Title
On tubular neighborhoods of piecewise linear and topological manifolds.

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Publication Date
1968

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Journal Title: Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967 Journal Volume:
Journal Issue:
Journal Month/Date:
Journal Year: 1968
Pages: 63-80
TN: 2574873

Call Number: QA611 C64 1967
Location: Math

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On Tubular Neighborhoods of Piecewise Linear and Topological Manifolds

MORRIS W. HIRSCH

In geometrical and topological problems we are frequently pre-ented with a pair (X, A) of spaces, and the question arises: What kind of neighborhoods has A in X? In general topology this question leads o the theory of neighborhood retracts; in differential topology all uestions are resolved by the tubular neighborhood theorem. In the ase of topological and/or piecewise linear manifolds the situation is nore difficult. In this article we shall discuss recent positive and negative esults on the existence and uniqueness of tubular neighborhoods of opological and piecewise linear manifolds.

Before going into technicalities, it should be mentioned that the *lock bundles* of Rourke and Sanderson, and independently Morlet, and ato, appear to be a satisfactory substitute for tubular neighborhoods n the piecewise linear category. Existence and uniqueness of normal lock bundles can be proved, as well as the transversality theorems eeded for surgery. There is an obstruction theory for the problem of iving a block bundle the structure of an open or closed cell bundle, or

vector bundle. The obstructions may be non-zero; there is no more reason to expect a random block bundle to have a bundle structure than to expect a random vector bundle to have a complex structure. The theory of tubular neighborhoods, then, for the piecewise linear category has an interest which is mainly technical. It is amusing to see how far existence theorems can be pushed, and what kinds of pathology can occur, but for general purposes, especially in connection with differential and algebraic topology, block bundles appear to be adequate.

DEFINITIONS

Three categories of manifolds will be considered: topological, piecewise linear (or PL), and smooth. For simplicity, manifolds are assumed to have empty boundaries unless the contrary is indicated. In each category the notion of bundle is defined, as well as that of microbundle; thanks to the Kister-Mazur theorem [19; 21] they are essentially equivalent.

Let (M, A) be a pair in one of the categories; M is a manifold and A a submanifold. An *open tube* for A in M is a bundle $p: E \to A$ such that $E \subset M$ is a neighborhood of A, p is a retraction, and the fibres are open k-cells where $k = \dim M - \dim A$. A *closed tube* is a similar bundle whose fibres are closed cells.

Let A_i be a submanifold of M for $i = 1, 2$, and let $p_i: E_i \to A_i$ be open tubes. These tubes are *homeomorphic* if there is an isomorphism $f: (M, A_1) \to (M, A_2)$ in the category and a neighborhood U of A in E_1 such that $fU \subset E_2$, and the following diagram commutes:

$$\begin{array}{ccc} E_1 \supset U & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ A_1 & \xrightarrow[f]{} & A_2 \end{array}$$

For closed tubes it is required that $fE_1 = E_2$. Two tubes are *isotopic* if they are homeomorphic by a homeomorphism isotopic to the identity. If $A_1 = A_2$ it is required that the isotopy be fixed on A_1.

THEOREMS

The classical result for smooth manifolds is the following.

TUBULAR NEIGHBORHOOD THEOREM FOR SMOOTH MANIFOLDS. In the smooth category every submanifold has a closed tube (and therefore also an open tube). If the submanifold is a closed subset, any two closed or open tubes are isotopic.

In the smooth category it is easy to see that a cell bundle has an essentially unique linear structure. For this reason open tubes are considered as vector bundles and closed tubes as orthogonal disk bundles.

In the topological category, the tubular neighborhood theorem is stably true:

STABLE TUBULAR NEIGHBORHOOD THEOREM FOR TOPOLOGICAL MANIFOLDS. If (M, A) is a topological manifold pair, there is an integer q depending only on $\dim A$ such that $A \times O$ has an open tube in $M \times R^q$. If E and E' are open tubes for A, then $E \times R^q$ and $E' \times R^q$ are isotopic tubes on $A \times O$ in $M \times R^q$.

The existence part of this theorem is due to Milnor [24]. Uniqueness up to isotopy is essentially proved in Lashof-Rothenberg [23]; see also Hirsch [12]. To get the full strength of an ambient isotopy some unpublished work of the author is needed. In [12] it is shown that one may take $q = (\dim A + 1)^2$, or if M is smoothable, $q = 4 \dim A - 1$. No doubt better estimates are possible.

The lack of a true tubular neighborhood theorem in the topological category is due not to the ignorance of topologists, but rather to the following counter-example of Rourke and Sanderson [32]:

THEOREM. There is a PL embedding $S^{19} \subset S^{19} \times S^9$ having no topological open tube. The standard $S^{18} \subset R^{27}$ has two closed PL tubes whose corresponding open tubes are not homeomorphic as topological open tubes; as abstract bundles they are trivial.

Not much more is known in the topological category. The collaring theorem of Brown [3] can be interpreted as a tubular neighborhood theorem for the case A has codimension 1. At the other extreme, $\dim A = 1$, some information can be deduced from Brown and Gluck [5].

