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ON TRUNCATIONS FOR WEAKLY ERGODIC INHOMOGENEOUS BIRTH AND DEATH PROCESSES

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We investigate a class of exponentially weakly ergodic inhomogeneous birth and death processes. We consider special transformations of the reduced intensity matrix of the process and obtain uniform (in time) error bounds of truncations. Our approach also guarantees that we can find limiting characteristics approximately with an arbitrarily fixed error. As an example, we obtain the respective bounds of the truncation error for an $M_t/M_t/S$ queue for *any* number of servers S. Arbitrary intensity functions instead of periodic ones can be considered in the same manner.

Keywords: birth and death process, weak ergodicity, truncation, forward Kolmogorov system, nonstationary Markovian queueing models.

1. Introduction

It is well known that explicit expressions for probability characteristics of stochastic models can be found only in a few special cases. In view of this, the study of the rate of convergence as time $t \to \infty$ to the steady state of a process is one of two main problems for obtaining the limiting behaviour of the process. If the model is Markovian and stationary in time, then the stationary limiting characteristics usually give usually, sufficient or almost sufficient information about the model. On the other hand, if we deal with an inhomogeneous Markovian model, then we must approximately calculate, in addition to that, the limiting probability characteristics of the process. The problem of existence and construction of limiting characteristics for inhomogeneous (in time) birth and death processes is important for queueing applications (see, e.g., Di Crescenzo and Nobile, 1995; Di Crescenzo *et al.*, 2003; 2012; Granovsky and Zeifman, 2004; Mandelbaum and Massey, 1995; Massey and Whitt, 1994; Massey and Pender, 2013; Olwal *et al.*, 2012; Tan *et al.*, 2013; Zeifman *et al.*, 2006). This is the second main problem. A general approach and related bounds for the study on the convergence rate was considered by Zeifman (1995a), who also first mentioned computation of the limiting characteristics for the process via truncations (Zeifman, 1988), and later considered it in detail (Zeifman *et al*., 2006). The first results for more general Markovian queueing models have been obtained recently by Zeifman *et al*. (2014).

About two decades ago Vladimir V. Kalashnikov suggested that in some cases one can obtain uniform (in time) error bounds of truncation. Here we prove this conjecture.

The paper is organized as follows. Basic notions are recalled below in Section 1. Auxiliary statements are considered in Section 2. The main result is proved in

Section 3. In Section 4 we consider application of our bounds for the $M_t/M_t/S$ queueing model. Finally, in Section 5 we consider a specific queueing example.

Let $X = X(t),\ t \geq 0$ be a Birth and Death Process (BDP) with birth and death rates $\lambda_n(t),\ \mu_n(t)$, respectively. Let $p_{ij}(s,t) = \Pr\{X(t) = j \,|X(s) = i\}$ for $i, j \geq 0,\ 0 \leq s \leq t$, be the transition probability functions of the process $X = X(t)$, and let $p_i(t) = \Pr\{X(t) = i\}$ be the state probabilities.

We assume throughout the paper that

$$\Pr\left(X\left(t+h\right) = j / X\left(t\right) = i\right) = \begin{cases} q_{ij}\left(t\right)h + \alpha_{ij}\left(t,h\right) & \text{if } j \neq i, \\ 1 - \sum\limits_{k \neq i} q_{ik}\left(t\right)h + \alpha_i\left(t,h\right) & \text{if } j = i, \end{cases} \tag{1}$$

where all $\alpha_i(t,h)$ are $o(h)$ uniformly in i, i.e., $\sup_i |\alpha_i(t,h)| = o(h)$. Here all $q_{i,i+1}\left(t\right) = \lambda_i(t)$, $q_{i,i-1}\left(t\right) = \mu_i(t)$ for any $i > 0$, and all other $q_{ij}(t) \equiv 0$.

The probabilistic dynamics of the process are represented by the forward Kolmogorov system of differential equations:

$$\begin{cases} \dfrac{\mathrm{d}p_0}{\mathrm{d}t} = -\lambda_0(t)p_0 + \mu_1(t)p_1, \\ \dfrac{\mathrm{d}p_k}{\mathrm{d}t} = \lambda_{k-1}(t)p_{k-1} - \left(\lambda_k(t) + \mu_k(t)\right)p_k \\ +\mu_{k+1}(t)p_{k+1}, \quad k \geq 1. \end{cases} \tag{2}$$

By $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T,\ t \geq 0$, we denote the column vector of state probabilities and by $A(t) = (a_{ij}(t)),\ t \geq 0$, the matrix related to (2). One can see that $A\left(t\right) = Q^T\left(t\right)$, where $Q(t)$ is the intensity (or infinitesimal) matrix for $X(t)$.

We assume that all birth and death intensity functions $\lambda_i(t)$ and $\mu_i(t)$ are linear combinations of a finite number of functions which are locally integrable on $[0,\infty)$. Moreover, we assume that

$$\lambda_n(t) \leq L < \infty, \quad \mu_n(t) \leq M < \infty, \tag{3}$$

for almost all $t \geq 0$. Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i.e., $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{\infty}$.

Let Ω be a set all stochastic vectors, i.e., l_1 vectors with nonnegative coordinates and the unit norm. Then we have

$$\|A(t)\| \leq 2\sup(\lambda_k(t) + \mu_k(t)) \leq 2\left(L + M\right)$$

for almost all $t \geq 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0;\infty)$.

Therefore we can consider the system (2) as the differential equation

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = A\left(t\right)\mathbf{p}, \quad \mathbf{p} = \mathbf{p}(t), \quad t \geq 0, \tag{4}$$

in the space l_1 with a bounded operator function $A(t)$.

It is well known (Daleckij and Krein, 1974) that the Cauchy problem for the differential equation (1) has unique solutions for arbitrary initial conditions, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geq s \geq 0$.

Therefore, we can apply the general approach to employ the logarithmic norm of a matrix for the study of the stability of a Kolmogorov system of differential equations associated with nonhomogeneous Markov chains. The method is based on the following two components: the logarithmic norm of a linear operator and a special similarity transformation of the matrix of intensities of the Markov chain considered, see the corresponding definitions, bounds, references and other details in the works of Van Doorn *et al*. (2010), Granovsky and Zeifman (2004), Zeifman (1985; 1995b; 1995a) or Zeifman *et al*. (2006).

Definition 1. A Markov chain $X(t)$ is called *weakly ergodic* if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$. Here $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (4).

Set $E_k(t) = E\{X(t)\,|X(0) = k\}$ (then the corresponding initial condition of the system (4) is the k-th unit vector $\mathbf{e_k}$).

Definition 2. Let $X(t)$ be a Markov chain. Then $\varphi(t)$ is called the *limiting mean* of $X(t)$ if

$$\lim_{t\to\infty}\left(\varphi(t) - E_k(t)\right) = 0$$

for any k.

2. Auxiliary notions and results

Consider an increasing sequence of positive numbers $\{d_i\},\ i = 1, 2, \dots,\ d_1 = 1$, and the corresponding triangular matrix

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5}$$

Let l_{1D} be the space of sequences

$$l_{1D} = \left\{\mathbf{z} = (p_1, p_2, \dots)^T : \|\mathbf{z}\|_{1D} \equiv \|D\mathbf{z}\| < \infty\right\}.$$

We also introduce the auxiliary space of sequences l_{1E} as

$$l_{1E} = \left\{\mathbf{z} = (p_1, p_2, \cdots)^T : \|\mathbf{z}\|_{1E} \equiv \sum k|p_k| < \infty\right\}.$$

Set

$$d = \inf_{i\geq 1} d_i = 1, \quad W = \inf_{i\geq 1}\frac{d_i}{i}, \quad g_i = \sum_{n=1}^{i} d_n.$$

Consider the following expressions:

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), \quad k \geq 0, \tag{6}$$

and

$$\alpha(t) = \inf_{k\geq 0} \alpha_k(t). \tag{7}$$

At first, we recall the definition of the logarithmic norm and the related bound (for details, see Van Doorn *et al*., 2010; Granovsky and Zeifman, 2004; Zeifman *et al*., 2006)

Let $B(t)$, $t \geq 0$ be a one-parameter family of bounded linear operators on a Banach space $\mathcal{B}$ and let I denote the identity operator. For a given $t \geq 0$, the number

$$\gamma(B(t))_{\mathcal{B}} = \lim_{h\to+0} \frac{\|I + hB(t)\| - 1}{h}$$

is called the *logarithmic norm* of the operator $B(t)$.

If $\mathcal{B}$ is an $(N+1)$-dimensional vector space with the l_1-norm, so that the operator $B(t)$ is given by the matrix $B(t) = (b_{ij}(t))_{i,j=0}^{N}$, $t \geq 0$, then the logarithmic norm of $B(t)$ can be found explicitly:

$$\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i\neq j} |b_{ij}(t)| \right), \quad t \geq 0.$$

On the other hand, the logarithmic norm of the operator $B(t)$ is related to the Cauchy operator $V(t,s)$ of the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = B(t)\mathbf{x}, \quad t \geq 0,$$

in the following way:

$$\gamma(B(t))_{\mathcal{B}} = \lim_{h\to+0} \frac{\|V(t+h,t)\| - 1}{h}, \quad t \geq 0.$$

From the latter, one can deduce the following bounds of the Cauchy operator $V(t,s)$:

$$\|V(t,s)\|_{\mathcal{B}} \leq e^{\int_s^t \gamma(B(\tau))\,\mathrm{d}\tau}, \quad 0 \leq s \leq t.$$

Recall now the following general statement.

