n, such that $1 \leqq n \leqq N$, and such that there exists a real number x with $p_n(x) = a_n$. Let B denote those integers n, $1 \leqq n \leqq N$, such that $n \notin G$.

Thus

$$\sum_{n \in G} a_n + \sum_{n \in B} a_n = \sum a_n ;$$

hence either

$$\sum_{n \in G} a_n \quad \text{or} \quad \sum_{n \in B} a_n \geqq \tfrac{1}{2} \sum a_n .$$

(We use the un-indexed symbol "$\sum$" to mean, the indices run from 1 to N.) Suppose the former; then $\sum_{n \in B} a_n \leqq \frac{1}{2} \sum a_n$.

Now, we claim that a sequence $I_1 \supset I_2 \supset \cdots \supset I_N$ of bounded closed intervals may be chosen so that for all n, $1 \leqq n \leqq N$,

$$(*) \qquad \begin{cases} \text{if} \quad n \in G, \quad \text{then} \quad p_n(x) \geqq \tfrac{2}{3}a_n \quad \text{for all} \quad x \in I_n \, ; \\ \text{if} \quad n \in B, \quad \text{then} \quad |p_n(x)| \leqq \tfrac{1}{3}a_n \quad \text{for all} \quad x \in I_n \, . \end{cases}$$

To see this, we observe first that if y belongs to an interval containing x as an end point, of length $1/3K_n$, then

$$(**) \qquad \begin{cases} \text{if} \quad p_n(x) = a_n \, , \quad \text{then} \quad p_n(y) \geqq \tfrac{2}{3}a_n \\ \text{if} \quad p_n(x) = 0, \quad \text{then} \quad |p_n(y)| \leqq \tfrac{1}{3}a_n \, . \end{cases}$$

((**) follows immediately from (1).)

Now, if $1 \in G$, choose x so that $p_1(x) = a_1$; if $1 \in B$, choose x, so that $p_1(x) = 0$, which we can do by (3). Now let I_1 be a closed interval of length $1/3K_1$ with x as an end point. Then I_1 satisfies (*) by (**).

Suppose I_l chosen, with length $I_l = 1/3K_l$. Now since period $p_{l+1} \leqq 1/3K_l$, all values of range $|p_{l+1}|$ are taken on in I_l (of course, p_{l+1} is continuous by (1)).

If $l + 1 \in G$, choose $x_o \in I_l$, with $p_{l+1}(x_o) = a_{l+1}$.

If $l + 1 \in B$, choose $x_o \in I_l$, with $p_{l+1}(x_o) = 0$. Now let I_{l+1} be one of the closed intervals of length $1/3K_{l+1}$, which has x_o as an endpoint, and which is contained in I_l. (Since $1/3K_{l+1} \leqq \frac{1}{2}$ length $I_l = (1/2)(1/3K_l)$ by (4), either

$$\left[x_o \, , x_o + \frac{1}{3K_{l+1}}\right] \subset I_l \quad \text{or} \quad \left[x_o - \frac{1}{3K_{l+1}} \, , x_o\right] \subset I_l \, .$$

Again, I_{l+1} satisfies (*) by (**); and we have that length $I_{l+1} = (1/3K_{l+1})$.

Thus the nested sequence $I_1 \supset \cdots \supset I_N$ has been constructed satisfying (*). Now, let $y_o \in I_N$. Then,

$$\sum p_n(y_o) = \sum_{n \in G} p_n(y_o) + \sum_{n \in B} p_n(y_o) \geqq \tfrac{2}{3} \sum_{n \in G} a_n - \tfrac{1}{3} \sum_{n \in B} a_n$$
$$\geqq \tfrac{2}{3} \cdot \tfrac{1}{2} \sum a_n - \tfrac{1}{3} \cdot \tfrac{1}{2} \sum a_n = \tfrac{1}{6} \sum a_n = \tfrac{1}{6} \sum ||p_n||_\infty \, .$$

Thus we have demonstrated that

$$\sum_{n=1}^{N} ||p_n||_\infty \leqq 6 \left|\left| \sum_{n=1}^{N} p_n \right|\right|_\infty$$

under the assumption that $\sum_{n \in G} ||p_n||_\infty \geqq \frac{1}{2} \sum ||p_n||_\infty$. However, if $\sum_{n \in G} ||p_n||_\infty < \frac{1}{2} \sum ||p_n||_\infty$, we have merely to replace p_n by $-p_n$ throughout. Q.E.D.