Haefliger and Wall [11] proved that in the PL category the tubular neighborhood theorem is true in the stable range. We shall give a proof of the following slight sharpening of their result:

THEOREM 1. Let $M^m \subset V^{m+k}$ be a PL submanifold. Then:

(a) M has a PL open tube provided $k \geq \max(m-1, \frac{1}{2}(m+3)]$.
(b) Any two PL open tubes for M are isotopic provided $k \geq \max(m, \frac{1}{2}(m+4))$.
(c) M has a PL closed tube provided $k \geq \max[m, \frac{1}{2}(m+3)]$. In fact every open tube contains a closed subtube.
(d) Any two PL closed tubes are isotopic provided $k \geq \max(m+1, \frac{1}{2}(m+4)]$. As an amusing consequence of (c) we note:

COROLLARY. Every PL manifold has a tangent disk bundle. In case A is a sphere, the dimension restrictions can be weakened:

THEOREM 2. Let $S^m \subset V^{m+k}$ be a PL sphere in a PL manifold. Then S has a PL open tube provided either:

(a) $k \geq \max[m-2, \frac{1}{2}(m+3)]$

or

(b) $k \geq m-3$ and $m \geq 12$.

It is taken for granted that a vector bundle has a trivial line bundle as a Whitney summand if and only if it has a non-zero section. This is not the case in the PL category, unless restrictions are placed on the dimension.

THEOREM 3. Let ξ be a PL bundle over a polyhedron of dimension m, with fibre R^k, having a distinguished zero section. If $k \geq m$ then ξ is a Whitney sum $\eta \oplus \epsilon^1$ if and only if ξ has a non-zero section. If $k \geq m+1$, homotopy classes of such splittings correspond bijectively to homotopy classes of non-zero sections.

We shall construct the following low-dimensional counter-examples to a closed tubular neighborhood theorem:

THEOREM 4. There exist PL submanifolds $M^4 \subset S^7$ and $S^4 \subset M^7$ having no topological closed tubes.

Explicit descriptions of such examples are possible, although we do not give them.

Without going into detail, let us mention some other types of pathology that are known to occur. An open tube might contain no closed subtube [36], or two non-isotopic closed subtubes [34]. A submanifold can have two closed subtubes which are not isomorphic as abstract bundles [22]. It is not known whether this can happen with open tubes, however.

In the next two sections we give some of the geometrical details of the proofs of Theorems 4 and 2. In the last section computational proofs to Theorems 1, 2, and 3 are given. The one new geometrical fact used for these results is the surjectivity of the map $\pi_i(\mathrm{PL}_k, O_k) \to \Gamma_i$ for $k \geq i - 1$, and $k \geq i - 2$ if $i \geq 11$; this last relies on a difficult theorem of J. Cerf. It is remarkable that the simple geometrical fact expressed in Theorem 4 requires for its proof some of the deepest results of differential topology. These include Haefliger's computation of the group of smooth 3-knots in S^6; Kneser's theorem that the orthogonal group is a deformation retract of the homeomorphism group of S^2; and Novikov's theorem on the topological invariance of rational Pontryagin classes!

PROOF OF THEOREM 4

A *Haefliger knot* is an oriented smooth submanifold $T^3 \subset S^6$ which is diffeomorphic to the three sphere S^3. A *framed* Haefliger knot is a pair (T^3, F) where T^3 is a Haefliger knot and $F: T^3 \times D^3 \to S^6$ is a framing of its normal bundle. That is, F is a smooth embedding such that $F(x, O) = x$ for all $x \in T^3$. (Here D^3 is the unit disk in R^3.) Every Haefliger knot can be framed.

The set of diffeotopy classes of framed Haefliger knots is classified by two integer invariants which we shall denote by p and L. To define $p(T, F)$, let $M = M^7(T, F)$ be the smooth 7-manifold obtained by attaching the handle $D^4 \times D^3$ to D^7 by $F': S^3 \times D^3 \to S^6 = \partial D^7$, where F' corresponds to F via an orientation preserving diffeomorphism of T and S^3. Then M has the homology type of S^4, and $H^4(M)$ is infinite cyclic with a distinguished generator; we identify $H^4(M)$ with Z. Define $p(T, F)$ to be the first Pontryagin class $p_1(M)$. It turns out that $p(T, F)$ is equal to twice the obstruction to trivializing the tangent bundle of M; this obstruction lies in $H^4[M; \pi_3(SO_7)] = \mathbf{Z}$.

The other invariant $L(T, F) \in Z$ is the linking number of $F(T \times x)$ and T in S^6, where $x \in \partial D^3$ is an arbitrary point. An alternative description of L is the following. Let $S^4 \subset M$ be the union $CT \cup O \times D^4$

where $CT \subset D^7$ is the cone on T' and $O \times D^4$ is the core of the handle $D^3 \times D^4$. Then $L(T, F)$ can be identified with the obstruction to deforming S^4 into ∂M. (If the inclusion $\partial M \to M$ is made into a fibration, the fibre is homotopically S^2, and the obstruction to a section lies in $H^4[M; \pi_3(S^2)] = \mathbf{Z}$.)

THEOREM 5 (Haefliger).