Theorem 1. *Let a BDP with the rates $\lambda_k(t)$ and $\mu_k(t)$ be given. Assume that there exists a sequence $\{d_i\}$ such that*

$$\int_0^\infty \alpha(t)\,\mathrm{d}t = +\infty. \tag{8}$$

Then $X(t)$ is weakly ergodic, and the following bounds hold:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \leq e^{-\int_s^t \alpha(\tau)\,\mathrm{d}\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \tag{9}$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-\int_s^t \alpha(\tau)\,\mathrm{d}\tau} \sum_{i\geq 1} g_i |p_i^*(s) - p_i^{**}(s)|, \tag{10}$$

for any $t \geq s \geq 0$ and any initial conditions $\mathbf{p}^*(s)$ and $\mathbf{p}^{**}(s)$.

Proof. The property $\mathbf{p}(t) \in \Omega$ for any $t \geq 0$ allows putting $p_0(t) = 1 - \sum_{i\geq 1} p_i(t)$. Then from (4) we obtain the following system:

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{11}$$

where

$$\mathbf{z}(\mathbf{t}) = (p_1(t), p_2(t), \dots)^T,$$

$$\mathbf{f}(\mathbf{t}) = (\lambda_0(t), 0, 0, \dots)^T,$$

$$B(t) = (b_{ij}(t))_{i,j=1}^{\infty}$$

and

$$b_{ij} = \begin{cases} -(\lambda_0 + \lambda_1 + \mu_1) & \text{if } i = j = 1, \\ \mu_2 - \lambda_0 & \text{if } i = 1, j = 2, \\ -\lambda_0 & \text{if } i = 1, j > 2, \\ -(\lambda_j + \mu_j) & \text{if } i = j > 1, \\ \mu_j & \text{if } i = j - 1 > 1, \\ \lambda_j & \text{if } i = j + 1 > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

This is a linear non-homogeneous differential system the solution of which can be written as

$$\mathbf{z}(t) = V(t,0)\mathbf{z}(0) + \int_0^t V(t,\tau)\mathbf{f}(\tau)\,\mathrm{d}\tau, \tag{13}$$

where $V(t,z)$ is the Cauchy operator of (11) (see, e.g., Zeifman, 1995a).

Consider Eqn. (11) in the space l_{1D}. We have $\|\mathbf{f}(t)\|_{1D} = d_1\lambda_0(t) \leq L$ for almost all $t \geq 0$. On the

other hand,

$$DBD^{-1} = \begin{pmatrix} -(\lambda_0+\mu_1) & \frac{d_1}{d_2}\mu_2 & & & \\ \frac{d_2}{d_1}\lambda_1 & -(\lambda_1+\mu_2) & & & \\ 0 & \frac{d_3}{d_2}\lambda_2 & & & \\ & \ddots & & & \\ & & & & \\ 0 & \ddots & & & \\ \frac{d_2}{d_3}\mu_3 & 0 & \ddots & & \\ -(\lambda_2+\mu_3) & \frac{d_3}{d_4}\mu_4 & 0 & & \\ \ddots & \ddots & & & \\ \ddots & \ddots & \ddots & & \end{pmatrix}. \tag{14}$$

Hence

$$\begin{aligned} &\|B(t)\|_{1D} \\ &= \sup_{k\geq 0}\Big(\lambda_k(t)+\mu_{k+1}(t)+\frac{d_{k+1}}{d_k}\lambda_{k+1}(t) \\ &\quad + \frac{d_{k-1}}{d_k}\mu_k(t)\Big) \\ &\leq \sup_{k\geq 0}\Big(\big(-\lambda_k(t)-\mu_{k+1}(t)+\frac{d_{k+1}}{d_k}\lambda_{k+1}(t) \\ &\quad + \frac{d_{k-1}}{d_k}\mu_k(t)\big)+2\lambda_k(t)+2\mu_{k+1}(t)\Big) \\ &\leq 2\,(L+M)-\alpha(t) \end{aligned} \tag{15}$$

for almost all $t \geq 0$.

Then $\mathbf{f}(t)$ and $B(t)$ are bounded and locally integrable on $[0,\infty)$ as a vector function and an operator function in l_{1D}, respectively.

Now we have the following bound for the logarithmic norm $\gamma\,(B(t))$ in l_{1D}:

$$\begin{aligned} \gamma\,(B)_{1D} &= \gamma\left(DB(t)D^{-1}\right)_1 \\ &= \sup_{i\geq 0}\Big(\frac{d_{i+1}}{d_i}\lambda_{i+1}(t)+\frac{d_{i-1}}{d_i}\mu_i(t) \\ &\quad -\big(\lambda_i(t)+\mu_{i+1}(t)\big)\Big) \\ &= -\inf_{k\geq 0}\left(\alpha_k\,(t)\right) = -\alpha(t), \end{aligned} \tag{16}$$

in accordance with (7). Hence

$$\|V(t,s)\|_{1D} \leq e^{-\int\limits_s^t \alpha(\tau)\,\mathrm{d}\tau}. \tag{17}$$

Therefore, the bound (9) holds.

On the other hand, we have

$$\begin{aligned} &\|\mathbf{z}\| \\ &\leq \sum_{i\geq 1} d_i|p_i| \\ &= d_1\left(\Big|\sum_{i\geq 1}p_i+\sum_{i\geq 2}-p_i\Big|\right) \\ &\quad + d_2\left(\Big|\sum_{i\geq 2}p_i+\sum_{i\geq 3}-p_i\Big|\right)+\dots \\ &\leq d_1\Big|\sum_{i\geq 1}p_i\Big|+2d_2\Big|\sum_{i\geq 2}p_i\Big|+\dots \leq 2\|\mathbf{z}\|_{1D}, \end{aligned} \tag{18}$$

and $\|\mathbf{p}^* - \mathbf{p}^{**}\| \leq 2\|\mathbf{z}\|$ for any $\mathbf{p}^*, \mathbf{p}^{**}$ and the corresponding $\mathbf{z}$. Hence the bound (10) holds. ■

Corollary 1. *Let, in addition, the numbers d_i grow sufficiently fast so that $W > 0$. Then $X(t)$ has the limiting mean, say $\phi(t)$, and the following bound holds:*

$$|\phi(t) - E_k(t)| \leq \frac{4}{W}e^{-\int\limits_0^t \alpha(\tau)\,\mathrm{d}\tau}\|\mathbf{p}(0)-\mathbf{e_k}\|_{1D}. \tag{19}$$

Proof. The bound (19) follows from (9) and from the inequality

$$\begin{aligned} \|\mathbf{z}\|_{1D} &= d_1\Big|\sum_{i=1}^{\infty}p_i\Big|+d_2\left|\sum_{i=2}^{\infty}p_i\right|+\dots \\ &\geq W\sum_{k\geq 1}k\Big|\sum_{i\geq k}p_i\Big| \geq \frac{W}{2}\|\mathbf{z}\|_{1E}. \end{aligned} \tag{20}$$

■

3. Truncations

Now we consider the family of "truncated" processes $X_N(t)$ on the state space $E_N = \{0,1,\dots,N\}$, where the birth rates are $\lambda_n(t)$, $n \in E_{N-1}$ and the death rates are $\mu_n(t)$, $n \in E_N$ (and with the intensity matrix A_N). The truncated process has the vector of probabilities governed by the forward Kolmogorov differential system

$$\frac{\mathrm{d}\mathbf{p}_N}{\mathrm{d}t} = A_N(t)\mathbf{p}_N. \tag{21}$$

Below we will identify the finite vector with entries $(a_1,\dots,a_N)$ and the infinite vector with the same first N coordinates and the others equal to zero. The same identification will be assumed also for the rate matrix A_N, triangular matrix D_N, and so on.

Theorem 2. *Let a BDP with the rates $\lambda_k(t)$ and $\mu_k(t)$ be given. Assume that there exists a sequence $\{d_i\}$ such that*

$$\int\limits_0^{\infty}\alpha(t)\,\mathrm{d}t = +\infty. \tag{22}$$

Then $X_N(t)$ is weakly ergodic for any N, and the following bounds hold:

$$\|\mathbf{p}^*{}_N(t) - \mathbf{p}^{**}{}_N(t)\|_{1D} \le e^{-\int_s^t \alpha(\tau)\,\mathrm{d}\tau} \|\mathbf{p}^*{}_N(s) - \mathbf{p}^{**}{}_N(s)\|_{1D}, \quad (23)$$

$$\|\mathbf{p}^*{}_N(t) - \mathbf{p}^{**}{}_N(t)\| \le 4e^{-\int_s^t \alpha(\tau)\,d\tau} \sum_{i\ge 1} g_i |p^*_{i,N}(s) - p^{**}_{i,N}(s)| \le 8g_N e^{-\int_s^t \alpha(\tau)\,d\tau}, \quad (24)$$

for any $t \ge s \ge 0$ and any initial conditions $\mathbf{p}^*{}_N(s)$ and $\mathbf{p}^{**}{}_N(s)$.