Remark. The above argument is valid under a weaker hypothesis than (2); all that is really used, is the fact that $|p_{n+1}|$ achieves both its minimum and maximum on any interval of length $1/3K_n$. This won't be used in the sequel, however.

Corollary 2. *Let $h_1, h_2, \cdots$ be a sequence of bounded complex valued periodic functions defined on the real line, and $K_1, K_2 \cdots$ a sequence of positive constants,* *so that for both* $p_n = \text{Re } h_n$ = *Real part of* h_n, *and* $p_n = \text{Im } h_n$ = *Imaginary part of* h_n, (1) *through* (4) *are satisfied, for all* n.

Conclusion: $\sum_{n=1}^{N} ||p_n||_\infty \leqq 12 \, ||\sum_{n=1}^{N} p_n||_\infty$ *for all* N.

Proof. Fix N; by Lemma 1, we have that

$$\sum ||\text{Re } h_n||_\infty \leqq 6 \, ||\sum \text{Re } h_n||_\infty$$

and

$$\sum ||\text{Im } h_n||_\infty \leqq 6 \, ||\sum \text{Im } h_n||_\infty .$$

But if f is any bounded complex function, then

$$||f||_\infty \leqq ||\text{Re } f||_\infty + ||\text{Im } f||_\infty \leqq 2 \, ||f||_\infty .$$

Thus

$$\begin{aligned} 2 \, ||\sum h_n||_\infty &\geqq ||\text{Re } \sum h_n||_\infty + ||\text{Im } \sum h_n||_\infty \\ &\geqq \tfrac{1}{6} \sum (||\text{Re } h_n||_\infty + ||\text{Im } h_n||_\infty) \\ &\geqq \tfrac{1}{6} \sum ||h_n||_\infty ; \end{aligned}$$

i.e.,

$$\sum_{n=1}^{N} ||h_n||_\infty \leqq 12 \left|\left| \sum_{n=1}^{N} h_n \right|\right|_\infty .$$

Q.E.D.

Our next result gives a class of sets E for which $L_E^\infty \subset C_E$; we first need the following notation: if S is a finite subset of the integers, let

$$M(S) = \max \{|j| : j \in S\}$$
$$m(S) = \min \{|j| : j \in S\}.$$

Theorem 3. *Let* $F_1, F_2, \cdots$ *be a sequence of finite disjoint subsets of the integers, and* $f_1, f_2, \cdots$ *a sequence of positive integers, satisfying the following two conditions for all numbers* n:

(a) $F_{n+1} \subset f_{n+1}\mathbf{Z}$ (*where* $\mathbf{Z}$ *denotes the integers*).

(b) $f_{n+1} \geqq 6\pi M(F_n)$.

Conclusion: Let $E = \bigcup_{n=1}^{\infty} F_n$; *then* $L_E^\infty \subset C_E$.

Proof. We may first suppose that $0 \notin F_n$ for all n. Indeed, our hypotheses imply that 0 belongs to at most one of the F_n's; but if E' has the property that $L_{E'}^\infty \subset C_{E'}$, then the union of E' with any finite set has this property also. Next, suppose $q_1, q_2, q_3, \cdots$ are given with $q_n \in C_{F_n}$ for all n. We claim that

$$\sum_{n=1}^{N} ||q_n||_\infty \leqq 12 \left|\left| \sum_{n=1}^{N} q_n \right|\right|_\infty$$

for all integers N.

Well, define $h_n(x) = q_n(e^{ix})$ for all real numbers x, and let $K_n = M(F_n)$. We have only to show that the h_n's and K_n's satisfy the hypotheses of Corollary 2. So, let p_n denote either $\text{Re } h_n$ or $\text{Im } h_n$. Then p_n is a trigonometric polynomial of degree at most K_n, whence by Bernstein's theorem (cf. page 11 of [3]),

$$|p_n'(x)| \leqq K_n \, ||p_n||_\infty \quad \text{for all } x;$$

thus (1) is satisfied.

By (a), h_{n+1} has $2\pi/f_{n+1}$ as a period, whence

$$\text{period } p_{n+1} \leqq \frac{1}{3M(F_n)} = \frac{1}{3K_n}$$

by (b), so (2) is satisfied. Since $0 \notin F_n$ for all n by assumption,

$$0 = \int_0^{2\pi} q_n(e^{ix})\,dx = 2\pi\hat{q}_n(0) = \int_0^{2\pi} h_n(x)\,dx.$$

Thus $\int_0^{2\pi} \operatorname{Re} h_n(x)\,dx = \int_0^{2\pi} \operatorname{Im} h_n(x)\,dx = 0$; thus (3) holds for p_n, since p_n is a continuous real function with $\int_0^{2\pi} p_n(x)\,dx = 0$.