(a) Two framed Haefliger knots are diffeotopic if and only if their p invariants are the same and their L invariants are the same.
(b) A Haefliger knot T is trivial (i.e., diffeotopic to $S^3 \subset S^6$) if and only if $p(T, F) = L(T, F) = O$ for some framing F.
(c) Two framings of T are homotopic if and only if their L invariants are the same.
(d) There exists a nontrivial Haefliger knot.

The proofs may be found in Haefliger [8; 10].

Using the topological invariance of the rational Pontryagin classes [28], it is easy to prove:

PROPOSITION 6. Let (T_i, F_i) be a framed Haefliger knot, $i = 0, 1$. Put $M_i = M^7(T_i, F_i)$. If there is a topological embedding $M_0 \subset M_1$ preserving the generator of H^4, then (T_0, F_0) and (T_1, F_1) have the same p and L invariants and are therefore diffeotopic.

A celebrated theorem of H. Kneser [20], combined with the Alexander trick [2], gives the following result:

PROPOSITION 7. The orthogonal group O_3 is a deformation retract of the group of homeomorphisms of D^3. Consequently every topological D^3 bundle carries an orthogonal structure.

The next theorem proves part of Theorem 4:

THEOREM 8. Let $T \subset S^6$ be a nontrivial Haefliger knot and let F be any framing of T. Then the 4-sphere $S^4 \subset M = M^7(T, F)$ has no topological closed tube.

Proof: If S^4 had a closed tube $p: E \to S^4$, then by Proposition 7 the bundle $\nu = (p, E, S^4)$ would have an orthogonal structure, and I

would be homeomorphic to an orthogonal D^3 bundle over S^4. The total space of such a bundle has the form $(D^4 \times D^3) \cup_G (D^4 \times D^3)$ where $G: S^3 \times D^3 \to S^3 \times D^3$ is given by $G(x, y) = [x, g_x(y)]$ for some map $g: S^3 \to O_3$. Considering $S^3 \times D^3$ as a tubular neighborhood of $S^3 \subset S^6$, this means that $E = M^7(S^3, G)$. By Proposition 6 this would make T a trivial knot. This proves Theorem 8.

The framing F of any T can be chosen so that $L(T, F) = O$; this makes $M^7(T, F)$ parallelizable and hence immersible in R^7 by [16; 29; 30]. Extending the definition of tube to immersions, we obtain:

THEOREM 9. There is a PL immersion $S^4 \to R^7$ having no topological closed tube.

From the immersion just described we get a PL immersion $S^4 \times I \to R^8$ which can be shown to be homotopic to an embedding in the category of PL immersions. Using block bundle theory [31] it can be proved that the resulting PL embedding $S^4 \times I \subset R^8$ has no closed PL tube, for if it did then S^4 would have a closed PL tube in R^7. This proves:

THEOREM 10. There is a PL embedding $S^4 \times I \subset R^8$ having no closed PL tube.

Such an embedding cannot be PD isotoped to a smooth embedding, even though for each $t \in I$, $S^4 \times t$ is unknotted (by Zeeman [35]) and hence smoothable.

For the next construction the following notation is used. Let D^7_+ and D^7_- be the two hemispheres of S^7 whose intersection is S^6. Let $T \subset S^6$ be a Haefliger knot bounding a smooth compact manifold $V^4 \subset D^7_+$. Let $\Delta^4 \subset D^7_-$ be the cone on T from the center of D. Put $N^4 = V \cup \Delta \subset S^7$. Then N is a PL submanifold in some smooth triangulation of S^7.

Part (b) of the next theorem completes the proof of Theorem 4.

THEOREM 11. Assume T is a nontrivial Haefliger knot. Then:

(a) $N^4 \subset S^7$ has no PL closed tube.

(b) If V is simply connected and has signature O, then N has no topological closed tube.

Remark: For every Haefliger knot there exists a manifold $V \subset D^7_+$ satisfying (b) (Haefliger [8]).

Proof: To prove (a) we appeal to the obstruction theory of Rourke and Sanderson [31]. The obstructions to a closed PL tube on N have coefficients in $\pi_i(\widetilde{\mathrm{PL}}_3, \overline{\mathrm{PL}}_3)$. (In [31] $\overline{\mathrm{PL}}$ is denoted by $\mathrm{PL}(I)$, in [11], by $\pi\Lambda$.) We may also consider the obstructions to finding a *pseudolinear* tube, i.e., a PL tube having a compatible orthogonal structure; these lie in $\pi_i(\widetilde{\mathrm{PL}}_3, O_3)$ and are related to the obstructions to a PL open tube by the homomorphism $\pi_i(\widetilde{\mathrm{PL}}_3, O_3) \to \pi_i(\widetilde{\mathrm{PL}}_3, \overline{\mathrm{PL}}_3)$ of the exact homotopy sequence of the triple $(\widetilde{\mathrm{PL}}_3, \overline{\mathrm{PL}}_3, O_3)$. By Hirsch [15], the group $\pi_i(\overline{\mathrm{PL}}, O_3)$ is isomorphic to $\pi_i(\mathrm{PL}_2, O_2)$, which vanishes by T. Akiba [1]. Therefore if N has a closed PL tube, it has a pseudolinear tube. By Rourke and Sanderson [31] the group $\pi_i(\widetilde{\mathrm{PL}}_3, O_3)$ is isomorphic to the group of smooth embeddings $S^i \to S^{i+3}$, which vanishes for $i \leq 2$. Therefore the only obstruction to a pseudolinear tube on N is the obstruction to extending the normal tube of the smooth embedding $V \subset D^7_+$ over $N \subset S^7$, considering it as a pseudolinear tube. This obstruction cannot vanish, since it is precisely the Haefliger knot we started with! Alternatively, if it vanished, then N would have a vector bundle neighborhood which was smoothly embedded in a neighborhood of V. A trivialization of this bundle over Δ could be isotoped in D^7_+ onto a smooth embedding keeping T fixed, by the product smoothing theorem [13; 17]. This would make T bound a smooth 4-cell in D^7_+, contradicting the nontriviality of T. This proves that N has no PL closed tube in S^7.