Proof. The property $\mathbf{p}_N(t) \in \Omega$ for any $t \ge 0$ allows setting $p_{0,N}(t) = 1 - \sum_{i\ge 1} p_{i,N}(t)$. Then from (21) we obtain the following system:

$$\frac{\mathrm{d}\mathbf{z}_N(t)}{\mathrm{d}t} = B_N(t)\mathbf{z}_N(t) + \mathbf{f}_N(t), \quad (25)$$

where $\mathbf{z_N(t)} = (p_{1,N}(t), \dots, p_{N,N}(t))^T$, $\mathbf{f_N(t)} = (\lambda_0(t), 0, \dots, 0)^T$, $B_N(t) = (b_{ij}(t))_{i,j=1}^N$ and

$$b_{ij} = \begin{cases} -(\lambda_0 + \lambda_1 + \mu_1) & \text{if } i = j = 1, \\ \mu_2 - \lambda_0 & \text{if } i = 1, j = 2, \\ -\lambda_0 & \text{if } i = 1, j > 2, \\ -(\lambda_j + \mu_j) & \text{if } i = j > 1, \\ \mu_j & \text{if } 1 < i = j - 1 \le N, \\ \lambda_j & \text{if } 1 < i = j + 1 \le N - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

The solution of Eqn. (25) can be written as

$$\mathbf{z}_N(t) = V_N(t,0)\mathbf{z}_N(0) + \int_0^t V_N(t,\tau)\mathbf{f}_N(\tau)\,\mathrm{d}\tau, \quad (27)$$

where $V_N(t,z)$ is the Cauchy operator of (25).

Now the following bound for the logarithmic norm $\gamma(B_N(t))$ in the $l_{1D,N}$ norm holds:

$$\begin{aligned} \gamma(B_N)_{1D,N} &= \gamma\left(D_N B_N(t) D_N^{-1}\right)_1 \\ &\le \max_{0\le i\le N-1} \left(\frac{d_{i+1}}{d_i}\lambda_{i+1}(t) + \frac{d_{i-1}}{d_i}\mu_i(t) - (\lambda_i(t) + \mu_{i+1}(t)) \right) \\ &\le \sup_{k\ge 0}(-\alpha_k(t)) = -\alpha(t), \end{aligned} \quad (28)$$

in accordance with (25). Hence

$$\|V_N(t,s)\|_{1D,N} \le e^{-\int_s^t \alpha(\tau)\,\mathrm{d}\tau}. \quad (29)$$

Our claim follows from this bound. ∎

Denote by $E_{k,N}(t) = E\{X_N(t)\,|X_N(0) = k\}$ the mathematical expectation of the truncated process under the initial condition $\mathbf{e_k}$.

Corollary 2. *Under the assumptions of Theorem 2 the truncated process $X_N(t)$ has the limiting mean for any N, say $\phi_N(t)$, and the following bounds hold:*

$$|\phi_N(t) - E_{k,N}(t)| \le \frac{4}{W} e^{-\int_0^t \alpha(\tau)\,\mathrm{d}\tau} \|\mathbf{p}_N(0) - \mathbf{e_k}\|_{1D,N}, \quad (30)$$

and

$$|\phi_N(t) - E_{k,N}(t)| \le 8Ng_N, \quad (31)$$

for any k and any $t \ge 0$.

Now if we suppose in addition that there exist positive R and a such that

$$e^{-\int_s^t \alpha(u)\,\mathrm{d}u} \le Re^{-a(t-s)} \quad (32)$$

for any $0 \le s \le t$, then the process $X(t)$ is *exponentially* weakly ergodic.

Moreover, consider a "new" sequence $\{d_i^*\}$ such that $d_i^* = d_i^2$, and the correspondent inequality

$$e^{-\int_s^t \alpha^*(u)\,\mathrm{d}u} \le R^* e^{-a^*(t-s)} \quad (33)$$

for any $0 \le s \le t$.

Theorem 3. *Let the assumptions of Theorem 2 hold. In addition, let the inequalities (32) and (33) hold for some positive R, R^*, a, a^*. Then we have the following bounds of truncation:*

$$\|\mathbf{p}(t) - \mathbf{p}_N(t)\| \le \frac{8LRR^*(L+M)}{aa^*d_N}, \quad (34)$$

for $\mathbf{p}(0) = \mathbf{p}_N(0) = \mathbf{e_0}$,

$$|E_0(t) - E_{0,N}(t)| \le \frac{4LRR^*(L+M)}{aa^*d_N W}. \quad (35)$$

Proof. Consider the correspondent non-homogeneous equation for $X(t)$ (11) in the form

$$\frac{\mathrm{d}\mathbf{z}(t)}{\mathrm{d}t} = B_N(t)\mathbf{z}(t) + (B(t) - B_N(t))\,\mathbf{z}(t) + \mathbf{f}(t). \quad (36)$$

Then we have

$$\mathbf{z}(t) = V_N(t)\mathbf{z}(0) + \int_0^t V_N(t,\tau)(B(\tau) - B_N(\tau))\mathbf{z}(\tau)\,\mathrm{d}\tau + \int_0^t V_N(t,\tau)\mathbf{f}(\tau)\,\mathrm{d}\tau. \quad (37)$$

We note that the following equality holds:

$$V_N(t,s)\mathbf{f}(s) = V_N(t,s)\mathbf{f_N}(s),$$

for any $0 \le s \le t$. Hence we have

$$\begin{aligned}\mathbf{z}(t) - \mathbf{z}_N(t) &\\ = \int_0^t V_N(t,\tau)(B(\tau) - B_N(\tau))\mathbf{z}(\tau)\,\mathrm{d}\tau &\end{aligned} \quad (38)$$

if $\mathbf{p}(0) = \mathbf{p}_N(0)$.

Write $B(t) - B_N(t) = \left(b^*_{ij}(t)\right)_{i,j=1}^{\infty}$. Then

$$b^*_{ij} = \begin{cases} 0 & \text{if } i = 1, j \le N, \\ -\lambda_0 & \text{if } i = 1, j > N, \\ 0 & \text{if } 2 \le i \le N-1, j \le 1, \\ -\lambda_N & \text{if } i = j = N, \\ -(\lambda_j + \mu_j) & \text{if } i = j > N, \\ \mu_j & \text{if } i \ge N, j = i+1 \\ \lambda_j & \text{if } N > i, j = i-1, \\ 0 & \text{otherwise.} \end{cases} \quad (39)$$

Hence

$$(B(\tau) - B_N(\tau))\,\mathbf{z}(\tau) = (r_1(\tau), r_2(\tau), \dots)^T, \quad (40)$$

where

$$r_i = \begin{cases} -\lambda_0 \sum_{i>N} p_i & \text{if } i = 1, \\ 0 & \text{if } 2 \le i \le N-1, \\ -\lambda_N p_N + \mu_{N+1} p_{N+1} & \text{if } i = N, \\ \lambda_{i-1} p_{i-1} - (\lambda_i + \mu_i) p_i & \\ +\mu_{i+1} p_{i+1} & \text{if } i > N. \end{cases} \quad (41)$$

Now we have

$$D\,(B(\tau) - B_N(\tau))\,\mathbf{z}(\tau) = (r^*_1(\tau), r^*_2(\tau), \dots)^T, \quad (42)$$

where

$$r^*_i = \begin{cases} -d_1\lambda_0 \sum_{i>N} p_i & \text{if } i = 1, \\ 0 & \text{if } 2 \le i \le N, \\ d_i\,(\lambda_i p_j - \mu_{i+1} p_{i+1}) & \text{if } i > N, \end{cases} \quad (43)$$

and

$$\begin{aligned}\|(B(\tau) - B_N(\tau))\mathbf{z}(\tau)\|_{1D} &\\ = \sum_i |r^*_i(\tau)| &\\ \le 2(L+M)\sum_{i=N}^{\infty} d_i p_i(\tau). &\end{aligned} \quad (44)$$

Therefore, in the l_{1D}-norm the following bound holds:

$$\begin{aligned}&\|\mathbf{z}(t) - \mathbf{z}_N(t)\| \\ &\le \int_0^t \|V_N(t,\tau)\|\|(B(\tau) - B_N(\tau))\mathbf{z}(\tau)\|\,\mathrm{d}\tau. \\ &\le 2(L+M)\int_0^t Re^{-a(t-\tau)} \sum_{i=N}^{\infty} \mathrm{d}_i p_i(\tau)\,\mathrm{d}\tau \\ &\le \frac{2R(L+M)}{a} \sup_{0\le\tau\le t} \sum_{i=N}^{\infty} \mathrm{d}_i p_i(\tau).\end{aligned} \quad (45)$$

For estimating the right-hand side of (45) we consider now the upper and lower bounds for $\|\mathbf{z}(t)\|_{1D^*}$. Firstly, one has

$$\begin{aligned}&\|\mathbf{z}(t)\|_{1D^*} \\ &\le \|V(t)\|_{1D^*}\|\mathbf{z}(0)\|_{1D^*} \\ &\quad + \int_0^t \|V(t,\tau)\|_{1D^*}\|\mathbf{f}(\tau)\|_{1D^*}\,\mathrm{d}\tau \\ &\le R^* e^{-a^* t}\|\mathbf{z}(0)\|_{1D^*} + \frac{LR^*}{a^*}\end{aligned} \quad (46)$$

since $\lambda_0(t) \le L$ for almost all $t \ge 0$.

Set $X(0) = 0$. Then $\mathbf{p}(0) = (1,0,0,\dots)^T$, $\mathbf{z}(0) = \mathbf{0}$, and hence

$$\|\mathbf{z}(t)\|_{1D^*} \le \frac{LR^*}{a^*} \quad (47)$$

for any $t \ge 0$.