Finally, again since $0 \notin F_n$ for all n, (a) implies that $m(F_{n+1}) \geqq f_{n+1}$, whence by (b)

$$6\pi M(F_n) = 6\pi K_n \leqq f_{n+1} \leqq m(F_{n+1}) \leqq M(F_{n+1}) = K_{n+1},$$

so (4) is satisfied.

We note for later use, that the last inequality shows that

$$M(F_n) < m(F_{n+1}) \quad \text{for all } n.$$

Now suppose that $f \in L_E^\infty$. Let f_n be defined by

$$f_n(e^{ix}) = \sum_{j \in F_n} \hat{f}(j)e^{ijx} \quad \text{for all } x.$$

Fix N an integer, and choose q a trigonometric polynomial such that

$$q \in C_E, \qquad \|q\|_\infty \leqq 2\,\|f\|_\infty,$$

and

$$\hat{q}(j) = \hat{f}(j) \quad \text{for all} \quad |j| \leqq M(F_N).$$

(Indeed, if we let $K_n(x)$ be the n^{th} Fejer kernel defined by

$$K_n(x) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right)e^{ijx},$$

and set $V_n(x) = 2K_{2n-1}(x) - K_{n-1}(x)$, then q may be given as

$$q(e^{iy}) = \frac{1}{2\pi}\int_0^{2\pi} V_{M(F_N)}(y - x)f(e^{ix})\,dx.)$$

Now let

$$q_n(e^{ix}) = \sum_{j \in F_n} \hat{q}(j)e^{ijx}$$

for all n. We thus have that $q = \sum q_n$, with only finitely many terms being non-zero, since q is a trigonometic polynomial; moreover, $q_n \in C_{F_n}$ for all n. Finally, since $M(F_n) < m(F_{n+1})$, we have that

$$F_n \subset \{j\colon |j| \leqq M(F_N)\}$$

for all $n \leqq N$, whence $f_n = q_n$ for all $n \leqq N$.

We thus have that

$$\sum_{n=1}^{N} \|f_n\|_\infty \leqq \sum \|q_n\|_\infty \leqq 12 \left\|\sum q_n\right\|_\infty = 12 \|q\|_\infty \leqq 24 \|f\|_\infty .$$

Thus since N was arbitrary, we have that

$$\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty .$$

So the series $\sum_{n=1}^{\infty} f_n(e^{ix})$ converges uniformly, to a continuous function h, say. Then $\hat{h}(j) = \hat{f}(j)$ for all integers j, whence $h = f$ by uniqueness of Fourier coefficients. Q.E.D.

Remark. We have actually shown that if $q \in C_E$, then

$$\sum_{n=1}^{\infty} \|q_n\|_\infty \leqq 12 \|q\|_\infty ,$$

where E is the set given in the statement of the theorem, and

$$q_n = \sum_{j \in F_n} \hat{q}(j) e^{ijx}$$

for all n. For at the beginning of the proof, we showed that this was true if $q \in C_E$, q a trigonometric polynomial; (for then there would exist an integer N so that $q = \sum_{n=1}^{N} q_n$); but the trigonometric polynomials in C_E are dense in C_E.

Corollary 4. *Let* $E^1 = \bigcup_{n=0}^{\infty} (19)^n n! E_{n+1}$, $E^2 = \bigcup_{n=1}^{\infty} (2n)! E_{2n}$. *Then* $L_{E^i}^{\infty} \subset C_{E^i}$, *but* E^i *is not a Sidon set (i.e.,* $C_{E^i} \not\subset A$*), for* $i = 1, 2$.

Proof. If $f_1, f_2, \cdots$ is any sequence of integers with $\limsup |f_n| > 0$, then $E = \sum_{n=1}^{\infty} f_n E_n$ cannot be a Sidon set. Indeed, no Sidon set can contain finite arithmetic progressions of arbitrarily long length (*cf.* p. 214 of [2]). So, the f_n's need only be chosen so that setting

$$F_{n+1} = f_n E_n ,$$

condition (b) of Theorem 3 is satisfied. ((a) holds automatically, since $f_n E_n \subset f_n \mathbf{Z}$.)

This reduces to $f_{n+1} \geqq 6\pi n f_n$. This condition is of course satisfied if $f_{n+1} = (19)^n n!$. For the case of E^2, we simply let $F_n = (2n)! E_{2n}$ and $f_n = (2n)!$, and again apply Theorem 3. Q.E.D.

References

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University of California at Berkeley

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