To prove part (b), assume N has a closed tube $p: E \to N$, where (p, E, N) is a D^3 bundle ν. By Proposition 7 we endow ν with an orthogonal bundle structure. The first step is to prove that ν is trivial as an orthogonal bundle.

Let E_O be the interior of E. Observe that E_O has two differential structures: α, making E into a smooth vector bundle over a compatible smoothing of N (see Cairns [6], or Munkres [26; 27]); and the smoothing β induced by the inclusion $E_O \subset S^7$. The topological invariance of rational Pontryagin classes shows that $p_1(E, \alpha) = p_1(E, \beta)$, which is O because β is parallelizable. (We use here the assumption that $\pi_1(V) = 0$ to ensure that N is orientable and consequently the integer Pontryagin class is also invariant. This is because $H^4(E_O) = \mathbf{Z}$.) Since N has signature O, it follows that $p_1(N) = O$, and therefore $p_1(\nu) = O$. Similarly the Stiefel-Whitney classes of ν are O.

At this point we have $p_1(\nu) = w_1(\nu) = w_2(\nu) = O$, and also the euler class of ν is O since ν is a (topological) normal bundle of an orientable manifold (or because $H^3(N) = O$). This information suffices to make ν trivial. (To see this, observe that ν is stably trivial and hence trivial over V. Therefore ν is induced from a bundle over S^4 having O Pontryagin class. Such a bundle must be trivial.)

Let $F: N \times D^3 \to S^7$ be a trivialization of ν, so that $F(x, O) = x$ and $F(N \times D^3) = E$. Consider F as defining a homotopy trivialization of the sphere bundle ∂E, i.e., a homotopy equivalence $N \times S^2 \to \partial E$ which commutes up to homotopy with projection onto N. It is easy to find a homotopy $F_t: N \times S^2 \to E - N$ such that:

(1) $F_O = F|N \times S^2$.
(2) F_1 and $F_1|T \times S^2$ are homotopy trivializations of the normal sphere bundles ξ and η of $V \subset D^7_+$ and $T \subset S^6$, respectively.
(3) $F_1(\Delta \times S^2) \subset D^7_- - \Delta$.

It follows that if $x_O \in S^2$, then

(4) $F_1: T \times x_O \to S^6 - T$ is null homotopic.

This is because the inclusion $S^6 - T \to D^7_- - \Delta$ is a homotopy equivalence, and $F_1(T \times x_O)$ bounds the cell $F_1(\Delta \times x_O)$ in $D^7_- - \Delta$.

Now choose a smooth embedding $H: V \times D^3 \to D^7_+$ which trivializes the orthogonal normal disk bundle $\hat{\xi}$ of V in D^7_+. Thus $H(x, O) = x$, and we may assume $H(T \times D^3) \subset S^6$. Next we show:

(5) $H: T \times x_O \to S^6 - T$ is null homotopic. This is done by proving that the two homotopy trivializations, $F_1|T \times S^2$ and $H|T \times S^2$, of the normal sphere bundle η of $T \subset S^6$, are homotopic through homotopy trivializations.

Let G_3 be the h-space of homotopy equivalences of S^2. The two homotopy trivializations differ by a map $\varphi: V \to G_3$. We must prove that the composition $\psi: T \subset V \to G_3$ is null homotopic. Up to homotopy type V is a union of a finite set of 2-spheres having a point in common but otherwise disjoint. (Here we use the assumption $\pi_1(V) = O$.) Since V has trivial normal bundle and N has signature O, it follows that the tangent bundle of N is stably trivial and therefore the inclusion $T \to V$ is stably null homotopic. This implies that the inclusion is homotopic to a sum of Whitehead products. Since these are known to vanish in a h-space, it follows that $\psi: T \to G_3$ is null homotopic, proving (5).

Part (b) of Theorem 11 is now proved as follows. Form the smooth manifold $W^7 = D^7_+ \bigcup_{H|S^3\times D^3} B^4 \times D^3$, where B is a smooth 4-ball bounded by T (but otherwise disjoint from S^7). Contained in W is a homeomorphic copy Y of $N \times D^3$. Novikov's theorem shows that $p(Y) = O$. Therefore

(6) $p_1(W) = O$.