On the other hand, all $p_i(t) \ge 0$. Therefore

$$\begin{aligned}&\|\mathbf{z}(t)\|_{1D^*} \\ &\quad = \sum_{i\ge1} p_i(t) \sum_{k=1}^{i} d_k^2 \ge \sum_{i\ge N} d_i^2 p_i(t) \\ &\quad \ge d_N \sum_{i\ge N} d_i p_i(t),\end{aligned} \quad (48)$$

and

$$\sum_{i=N}^{\infty} d_i p_i(t) \le \frac{\|\mathbf{z}(t)\|_{1D^*}}{d_N} \le \frac{LR^*}{a^* d_N}, \quad (49)$$

for any $t \ge 0$.

Finally, we have

$$\|\mathbf{z}(t) - \mathbf{z}_N(t)\|_{1D} \le \frac{2LRR^*(L+M)}{aa^* d_N}, \quad (50)$$

for $\mathbf{p}(0) = \mathbf{p}_N(0) = \mathbf{e_0}$, and the first bound of our theorem follows now from the inequality $d\|\mathbf{z}\| \le 2\|\mathbf{z}\|_{1D}$.

The inequality

$$\begin{aligned}\|\mathbf{z}\|_{1D} &= d_1\Big|\sum_{i=1}^{\infty} p_i\Big| + d_2\Big|\sum_{i=2}^{\infty} p_i\Big| + \dots \\ &\ge W\sum_{k\ge1} k\Big|\sum_{i\ge k} p_i\Big| \ge \frac{W}{2}\|\mathbf{z}\|_{1E}\end{aligned} \quad (51)$$

implies the second bound. ■

Denote by $\mathbf{p}(t,k)$ the probability distribution of $X(t)$ with the initial condition $\mathbf{p}(0,k) = \mathbf{e_k}$, i.e., with the initial condition $X(0) = k$.

Theorem 4. *Under the assumptions of Theorem 3, the following bounds hold:*

$$\|\mathbf{p}(t,k) - \mathbf{p}_N(t,0)\| \le 8Rg_k e^{-at} + \frac{8LRR^*(L+M)}{aa^*d_N}, \quad (52)$$

$$\|\mathbf{p}(t,k) - \mathbf{p}_N(t,0)\| \le 8R^* g_k^* e^{-a^*t} + \frac{8LRR^*(L+M)}{aa^*d_N}, \quad (53)$$

and

$$|E_k(t) - E_{0,N}(t)| \le \frac{4Rg_k}{W} e^{-at} + \frac{4LRR^*(L+M)}{aa^*d_N W}, \quad (54)$$

$$|E_k(t) - E_{0,N}(t)| \le \frac{4R^* g_k^*}{W^*} e^{-a^*t} + \frac{4LRR^*(L+M)}{aa^*d_N W}. \quad (55)$$

Proof. The claim follows from the inequalities (10), (19), (34) and (35). ■

We formulate separately the correspondent statement for the important case of periodic (1-periodic, for definiteness) intensities.

Note firstly that in this situation the condition (22) implies the inequality (32) for

$$a = \int_0^1 \alpha(t)\,\mathrm{d}t, \quad R = e^K, \quad K = \sup_{|t-s|\le 1} \int_s^t \alpha(u)\,\mathrm{d}u.$$

Similarly, the assumption

$$\int_0^\infty \alpha^*(t)\,\mathrm{d}t = \infty$$

implies the inequality (33) for

$$a^* = \int_0^1 \alpha^*(t)\,\mathrm{d}t,$$
$$R^* = e^{K^*},$$
$$K^* = \sup_{|t-s|\le 1} \int_s^t \alpha^*(u)\,\mathrm{d}u.$$

Theorem 5. *Under the assumptions of Theorem 3, let the birth and death intensities be* 1*-periodic. Then there exist a limiting* 1*-periodic probability distribution* $\pi(t)$*, say, and the respective limiting* 1*-periodic mean* $\phi(t)$*, and the following bounds hold:*

$$\|\pi(t) - \mathbf{p}_N(t,0)\| \le \frac{4LR^2}{a} e^{-at} + \frac{8LRR^*(L+M)}{aa^*d_N}, \quad (56)$$

$$\|\pi(t) - \mathbf{p}_N(t,0)\| \le \frac{4LR^{*2}}{a^*} e^{-a^*t} + \frac{8LRR^*(L+M)}{aa^*d_N}, \quad (57)$$

and

$$|\phi(t) - E_{0,N}(t)| \le \frac{4LR^2}{aW} e^{-at} + \frac{4LRR^*(L+M)}{aa^*d_N W}. \quad (58)$$

$$|\phi(t) - E_{0,N}(t)| \le \frac{4LR^{*2}}{a^*W^*} e^{-a^*t} + \frac{4LRR^*(L+M)}{aa^*d_N W}. \quad (59)$$

Proof. The existence of the 1-periodic solution $\pi(t)$ of the forward Kolmogorov system follows from the exponential ergodicity in l_{1D}-norm or l_{1D^*}-norm, respectively. For obtaining the first summands in all inequalities, we consider the respective expression on the right-hand side of (9). Namely, we have, instead of $\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}$, the following quantity:

$$\begin{aligned}\|\pi(0) - \mathbf{e}_0\|_{1D} &= \|\pi(0)\|_{1D} \\ &= \|\mathbf{q}(0)\|_{1D} \\ &\le \limsup_{t\to\infty} \|\mathbf{q}(t)\|_{1D}, \end{aligned} \quad (60)$$

where $\pi(t)$ and $\mathbf{q}(t)$ are the correspondent solutions of (4) and (11). Now we have, instead of (46), the following bound:

$$\begin{aligned}&\|\mathbf{q}(t)\|_{1D} \\ &\le \|V(t)\|_{1D}\|\mathbf{q}(0)\|_{1D} \\ &\quad + \int_0^t \|V(t,\tau)\|_{1D}\|\mathbf{f}(\tau)\|_{1D}\,\mathrm{d}\tau \\ &\le Re^{-at}\|\mathbf{q}(0)\|_{1D} + \frac{LR}{a}, \end{aligned} \quad (61)$$

and

$$\|\pi(0) - \mathbf{e}_0\|_{1D} \le \limsup_{t\to\infty} \|\mathbf{q}(t)\|_{1D} \le \frac{LR}{a}. \quad (62)$$

Similarly, (46) implies the bound

$$\|\pi(0) - \mathbf{e}_0\|_{1D^*} \le \limsup_{t\to\infty} \|\mathbf{q}(t)\|_{1D^*} \le \frac{LR^*}{a^*}. \quad (63)$$

Finally, the claim follows from the inequalities (10), (19), (34), (35), and (62), (63). ■

Remark 1. The best previous bounds contain an additional factor of t on the right-hand sides of (34) and (35), (see Zeifman *et al*., 2006).

4. $M_t/M_t/S$ queue

In this section we deal with a classic queueing model. An $M_t/M_t/S$ is a stochastic process whose values correspond to the numbers of customers in the system, including any currently in service. Arrivals occur at the rate $\lambda(t)$ according to an inhomogeneous Poisson process and move the process from state i to $i+1$. Service times have a nonstationary exponential distribution with the parameter $\mu(t)$ in the $M_t/M_t/S$ queue. There are S servers which serve from the front of the queue. If there are less than S jobs, some of the servers will be idle. If there are more than S jobs, the jobs queue in a buffer. The buffer is of infinite size, so there is no limit on the number of customers it can contain. Let $X = X(t)$, $t \geq 0$ be a queue-length process for the $M_t/M_t/S$ queue. This is a BDP with the birth and death rates $\lambda_n(t) = \lambda(t)$ and $\mu_n(t) = \min(n, S)\,\mu(t)$, respectively. There are a number of investigations of this model in a general situation, and, especially, in the case of periodic intensities and for the simplest $M_t/M_t/1$ model (see Di Crescenzo and Nobile, 1995; Knessl, 2000; Knessl and Yang, 2002; Mandelbaum and Massey, 1995; Margolius, 2007a; 2007b; Massey and Whitt, 1994; Massey, 2002; Zhang and Coyle, 1991).

Bounds on the rate of convergence, truncations and stability for this process were obtained by Granovsky and Zeifman (2004), Zeifman (1995b; 1995a), Zeifman *et al*. (2006) as well as Zeifman and Korotysheva (2012). Here we improve estimates of the truncation error.

Assume that there exist $\delta > 1$, a function $\theta^*(t)$ and positive numbers R^* and a^* such that

$$S\mu(t) - \delta^2\lambda(t) \geq \theta^*(t) \tag{64}$$

and

$$e^{-\int_s^t \left(1-\delta^{-2}\right)\theta^*(u)\,\mathrm{d}u} \leq R^* e^{-a^*(t-s)}, \tag{65}$$

for any s and t, $0 \leq s \leq t$.

One can see that for some function $\theta(t)$ and positive numbers R and a the corresponding bounds

$$S\mu(t) - \delta\lambda(t) \geq \theta(t) \tag{66}$$

and

$$e^{-\int_s^t \left(1-\delta^{-1}\right)\theta(u)\,\mathrm{d}u} \leq Re^{-a(t-s)}, \tag{67}$$

for any s and t, $0 \leq s \leq t$, hold, too.

