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ON TUCKER'S KEY THEOREM

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ABSTRACT. A new proof of a (slightly extended) geometric version of Tucker's fundamental result is given.

1. INTRODUCTION.

A classical result of A. W. Tucker (9) states that the dual systems

$$Ax = 0, \quad x \geqq 0$$

and

$$A^T y = 0$$

have solutions x^o and y^o such that $A^T y^o + x^o$ is positive.

In this note we suggest a new proof of a (slightly extended) geometric version of this fundamental result, which was observed, e.g. (6), to be a key to duality theory.

2. MAIN RESULTS.

We use L and S to denote convex cones in C^n, i.e., subsets of the n-dimensional unitarian space which are closed under addition and under multiplication by a nonnegative scalar. For a nonempty set $T \subseteq C^n$, T^* denotes the closed convex cone $\{x \in C^n;\ \mathrm{Re}(x,T) \geqq 0\}$.

We shall make use of the following identities:

$$K^* = (c\ell K)^*, \tag{2.1}$$

$$K^{**} = c\ell K, \tag{2.2}$$

$$(K_1 + K_2)^* = K_1^* \cap K_2^*, \tag{2.3}$$

satisfied by the convex cones K, K_1 and K_2. For these and other basic results on convex cones the reader is referred to (2) and (4).

Consider the following intersection.

$$I(L,S) = S \cap S^* \cap (L \cap S)^* \cap (-L \cap S)^*) \cap (L-S).$$

The proof of the main result is based on the fact that this intersection consists only of the origin.

<u>LEMMA</u>. $I(L,S) = \{0\}$.

<u>PROOF</u>. Let $x \in I(L,S)$. Then $x \in L - S$ and there exists an $s \in S$ such that $x + s \in L$.

Now, $x \in S \Rightarrow x + s \in S \Rightarrow x + s \in L \cap S$.

On the other hand, $x \in (L \cap S)^* \cap (-(L \cap S)^*$.

Thus $\mathrm{Re}(x,x+s) = 0$.

Still more, $x \in S^* \Rightarrow \mathrm{Re}(x,s) \geqq 0$. Thus $||x||^2 \leqq 0$, but this is possible only when $x = 0$, which was to be proved

The intersection $S \cap S^*$ is pointed. (It consists only of the origin if and only if S is a real subspace, e.g. (1), (5)). Thus $(S \cap S^*)^*$ is solid and the following theorem is meaningful.

<u>KEY THEOREM</u>. If (i) $L - S$ is closed or (iia) $L^* + S^*$ is closed and (iib) $c\ell(L \cap S) = c\ell\ L \cap c\ell\ S$, then

$$x \in L\cap S,\ v \in (S-L)^*,\qquad x + v \in \operatorname{int}(S\cap S^*)^*, \tag{2.4}$$

is consistent.

PROOF. The consistency of (2.4) is equivalent (by 2.3) to

$$(L\cap S + (-L^*)\cap S^*)\cap \operatorname{int}(S\cap S^*)^* \neq \emptyset, \tag{2.5}$$

The set $L\cap S + (-L^*)\cap S^*$ is convex. The set $\operatorname{int}(S\cap S^*)^*$ is the interior of a convex cone. Thus, e.g. (4), (2.5) is not true if and only if there exists a non-zero $z \in S\cap S^*$ such that $\operatorname{Re}(z, L\cap S + (-L^*)\cap S^*) \leqq 0$.
But $z \in S \Rightarrow \operatorname{Re}(z,(-L^*)\cap S^*) \geqq 0$ and $z \in S^* \Rightarrow \operatorname{Re}(z,L\cap S) \geqq 0$. Thus the negation of (2.5) is equivalent to the existence of a $0 \neq z \in S\cap S^*$ such that $\operatorname{Re}(z,L\cap S) = \operatorname{Re}(z,(-L^*)\cap S^*) = 0$. To show that this is impossible consider the intersection

$$I = S\cap S^*\cap(L\cap S)^*\cap(-(L\cap S)^*)\cap((-L^*)\cap S^*)^*\cap(L^*\cap(-S^*))^*.$$

By (2.3) and (2.2), $(L^*\cap(-S^*))^* = ((L-S)^*)^* = c\ell\,(L-S)$. By (2.1), (iib), (2.2) and (2.3), $-(L\cap S)^* = -(L^{**}\cap S^{**})^* = -(L^* + S^*)^{**} = -\,c\ell(L^*+S^*)$, and by (2.2), $S \subseteq S^{**}$.

Thus if $L - S$ is closed, $I \subseteq I(L,S) = \{0\}$ and if $L^* + S^*$ is closed $I \subseteq I(-L^*,S^*) = \{0\}$ so in both cases the proof is complete.

The assumptions made in the theorem suggest two interrelated open problems:
a) Is the theorem true without the assumptions?
b) For what convex cones L and S, both assumptions, (i) and (ii), do not hold? Notice that if L and S are polyhedral then all the assumptions hold. We remark that, in general, assumptions (iia) and (iib) are independent. Let S and L be closed convex cones in R^3 such that

$$S^* = \{\begin{pmatrix}x\\y\\z\end{pmatrix},\quad x \geqq 0,\quad z \geqq 0,\quad 2xz \geqq y^2\}$$

and L^* is the x-axis. Then obviously (iib) holds but, e.g.(2, p. 7), (iia) does not. Conversely, let

$$S = \{\binom{x}{y};\quad x > 0,\quad y > 0\}\cup\{0\}$$

and
$$L = \{\binom{x}{y};\quad x > 0,\quad y < 0\}\cup\{0\}$$

be (not closed) convex cones in R^2. Then (iia) holds but (iib) is false. In conclusion, we point out some special cases.

The real version of the theorem with $S = R^n_+$ is due to Epelman and Waksman (3).

Taking S to be polyhedral and L the null space of a matrix $A, L^* = L^+ = R(A^H)$ and replacing $v \in R(A^H) \cap S^*$ by $A^H y \in S^*$ one gets the Key Theorem of Abrams and Ben-Israel (1). As shown in (1), the theorem of Tucker is the real special case where $S = R^n_+$. Its complex extension, due to Levinson (8), is the special case where

$$S = T_\alpha = \{z \in C^n;\ |\arg z_i| \leqq \alpha_i\}$$

$$\alpha = (\alpha_i) \leqq \frac{\pi}{2} e,\quad \text{e-vector of ones}$$

and $S^* = T_{\frac{\pi}{2}e - \alpha}$.

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