Now consider the framed Haefliger knot $(T, H|T)$. By (5) and (6), $L(T, H|T) = O$, and also $p(T, H|T) = O$. Therefore T is trivial by Theorem 5(b). This completes the proof of Theorem 11(b).

Remark: By following Haefliger [8] specific examples of manifolds N as in Theorem 11 can be described. It would be interesting to discover whether such manifolds also admit *smooth* embeddings in S^7.

PROOF OF THEOREM 2

Experts are advised to skip to Theorem 12, below, on which the proof is based, and then go to the next section, where another proof of Theorem 2 is given by obstruction theory.

Let V^{m+k} be a PL manifold and $Q^m \subset V$ a PL sphere. The obstruction to a PL open tube for Q can be described as follows. Let B^m_+ and B^m_- be m-cells whose union is Q and whose intersection is a sphere $P^{m-1} \subset Q$. Let N^{m+k}_+ and N^{m+k}_- be regular neighborhoods of B_+ and B_- in V. Choose PL homeomorphisms $\varphi_+ : D^m \times D^k \to N_+$ and $\varphi_- : D^m \times D^k \to N_-$ in such a way that $\varphi_+(D^m \times O) = B_+$ and $\varphi_-(D^m \times O) = B_-$. We may assume further that $\varphi_+(S^{m-1} \times D^k) = \varphi_-(S^{m-1} \times D^k)$, where $S^{m-1} = \partial D^m$. Put $f = \varphi_-^{-1}\varphi_+ : S^{m-1} \times D^k \to S^{m-1} \times D^k$. Then f is a PL embedding; it is further assumed that $f|S^{m-1} \times O =$ identity. Let $S^{m-1} = A^{m-1}_+ \cup A^{m-1}_-$, where A_+ and A_- are hemispheres meeting in S^{m-2}. After an isotopy we may assume $f|A_+ \times D^k =$ identity. (Compare Hirsch [14].)

If f happens to be fibre preserving, that is, if $f(x \times D^k) \subset x \times D^k$ for all $x \in S^{m-1}$, then it is easy to see that Q has a normal microbundle, and hence an open tube. It turns out that it suffices for f to be isotopic, or merely concordant, to a fibre preserving homeomorphism (all maps being understood to be PL). Observe also that we may replace f by fg, where $g : S^{m-1} \times D^k \to S^{m-1} \times D^k$ is a fibre preserving homeomorphism.

The product smoothing theorem [13; 17] implies that f is PD isotopic to a smooth embedding $h : S^{m-1}_a \times D^k \to S^{m-1} \times D^k$, where

α is a compatible differential structure (or *smoothing*) of S^{m-1}. If it happens that α is diffeomorphic to the standard smoothing, and we are working in dimensions where all smooth knots are trivial, then $h(S^{m-1}_{\alpha} \times O)$ can be diffeotoped back onto $S^{m-1} \times O$, and the smooth tubular neighborhood theorem implies that the diffeotopy can be chosen to carry the fibres of the orthogonal disk bundle of $h(S^{m-1}_{\alpha} \times O)$ back onto the fibres $x \times D^k$ of $S^{m-1} \times D^k$. The diffeotopy can then be approximated by a PL isotopy. Finally we take advantage of another degree of freedom by observing that g need not be defined on all of $S^{m-1} \times D^k$. It suffices to find a fibre preserving PD embedding $g_0 : A_- \times D^k \to A_- \times D^k$ having the added property that in a neighborhood of $S^{m-2} \times O$, g is a diffeomorphism; for example, in such a neighborhood, $N \times D^k$, g_0 might have the form $g(x, y) = [x, u_x(y)]$, where $u : N \to O_k$ is a smooth map into the orthogonal group. In that case the smoothing on $A_- \times D^k$ induced from the standard smoothing by fg_0 will coincide with the standard smoothing in a neighborhood of $S^{m-2} \times D^k$, and the relative form of the product smoothing theorem can be applied.

In this way we see that $Q^m \subset V^{m+k}$ has an open PL tube provided the following conditions are satisfied:

(a) $k \geq \frac{1}{2}(m + 3)$ (so that Haefliger's unknotting theorems can be applied);

(b) There is a PD fibre preserving embedding $g : A_- \times D^k \to A_- \times D^k$ which is a diffeomorphism in a neighborhood of $S^{m-1} \times O$, and which induces a smoothing of $A_- \times D$ which is PD isotopic to the standard smoothing keeping a neighborhood of $S^{m-2} \times O$ fixed.

The function which to g assigns a smoothing of S^{m-1} via the relative form of the product smoothing theorem gives rise to a homomorphism

$$\Phi : \Pi_{m-1}(\mathrm{PD}_k, O_k) \to \Gamma_{m-1} \approx \Pi_{m1}(\mathrm{PD}, O).$$

(Here Γ_{m-1} is the group of concordance classes (or, equivalently, isotopy or diffeomorphism classes) of smoothings of S^{m-1}; for the isomorphism $\Gamma^{v}_{m-1} \approx \Pi_{m-1}(\mathrm{PD}, O)$ see [13] or [31]. See [31] for the definition of PD_k, and other undefined objects.) Actually this map factors through Γ^{k}_{m-1}, the group of smooth embeddings of smoothings of S^{m-1} in S^{m-1+k}; in the range $k \geq \frac{1}{2}(m + 3)$, $\Gamma^{k}_{m-1} = \Gamma_{m-1}$ by Haefliger [9].