For $S = 1$ we can choose $\delta > 1$ arbitrarily, and if $S > 1$ we suppose $\delta^2 \leq S/(S-1)$. Note that this assumption is unnecessary. It is formulated here only for ease of computation. If we choose $\delta^2 > S/(S-1)$, then we obtain another formula for the bounding of $\alpha(t)$ instead of (71).

Set $d_k = \delta^{k-1}$. Then

$$g_k = \sum_{n=1}^{k} d_n = \frac{\delta^k - 1}{\delta - 1}.$$

Consider

$$\begin{aligned}\alpha_k(t) &= \lambda(t) + \min(k+1, S)\,\mu(t) \\ &\quad - \delta\lambda(t) - \frac{1}{\delta}\min(k, S)\,\mu(t), \quad k \geq 0.\end{aligned} \tag{68}$$

Then

$$\alpha_k(t) \geq \alpha_0(t) \geq \lambda(t) + \mu(t) - \delta\lambda(t) \tag{69}$$

for $k < S$, and

$$\alpha_k(t) \geq (S\mu(t) - \delta\lambda(t))\left(1 - \delta^{-1}\right) \tag{70}$$

for $k \geq S$.

Hence the inequality $\delta \leq \delta^2 \leq S/(S-1)$ implies

$$\begin{aligned}\alpha(t) &\geq \alpha_S(t) \\ &\geq (S\mu(t) - \delta\lambda(t))\left(1 - \delta^{-1}\right) \\ &\geq \left(1 - \delta^{-1}\right)\theta(t),\end{aligned} \tag{71}$$

and we obtain the following theorem.

Theorem 6. *The following bounds of truncation for the queue-length process of the $M_t/M_t/S$ queue hold:*

$$\|\mathbf{p}(t) - \mathbf{p}_N(t)\| \leq \frac{8LRR^*(L+M)}{aa^*d_N}, \tag{72}$$

for $\mathbf{p}(0) = \mathbf{p}_N(0) = \mathbf{e_0}$,

$$|E_0(t) - E_{0,N}(t)| \leq \frac{4LRR^*(L+M)}{aa^*d_N W}, \tag{73}$$

where $g_N = (\delta^N - 1)/(\delta - 1)$.

Moreover, Theorem 4 is valid for *both* sequences $\{d_i\}$ and $\{d_i^*\}$. Hence we obtain the following theorem.

Theorem 7. *The following bounds of truncation for the queue-length process of the $M_t/M_t/S$ queue hold:*

$$\begin{aligned}&\|\mathbf{p}(t,k) - \mathbf{p}_N(t,0)\| \\ &\qquad \leq 8Rg_k e^{-at} + \frac{8LRR^*(L+M)}{aa^*d_N},\end{aligned} \tag{74}$$

$$\begin{aligned}&\|\mathbf{p}(t,k) - \mathbf{p}_N(t,0)\| \\ &\qquad \leq 8R^* g_k^* e^{-a^*t} + \frac{8LRR^*(L+M)}{aa^*d_N},\end{aligned} \tag{75}$$

where $\mathbf{p}(0,k)=\mathbf{e_k}$, $\mathbf{p}_N(0,0)=\mathbf{e_0}$,

$$|E_k(t)-E_{0,N}(t)| \le \frac{4Rg_k}{W}e^{-at}+\frac{4LRR^*(L+M)}{aa^*d_NW}, \quad (76)$$

$$|E_k(t)-E_{0,N}(t)| \le \frac{4R^*g_k^*}{W^*}e^{-a^*t}+\frac{4LRR^*(L+M)}{aa^*d_NW}, \quad (77)$$

$$g_N=\frac{\delta^N-1}{\delta-1}, \quad g_N^*=\frac{\delta^{2N}-1}{\delta^2-1}.$$

Consider the case of 1-periodic arrival and service rates in greater detail. In this case, the conditions (64)–(67) appear substantially simpler. Namely, let the following assumption hold:

$$\int_0^1 \left(S\mu(t)-\lambda(t)\right)\,\mathrm{d}t>0. \quad (78)$$

Then

$$\int_0^1 S\mu(t)\,\mathrm{d}t>\int_0^1\lambda(t)\,\mathrm{d}t. \quad (79)$$

Hence, for a sufficiently small $\eta>0$,

$$\int_0^1 S\mu(t)\,\mathrm{d}t>(1+\eta)\int_0^1\lambda(t)\,\mathrm{d}t. \quad (80)$$

Therefore, setting $\delta_0^2=(1+\eta)>1$, we have

$$\int_0^1 \left(S\mu(t)-\delta_0^2\lambda(t)\right)\,\mathrm{d}t>0. \quad (81)$$

Then

$$\int_0^1 \left(S\mu(t)-\delta_0\lambda(t)\right)\,\mathrm{d}t>0, \quad (82)$$

and we can choose

$$\theta_0^*(t)=S\mu(t)-\delta_0^2\lambda(t), \quad (83)$$

and

$$\theta_0(t)=S\mu(t)-\delta_0\lambda(t). \quad (84)$$

Therefore, we obtain

$$\alpha_0(t)=\left(1-\delta_0^{-1}\right)\theta_0(t), \quad a_0=\int_0^1\alpha_0(t)\,\mathrm{d}t,$$
$$R_0=e^{K_0}, \quad K_0=\sup_{|t-s|\le1}\int_s^t\alpha_0(u)\,\mathrm{d}u. \quad (85)$$

Similarly, we have

$$\alpha_0^*(t)=\left(1-\delta_0^{-2}\right)\theta_0^*(t), \quad a_0^*=\int_0^1\alpha_0^*(t)\,\mathrm{d}t,$$
$$R_0^*=e^{K_0^*}, \quad K_0^*=\sup_{|t-s|\le1}\int_s^t\alpha_0^*(u)\,\mathrm{d}u. \quad (86)$$

Theorem 8. *Let the arrival and service rates in the $M_t/M_t/S$ queue be 1-periodic, and let (78) hold. Then there exist a limiting 1-periodic probability distribution $\pi(t)$, say, and the respective limiting 1-periodic mean $\phi(t)$ for the queue-length process $X(t)$, and the following bounds hold:*

$$\|\pi(t)-\mathbf{p}_N(t,0)\| \le \frac{4LR_0^2}{a_0}e^{-a_0t}+\frac{8LR_0R_0^*(L+M)}{a_0a_0^*d_N}, \quad (87)$$

$$\|\pi(t)-\mathbf{p}_N(t,0)\| \le \frac{4LR_0^{*2}}{a_0^*}e^{-a_0^*t}+\frac{8LR_0R_0^*(L+M)}{a_0a_0^*d_N}, \quad (88)$$

and

$$|\phi(t)-E_{0,N}(t)| \le \frac{4LR_0^2}{a_0W}e^{-a_0t}+\frac{4LR_0R_0^*(L+M)}{a_0a_0^*d_NW}, \quad (89)$$

$$|\phi(t)-E_{0,N}(t)| \le \frac{4LR_0^{*2}}{a_0^*W^*}e^{-a_0^*t}+\frac{4LR_0R_0^*(L+M)}{a_0a_0^*d_NW}. \quad (90)$$

5. Examples

5.1. Example 1. Let $X=X(t)$, $t\ge0$ be a queue-length process for a concrete $M_t/M_t/S$ with $S=2$, $\lambda(t)=1+\sin 0.02\pi t$, $\mu(t)=4+2\cos 0.02\pi t$. This example of a queueing system with periodical intensity functions with a sufficiently large period $T=100$ was considered by Zeifman *et al*. (2006). Now we can obtain essentially more accurate bounds.

Remark 2. This example deals with the situation that can happen if the intensities of arrivals and services have different periods. As this is so if these periods are commensurable, the limit characteristics will also be periodic, although with a rather large period. Here $L=2$, $M=6$.

Set $\delta_0=\sqrt{2}$. Then $d_{0,k}=2^{\frac{k-1}{2}}$ and $d_{0,k}^*=2^{k-1}$. We have

$$W=\inf_{i\ge1}\frac{d_i}{i}=\frac{2}{3},$$
$$W^*=\inf_{i\ge1}\frac{d_i^*}{i}=1,$$
$$g_N=\frac{2^{N/2}-1}{\sqrt{2}-1}, \quad g_N^*=2^N-1.$$

Further,

$$\alpha_0(t)=\left(1-1/\sqrt{2}\right)\left(2\mu(t)-\sqrt{2}\lambda(t)\right),$$

$$a_0 = 9 - 5\sqrt{2} > 1.5,$$

$$\begin{aligned}\alpha_0^*(t) &= 0.5\,(2\mu(t) - 2\lambda(t)) \\ &= 3 + 2\cos 0.02\pi t - \sin 0.02\pi t, \\ a^* &= 3.\end{aligned}$$

Hence we obtain

$$K_0 = \sup_{|t-s|\le 100} \int_s^t \alpha_0(u)\,\mathrm{d}u = \frac{100(3-\sqrt{2})}{\pi},$$

and

$$R_0 = \exp\frac{100(3-\sqrt{2})}{\pi} < e^{60}.$$

Similarly,

$$K_0^* = \sup_{|t-s|\le 100} \int_s^t \alpha_0^*(u)\,\mathrm{d}u = \frac{300}{\pi},$$

and

$$R_0^* \le e^{\frac{300}{\pi}} < e^{100}.$$

Therefore the inequalities (74) and (76) of Theorem 7 imply the following statement.