A test for whether $Q \subset V$ has a PL open tube is now seen to consist of the following steps:

(i) Define a PL embedding $f: S^{m-1} \times D^k \to S^{m-1} \times D^k$ by comparing product neighborhoods of the two hemispheres of Q.

(ii) Apply the product smoothing theorem to obtain a smoothing α of S^{m-1} such that the smoothing $S^{m-1}_\alpha \times D^k$ is isotopic to the smoothing induced by f.

(iii) If $[\alpha]$ is in the image of $\Phi = \Pi_{m-1}(\mathrm{PD}_k, O_k) \to \Gamma_{m-1}$, and if $k \geq \frac{1}{2}(m+3)$, then Q has a PL open tube.

Theorem 2(a) states that Q has such a tube provided $k \geq \max[m-2, \frac{1}{2}(m+3)]$. This is now seen to be a consequence of:

THEOREM 12. The homomorphism $\Phi: \Pi_n(\mathrm{PD}_k, O_k) \to \Gamma_n$ is surjective for $k \geq n-1$.

Proof (outline): It is enough to prove that $\Phi = \Pi_n(\mathrm{PD}_{n-1}, O_{n-1}) \to \Gamma_n$ is surjective. To do this we use Munkres' theorem [37] that every smoothing of S^n is obtained by glueing together two n-balls by an orientation preserving diffeomorphism of S^{n-1}. Such a diffeomorphism may be assumed fixed on a neighborhood of a hemisphere. If $D^{n-1} \subset S^{n-1}$ is the other hemisphere, let ∇_{n-1} denote the group of diffeomorphisms of D^{n-1} which are fixed in a neighborhood of S^{n-2}; give ∇_{n-1} the C^1 topology. Then there is defined a homomorphism $\theta: \Pi_0(\nabla_{n-1}) \to \Gamma_n$, which is surjective by Munkres' theorem. Theorem 12 is proved by factoring θ thus:

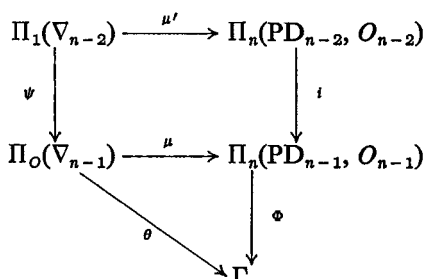
To define μ, for every diffeomorphism $g: D^{n-1} \to D^{n-1}$ in ∇_{n-1}, choose a PD isotopy g_t from g to the identity; assume g_t is fixed in a neighborhood of S^{n-2}. Any two such isotopies are themselves PD isotopic through similar isotopies. (These facts can easily be proved by first isotoping to a PL homeomorphism, or isotopy, and then applying Alexander's shrinking process.) Assume that $g_t = g_0$ for $0 \leq t \leq \frac{1}{3}$ and $g_t = g_1$ for $\frac{2}{3} \leq t \leq 1$. Define a level preserving PD homeomorphism $h: D^{n-1} \times I \to D^{n-1} \times I$ by $h(x, t) = [g_t(x), t]$. Then h is a diffeomorphism in a neighborhood of $\partial(D^{n-1} \times I)$. Extend h to a PD level preserving homeomorphism $H: R^{n-1} \times I \to R^{n-1} \times I$ by defining H to be the identity outside $D^{n-1} \times I$. Put $H(X, t) = [H_t(x), t]$.

Define $G: (D^{n-1} \times I) \times R^{n-1} \to (D^{n-1} \times I) \times R^{n-1}$ by the formula $G[(x, t), y] = [(x, t), H_t(x + y) - H_t(x)]$. For each (x, t) the restriction of G to $(x, t) \times R^{n-1}$ is the microbundle "differential" of the PD homeomorphism H_t of R^{n-1} at the point x of D^{n-1}. Since G preserves projection onto $D^{n-1} \times I$, G has the form $G(z, y) = [z, G_z(y)]$. For each $z \in D^{n-1} \times I$, G_z is a PD homeomorphism of R^{n-1}, and G_z is a diffeomorphism if $z \in \partial(D^{n-1} \times I)$. Since the orthogonal group O_{n-1} is a deformation retract of the group of diffeomorphism of R^{n-1} [33], G defines an element $\mu(G) \in \Pi_n(\mathrm{PD}_{n-1}, O_{n-1})$. It is not hard to see that depends only on the diffeotopy class of g, so that $\mu: \Pi_0(\nabla_{n-1}) \to \Pi_n(\mathrm{PD}_{n-1}, O_{n-1})$ is well defined. It is less obvious that $\theta = \Phi\mu$, but this can be proved using smoothing theory; I expect to give more details in another paper. In this way Theorem 12 is proved.

Remark: I have recently proved that $\Pi_i(\mathrm{PD}_n, O_n) \approx \Gamma_n$ for $i \geq n$. Most of the theorems in this paper can be improved by one dimension using this result. The proof will appear elsewhere.