Proposition 1. *The following bounds hold:*

$$\|\mathbf{p}(k,t) - \mathbf{p}_N(0,t)\| \le 2^{4+k/2}\cdot e^{60-1.5t} + \frac{e^{160}}{10\cdot 2^{\frac{N-17}{2}}}, \quad (91)$$

and

$$\begin{aligned}|E_k(t) - E_{0,N}(t)| &\le \frac{1+\sqrt{2}}{2}\|\mathbf{p}(k,t) - \mathbf{p}_N(0,t)\| \\ &< 2^{\frac{k+9}{2}}\cdot e^{60-1.5t} + \frac{e^{160}}{10\cdot 2^{\frac{N-18}{2}}},\end{aligned} \quad (92)$$

for any k, N and any $t \ge 0$.

Further, we can apply Theorem 8.

Proposition 2. *The queue-length process for the model has a limiting 1-periodic probability distribution $\pi(t)$, the respective limiting 1-periodic mean $\phi(t)$, and the following bounds hold:*

$$\|\pi(t) - \mathbf{p}_N(0,t)\| \le 16\cdot e^{119-1.5t} + \frac{e^{160}}{10\cdot 2^{\frac{N-17}{2}}}, \quad (93)$$

and

$$|\phi(t) - E_{0,N}(t)| \le 8\cdot e^{120-1.5t} + \frac{e^{160}}{10\cdot 2^{\frac{N-18}{2}}},$$

for any N and any $t \ge 0$.

These bounds can be used for the study and construction of limiting characteristics of the queue, as was firstly shown by Zeifman *et al.* (2006).

Particularly, for $N \ge 550$, $t \ge 150$ and the initial conditions $k \le 400$, we obtain error bounds for all characteristics less than 10^{-10}.

Note that the best of previously known estimates (see Zeifman *et al.*, 2006, Example 2(iii)) give us (even for $k = 0$) significantly poorer estimates $N = 945$ and interval $[418, 518]$, and the error bound 10^{-8}.

The behaviour of probabilistic characteristics of the queue-length process is shown in the figures below.

Remark 3. All the characteristics mentioned above (state probabilities, mathematical expectations, etc.) are found in one and the same way, namely, by solving the Cauchy problem with an appropriate initial condition for the truncated Kolmogorov system by the Runge–Kutta method of the fourth order of accuracy. Moreover, as was shown, the dimensionality of the corresponding system appears to be rather moderate. Further, if the figure depicts a characteristic on the interval $[a, a+T]$, then the Cauchy problem is solved on $[0, a+T]$. Furthermore, the figures show that the rate of convergence to the corresponding limit characteristics is sufficiently large so that the parameter a can be chosen not so large.

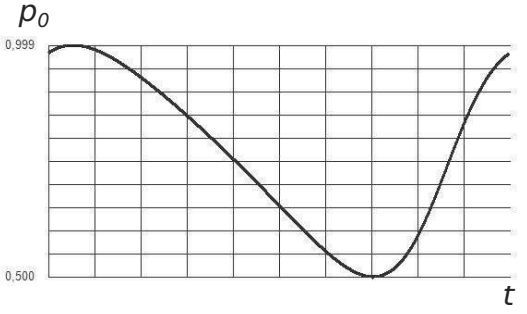

Fig. 1. Probability $\Pr(X(t) = 0 | X(0) = k)$ of an empty queue for $k = 0$ and $t \in [70, 170]$.

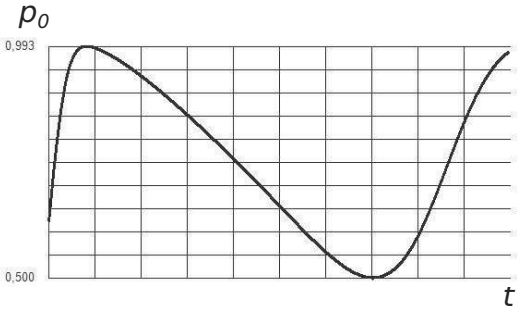

Fig. 2. Probability $\Pr(X(t) = 0 | X(0) = k)$ of an empty queue for $k = 400$ and $t \in [70, 170]$.

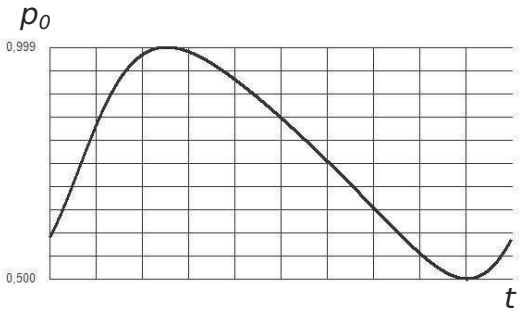

Fig. 3. Probability $\Pr(X(t) = 0|X(0) = k)$ of an empty queue for $k = 0$ and $t \in [150, 250]$.

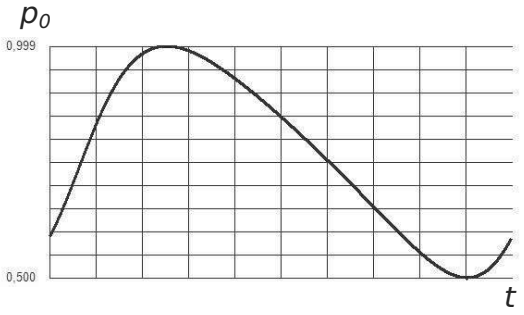

Fig. 4. Probability $\Pr(X(t) = 0|X(0) = k)$ of an empty queue for $k = 400$ and $t \in [150, 250]$.

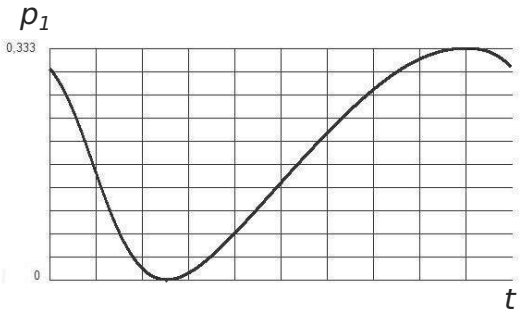

Fig. 5. Probability $\Pr(X(t) = 1|X(0) = k)$ for $k = 0$ and $t \in [150, 250]$.

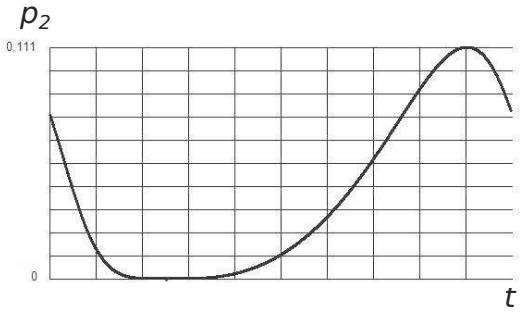

Fig. 6. Probability $\Pr(X(t) = 2|X(0) = k)$ for $k = 0$ and $t \in [150, 250]$.

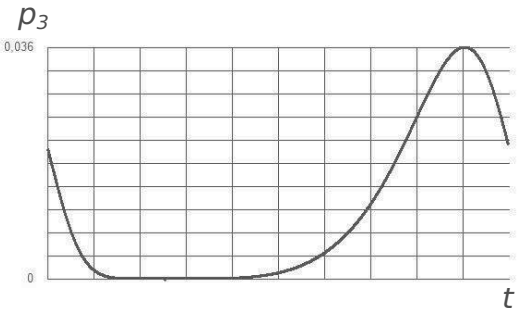

Fig. 7. Probability $\Pr(X(t) = 3|X(0) = k)$ for $k = 0$ and $t \in [150, 250]$.

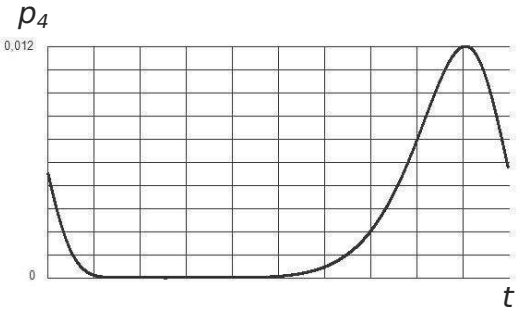

Fig. 8. Probability $\Pr(X(t) = 4|X(0) = k)$ for $k = 0$ and $t \in [150, 250]$.

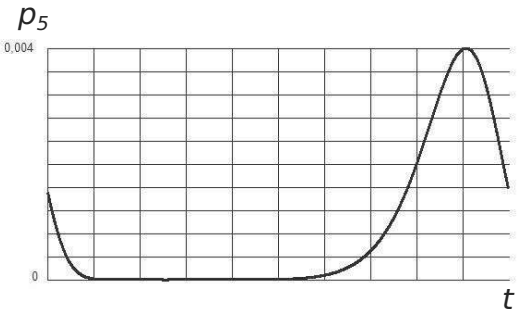

Fig. 9. Probability $\Pr(X(t) = 5|X(0) = k)$ for $k = 0$ and $t \in [150, 250]$.