Theorem 2(b) is proved by using the theorem of J. Cerf [7] implying that for $n \geq 11$; the homomorphism $\psi: \Pi_1(\nabla_{n-2}) \to \Pi_0(\nabla_{n-1})$ is surjective. (To define ψ, let f: $D^{n-2} \times I \to D^{n-2} \times I$ be a level preserving diffeomorphism, fixed in a neighborhood of the boundary, representing an element $\xi \in \Pi_1(\nabla_{n-2})$. Identify $D^{n-2} \times I$ with D^{n-1}, making f a diffeomorphism of D^{n-1} representing $\psi(\xi)$.)

There is a commutative diagram



where the definition of μ' is similar to that of μ, and i is induced by inclusion. Since ψ and θ are surjective, Theorem 2(b) is proved. We have also proved:

THEOREM 13. For $n \geq n - 2$ the homomorphism

$$\Phi: \Pi_n(\mathrm{PD}_k, O_k) \to \Gamma_n$$

is surjective.

PROOFS OF THEOREMS 1, 2, AND 3

The obstruction theory of Haefliger–Wall [11] and Rourke–Sanderson [31] will be applied. It is shown that the appropriate homotopy groups in which the obstructions take values are O. For Theorem 2 this is done in (9), below; for Theorem 1, in (10) and (13); and for Theorem 3, in (12).

The proofs are standard diagram chases based on the following facts:

(1) $\Pi_i(\mathrm{PL}_{k+1}, \mathrm{PL}_k) = O$ for $i \leq k-1$ (Haefliger–Wall [11]).

(2) The kernel of $\Pi_{k-1}(\mathrm{PL}_k) \to \Pi_{k-1}(\mathrm{PL}_{k+1})$ is the image of $\ker(\Pi_{k-1}(O_k) \to \Pi_{k-1}(O_{k+1}))$ under the homomorphism $\Pi_{k-1}(O_k) \to \Pi_{k-1}(\mathrm{PL}_k)$ (Haefliger–Wall [11]).

(3) $\Pi_i(\widetilde{\mathrm{PL}}_k, O_k) \approx \Gamma_i^k$ (Rourke–Sanderson [31]).

(4) $\Gamma_i^k \approx \Gamma_i$ for $k \geq \frac{1}{2}(i+4)$ (Haefliger [9]).

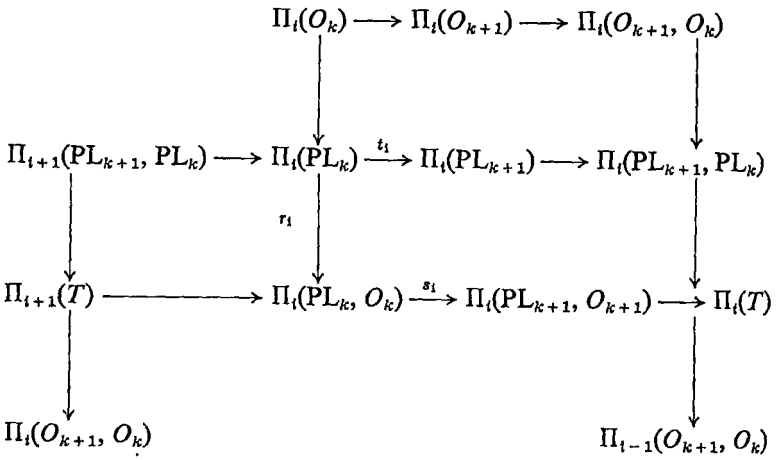
(5) $\Pi_i(\mathrm{PL}_k, O_k) \to \Gamma_i$ is surjective for $k \geq i-1$, and for $k \geq i-2$ if $i \geq 11$ (Theorems 12 and 13).

(6) $\mathrm{Lim}_{k\to\infty} \Pi_i(\mathrm{PL}_k, O_k) \approx \Gamma_i$ (Hirsch [13]).

From these we make the following deductions:

(7) $\Pi_i(\mathrm{PL}_k, O_k) \to \Pi_i(\mathrm{PL}_{k+1}, O_k)$ is bijective for $i \leq k-1$ and surjective for $i = k$.

Proof: Let T stand for the triad $(\mathrm{PL}_{k+1}, O_{k+1}, \mathrm{PL}_k)$ and consider the commuting diagram, with exact rows and columns:



From (1) and the last column we get $\Pi_i(T) = O$ for $i \leq k-1$; hence from the bottom row, s_i is bijective for $i \leq k-2$, and surjective for $i = k-1$. Since $\Pi_{k-1}(O_{k+1}, O_k) = O$, we have $\ker(s_{k-1}) \subset r_i \ker(t_i)$, which is 0 by (2). This proves the first statement of (7). The second now follows from (5) and (6).

(8) The homomorphism $\Pi_i(\mathrm{PL}_k, O_k) \to \Pi_i(\widetilde{\mathrm{PL}}_k, O_k)$ is bijective for $k \geq \max[i+1, \frac{1}{2}(i+4)]$. It is surjective for $k \geq \max[i-1, \frac{1}{2}(i+4)]$, and also for $k \geq i-2$ and $i \geq 11$.