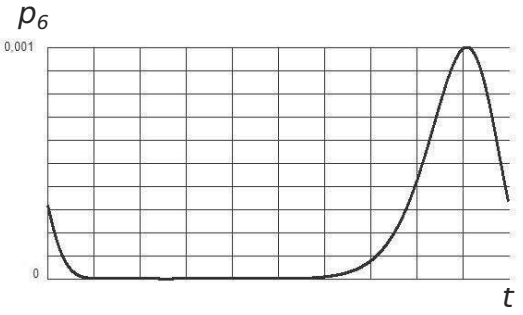

Fig. 10. Probability $\Pr(X(t) = 6|X(0) = k)$ for $k = 0$ and $t \in [150, 250]$.

Remark 4. One can see in Figs. 3 and 4 that they give us very good approximations of the real limiting probability of an empty queue.

Remark 5. In Figs. 5–10 one can see the limiting probability of the corresponding length of the queue.

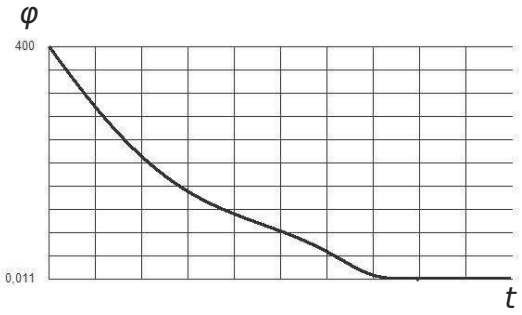

Fig. 11. Mathematical expectation of the length of the queue $E_{400}(t)$, $k = 400$ and $t \in [0, 100]$.

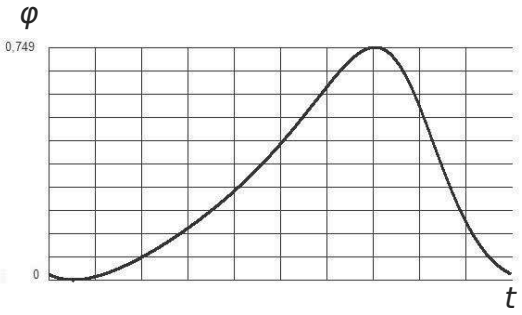

Fig. 12. Mathematical expectation of the length of the queue $E_0(t)$, $k = 0$ and $t \in [70, 170]$.

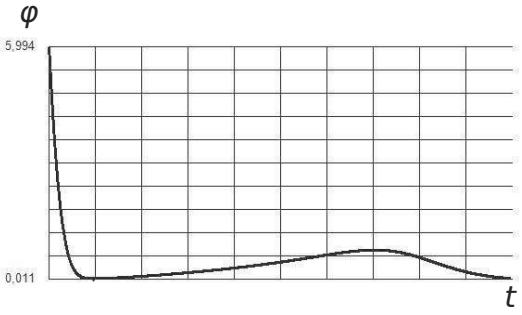

Fig. 13. Mathematical expectation of the length of the queue $E_{400}(t)$, $k = 400$ and $t \in [70, 170]$.

Remark 6. In Figs. 14 and 15 one can see that they give us very good approximation of the real limiting mean (mathematical expectation) of the length of a queue.

5.2. Example 2. Let $X = X(t)$, $t \geq 0$ be now a queue-length process for $M_t/M_t/S$ with $S = 10^{12}$,

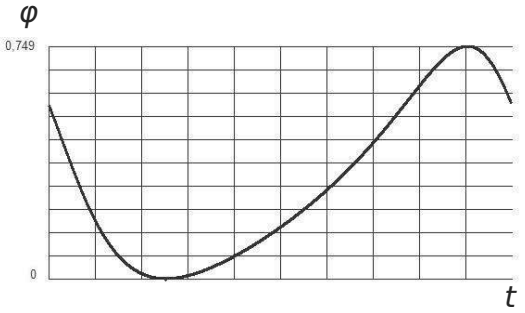

Fig. 14. Mathematical expectation of the length of the queue $E_0(t)$, $k = 0$ and $t \in [150, 250]$.

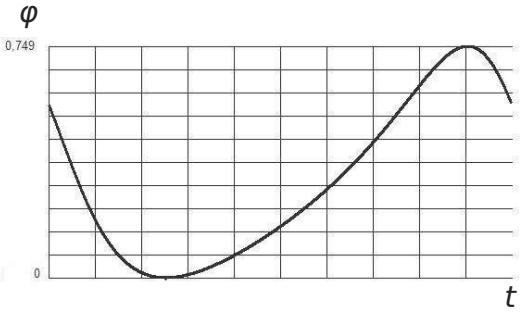

Fig. 15. Mathematical expectation of the length of the queue $E_{400}(t)$, $k = 400$ and $t \in [150, 250]$.

$\lambda(t) = 1 + \sin 2\pi t$, $\mu(t) = 3 + 2\cos 2\pi t$. This example and its analogue for a queueing system with group services was considered by Zeifman *et al.* (2013b).

Now we consider only an ordinary "classic" queueing model and obtain its main limiting characteristics. Here $L = 2$, $M = 5 \cdot 10^{12}$. Put $\delta_0 = \sqrt{2}$. Then $d_{0,k} = 2^{\frac{k-1}{2}}$ and $d^*_{0,k} = 2^{k-1}$. We have

$$W = \inf_{i \geq 1} \frac{d_i}{i} = \frac{2}{3},$$

$$W^* = \inf_{i \geq 1} \frac{d^*_i}{i} = 1,$$

$$g_N = \frac{2^{N/2} - 1}{\sqrt{2} - 1},$$

$$g^*_N = 2^N - 1.$$

Further,

$$\alpha_0(t) = \mu(t) - \left(\sqrt{2} - 1\right) \lambda(t),$$

$$a_0 = 4 - \sqrt{2} > 2,$$

$$\alpha^*_0(t) = \mu(t) - \lambda(t), \quad a^* = 2.$$

Hence we obtain

$$K_0 = \sup_{|t-s| \leq 100} \int_s^t \alpha_0(u)\, \mathrm{d}u = \frac{1 + \sqrt{2}}{\pi},$$

and

$$R_0 = \exp\left(\frac{1+\sqrt{2}}{\pi}\right) < e.$$

Similarly,

$$K_0^* = \sup_{|t-s|\le 100} \int_s^t \alpha_0^*(u)\,\mathrm{d}u = \frac{3}{\pi},$$

and $R_0^* \le e^{\frac{3}{\pi}} < e$.

Therefore, the inequalities (74) and (76) of Theorem 7 imply the following statement.

Proposition 3. *The following bounds hold:*

$$\|\mathbf{p}(k,t) - \mathbf{p}_N(0,t)\| \le 2^{k/2}\cdot e^{4-2t} + \frac{e^2\cdot 10^{13}}{2^{\frac{N-3}{2}}} \quad (94)$$

and

$$|E_k(t) - E_{0,N}(t)| \le 2^{(k+2)/2}\cdot e^{4-2t} + \frac{3e^2\cdot 10^{13}}{2^{\frac{N-1}{2}}} \quad (95)$$

for any k, N and any $t \ge 0$.

Further, Theorem 8 implies the following statement.

Proposition 4. *A queue-length process for the model has a limiting 1-periodic probability distribution $\pi(t)$, the respective limiting 1-periodic mean $\phi(t)$, and the following bounds hold:*

$$\|\pi(t) - \mathbf{p}_N(0,t)\| \le 4\cdot e^{2-2t} + \frac{e^2\cdot 10^{13}}{2^{\frac{N-3}{2}}} \quad (96)$$

and

$$|\phi(t) - E_{0,N}(t)| \le 6\cdot e^{2-2t} + \frac{3e^2\cdot 10^{13}}{2^{\frac{N-1}{2}}} \quad (97)$$

for any N and any $t \ge 0$.

Here for $N \ge 170$, $t \ge 85$, and the initial conditions $k \le N = 170$ we obtain error bounds for all characteristics less than 10^{-10}.

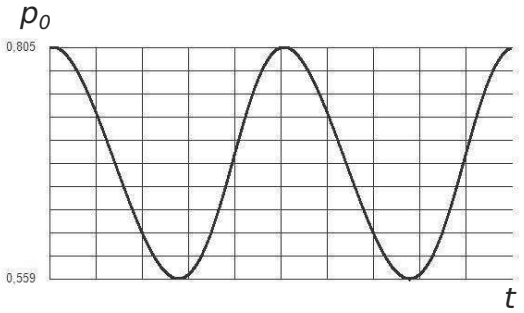

Fig. 16. Probability $\Pr(X(t)=0|X(0)=k)$ of an empty queue for $k=0$ and $t\in[3,5]$.

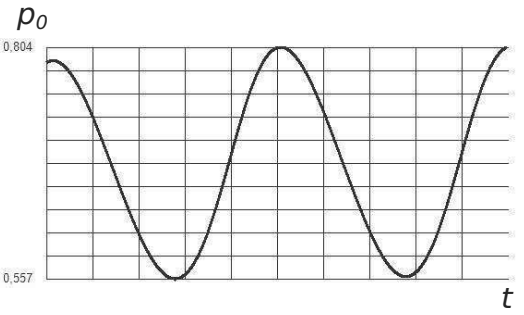

Fig. 17. Probability $\Pr(X(t)=0|X(0)=k)$ of an empty queue for $k=170$ and $t\in[3,5]$.

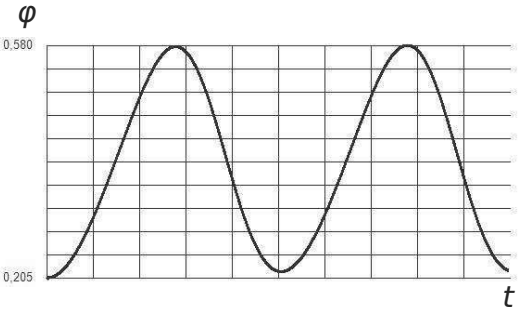

Fig. 18. Mathematical expectation of the length of the queue $E_k(t)$, $k=0$ and $t\in[1,3]$.

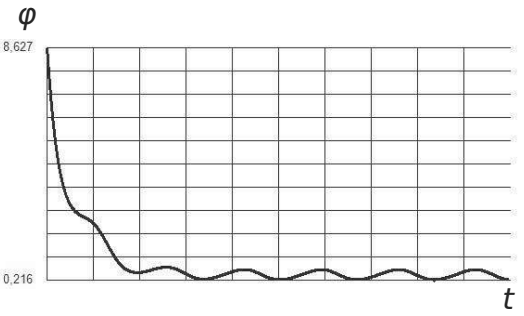

Fig. 19. Mathematical expectation of the length of the queue $E_k(t)$, $k=170$ and $t\in[1,7]$.

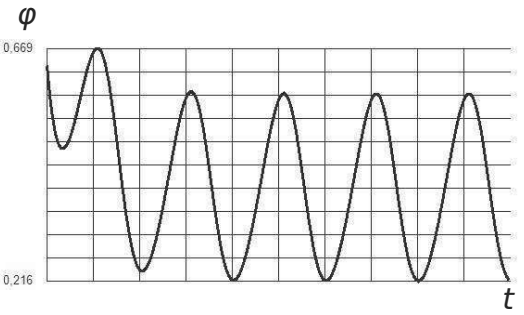

Fig. 20. Mathematical expectation of the length of the queue $E_k(t)$, $k=170$ and $t\in[2,7]$.

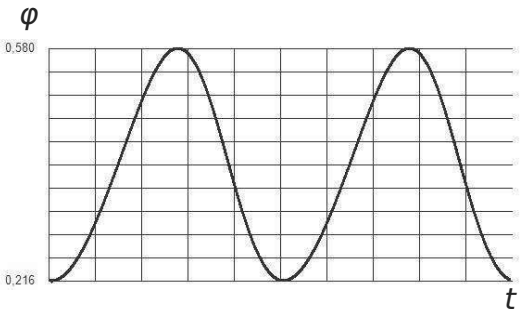

Fig. 21. Mathematical expectation of the length of the queue $E_k(t)$, $k = 0$ and $t \in [3, 5]$.

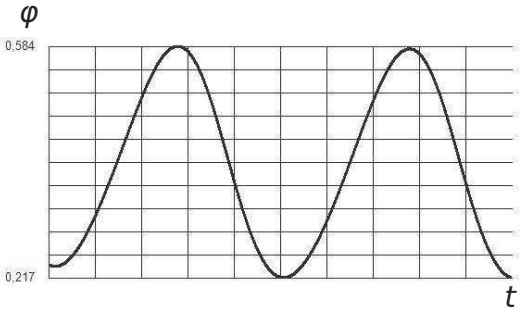

Fig. 22. Mathematical expectation of the length of the queue $E_k(t)$, $k = 170$ and $t \in [3, 5]$.

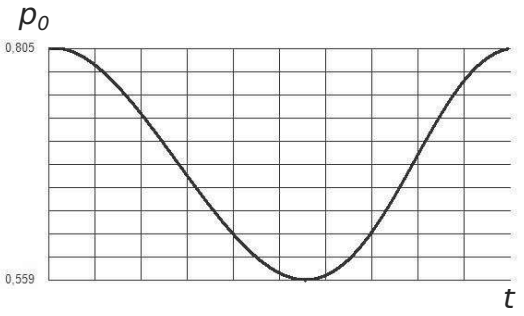

Fig. 23. Approximation of the limiting probability of an empty queue $\Pr(X(t) = 0|X(0) = k)$ for $k = 0$ and $t \in [85, 86]$.

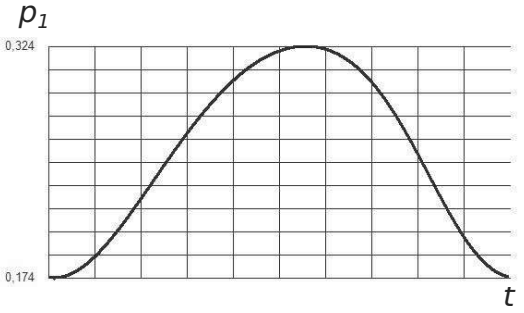

Fig. 24. Approximation of the limiting probability $\Pr(X(t) = 1|X(0) = k)$ for $k = 0$ and $t \in [85, 86]$.

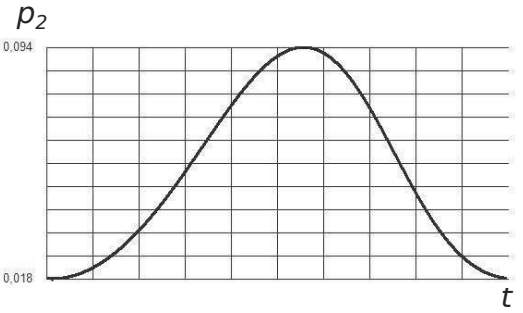

Fig. 25. Approximation of the limiting probability $\Pr(X(t) = 2|X(0) = k)$ for $k = 0$ and $t \in [85, 86]$.

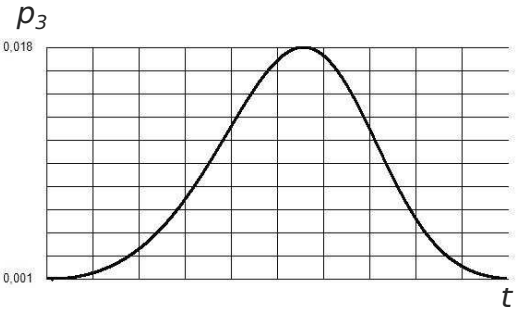

Fig. 26. Approximation of the limiting probability $\Pr(X(t) = 3|X(0) = k)$ for $k = 0$ and $t \in [85, 86]$.

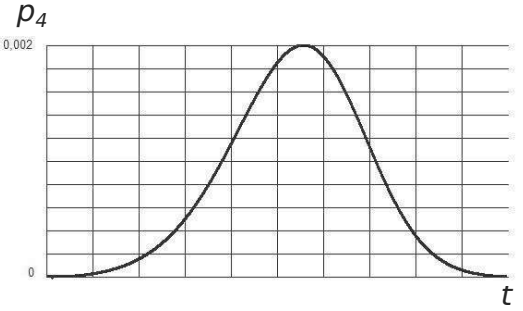

Fig. 27. Approximation of the limiting probability $\Pr(X(t) = 4|X(0) = k)$ for $k = 0$ and $t \in [85, 86]$.

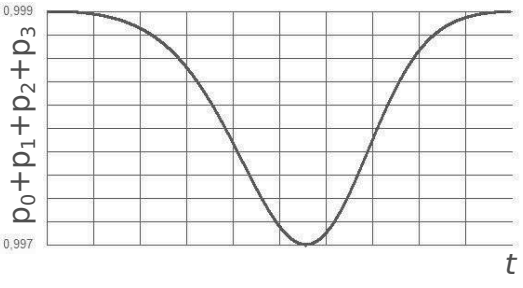

Fig. 28. Approximation of the limiting probability $\Pr(X(t) \leq 3|X(0) = k)$ for $k = 0$ and $t \in [85, 86]$.

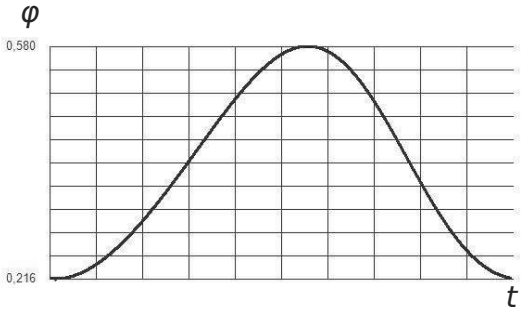

Fig. 29. Approximation of the limiting mathematical expectation of the length of the queue $\phi(t)$ by $E_k(t)$, $k = 0$ and $t \in [85, 86]$.

6. Conclusion

In this paper, we investigated a class of weakly ergodic inhomogeneous birth and death processes and obtained uniform (in time) error bounds of truncation. Our approach also guarantees that we can find limiting characteristics approximately with an arbitrary fixed error, see the detailed discussion by Zeifman *et al*. (2006). Moreover, we can find the limiting characteristics for *any* number of servers S; see the respective example for $S = 10^{12}$ by Zeifman *et al*. (2013b). Arbitrary intensity functions instead of periodic ones can be considered in the same manner. Finally, we would like to remark that all of our results can be applied to birth and death process with catastrophes; see perturbation bounds in the work of Zeifman and Korotysheva (2012) and the bounds on the rate of convergence given by Zeifman *et al*. (2013a).

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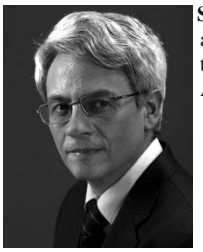

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