Proof: Apply (3) and (4) and (5).

(9) The homomorphism $\Pi_i(\widetilde{\mathrm{PL}}_k) \to \Pi_i(\widetilde{\mathrm{PL}}_k, \mathrm{PL}_k)$ is O if $k \geq \max[i-1, \frac{1}{2}(i+4)]$; also, if $k \geq i-2$ and $i \geq 11$.

Proof: Apply (8) and the following commuting diagram with exact row and column:

$$\begin{array}{ccccccc} & & & & \Pi_i(\widetilde{\mathrm{PL}}_k) & & \\ & & & & \downarrow & & \\ \Pi_i(\mathrm{PL}_k, O_k) & \longrightarrow & \Pi_i(\widetilde{\mathrm{PL}}_k, O_k) & \longrightarrow & \Pi_i(\widetilde{\mathrm{PL}}_k, \mathrm{PL}_k) & \longrightarrow & \Pi_{i-1}(\mathrm{PL}_k, O_k) \\ & & & & \downarrow & & \\ & & & & \Pi_{i-1}(\mathrm{PL}_k) & & \end{array}$$

This last result implies Theorem 2.

(10) $\Pi_i(\widetilde{\mathrm{PL}}_k, \mathrm{PL}_k) = O$ if $k \geq \max[i, \frac{1}{2}(i+4)]$.

Proof: Apply (8) and the exact homotopy sequence of the triple $(\mathrm{PL}_k, \mathrm{PL}_k, O_k)$.

This proves (a) and (b) of Theorem 1.

(11) $\Pi_k(O_{k+1}, O_k) \to \Pi_k(\mathrm{PL}_{k+1}, \mathrm{PL}_k)$ is bijective.

Proof: Let T stand for the triad $(\mathrm{PL}_{k+1}; O_{k+1}, \mathrm{PL}_k)$, and consider the commuting diagram with exact rows and columns:

$$\begin{array}{ccccc}
\Pi_i(O_k) & \longrightarrow & \Pi_i(O_{k+1}) & \longrightarrow & \Pi_i(O_{k+1}, O_k) \\
\downarrow & & \downarrow & & \downarrow \lambda_i \\
\Pi_i(\mathrm{PL}_k) & \longrightarrow & \Pi_i(\mathrm{PL}_{k+1}) & \longrightarrow & \Pi_i(\mathrm{PL}_{k+1}, \mathrm{PL}_k) \\
\downarrow & & \downarrow & & \downarrow \\
\Pi_i(\mathrm{PL}_k, O_k) & \longrightarrow & \Pi_i(\mathrm{PL}_{k+1}, O_{k+1}) & \longrightarrow & \Pi_i(T)
\end{array}$$

From (7) and the bottom row, $\Pi_i(T) = O$ for $i \leq k$. From the last column we get the surjectivity of λ_k. But λ_i is injective for all i, as can be seen from the commuting diagram:

$$\begin{array}{ccc}
\Pi_i(O_{k+1}, O_k) & \xrightarrow{\lambda_i} & \Pi_i(\mathrm{PL}_{k+1}, \mathrm{PL}_k) \\
\downarrow u & & \downarrow \\
\Pi_i(S^k) & \xleftarrow{v} & \Pi_i(G_{k+1}, F_k)
\end{array}$$

Here G_{k+1} is the space of homotopy equivalences of S^k and F_k is the subspace leaving the north pole fixed. The evaluation maps $O_{k+1} \to S^k$ and $G_{k+1} \to S^k$ induce isomorphisms u and v. Therefore λ_i is injective. This proves 11, and also:

(12) $\Pi_k(\mathrm{PL}_{k+1}, \mathrm{PL}_k) \to \Pi_k(G_{k+1}, F_k) \approx \Pi_k(S^k)$ is bijective. This proves Theorem 3.

(13) Let $\overline{\mathrm{PL}}_k$ denote the simplicial set based on PL homeomorphisms of I^k (denoted by $\mathrm{PL}_k(I)$ in (31) and by $\Pi\Lambda_k$ in (11)). Then $\Pi_i(\mathrm{PL}_k, \mathrm{PL}_k) = O$ for $i \leq k - 1$.

Proof: By Hirsch [15], $\Pi_j(\overline{\mathrm{PL}}_k, \mathrm{PL}_{k-1}) \approx \Pi_j(S^{k-1})$. By (1) and (11), $\Pi_j(\mathrm{PL}_k, \mathrm{PL}_{k-1}) \approx \Pi_j(S^{k-1})$ for $j \leq k - 1$; moreover $\Pi_{k-1}(\overline{\mathrm{PL}}_k, \mathrm{PL}_{k-1}) \to \Pi_{k-1}(\mathrm{PL}_k, \mathrm{PL}_{k-1})$ is easily seen to be bijective. The result now follows from the exact homotopy sequence of the triple $(\mathrm{PL}_k, \overline{\mathrm{PL}}_k, \mathrm{PL}_{k-1})$.

Note that (13) and (9) prove (c) and (d) of Theorem 1.

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DECEMBER 1967