

ON TUCKER'S KEY THEOREM

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ABSTRACT. A new proof of a (slightly extended) geometric version of Tucker's fundamental result is given.

1. INTRODUCTION.

A classical result of A. W. Tucker (9) states that the dual systems

$$Ax = 0, \quad x \geq 0$$

and

$$A^T y = 0$$

have solutions x^0 and y^0 such that $A^T y^0 + x^0$ is positive.

In this note we suggest a new proof of a (slightly extended) geometric version of this fundamental result, which was observed, e.g. (6), to be a key to duality theory.

2. MAIN RESULTS.

We use L and S to denote convex cones in C^n , i.e., subsets of the n -dimensional

unitarian space which are closed under addition and under multiplication by a nonnegative scalar. For a nonempty set $T \subseteq C^n$, T^* denotes the closed convex cone $\{x \in C^n; \operatorname{Re}(x, T) \geq 0\}$.

We shall make use of the following identities:

$$K^* = (c\ell K)^*, \quad (2.1)$$

$$K^{**} = c\ell K, \quad (2.2)$$

$$(K_1 + K_2)^* = K_1^* \cap K_2^*, \quad (2.3)$$

satisfied by the convex cones K , K_1 and K_2 . For these and other basic results on convex cones the reader is referred to (2) and (4).

Consider the following intersection.

$$I(L, S) = S \cap S^* \cap (L \cap S)^* \cap (-(L \cap S))^* \cap (L - S).$$

The proof of the main result is based on the fact that this intersection consists only of the origin.

LEMMA. $I(L, S) = \{0\}$.

PROOF. Let $x \in I(L, S)$. Then $x \in L - S$ and there exists an $s \in S$ such that $x + s \in L$.

$$\text{Now, } x \in S \Rightarrow x + s \in S \Rightarrow x + s \in L \cap S.$$

$$\text{On the other hand, } x \in (L \cap S)^* \cap (-(L \cap S))^*.$$

Thus $\operatorname{Re}(x, x+s) = 0$.

Still more, $x \in S^* \Rightarrow \operatorname{Re}(x, s) \geq 0$. Thus $\|x\|^2 \leq 0$, but this is possible only when $x = 0$, which was to be proved

The intersection $S \cap S^*$ is pointed. (It consists only of the origin if and only if S is a real subspace, e.g. (1), (5)). Thus $(S \cap S^*)^*$ is solid and the following theorem is meaningful.

KEY THEOREM. If (i) $L - S$ is closed or (ia) $L^* + S^*$ is closed and (iib) $c\ell(L \cap S) = c\ell L \cap c\ell S$, then

$$x \in L \cap S, v \in (S-L)^*, \quad x + v \in \text{int}(S \cap S^*)^*, \quad (2.4)$$

is consistent.

PROOF. The consistency of (2.4) is equivalent (by 2.3) to

$$(L \cap S + (-L^*) \cap S^*) \cap \text{int}(S \cap S^*)^* \neq \emptyset, \quad (2.5)$$

The set $L \cap S + (-L^*) \cap S^*$ is convex. The set $\text{int}(S \cap S^*)^*$ is the interior of a convex cone. Thus, e.g. (4), (2.5) is not true if and only if there exists a non-zero $z \in S \cap S^*$ such that $\text{Re}(z, L \cap S + (-L^*) \cap S^*) \leq 0$.

But $z \in S \Rightarrow \text{Re}(z, (-L^*) \cap S^*) \geq 0$ and $z \in S^* \Rightarrow \text{Re}(z, L \cap S) \geq 0$. Thus the negation of (2.5) is equivalent to the existence of a $0 \neq z \in S \cap S^*$ such that $\text{Re}(z, L \cap S) = \text{Re}(z, (-L^*) \cap S^*) = 0$. To show that this is impossible consider the intersection

$$I = S \cap S^* \cap (L \cap S)^* \cap ((-L \cap S)^*) \cap ((-L^*) \cap S^*)^* \cap (L^* \cap (-S^*))^*.$$

By (2.3) and (2.2), $(L^* \cap (-S^*))^* = ((L-S)^*)^* = \text{cl}(L-S)$. By (2.1), (iib),

(2.2) and (2.3), $-(L \cap S)^* = -(L^* \cap S^*)^* = -(L^* + S^*)^{**} = -\text{cl}(L^* + S^*)$, and by

(2.2), $S \subseteq S^{**}$.

Thus if $L - S$ is closed, $I \subseteq I(L, S) = \{0\}$ and if $L^* + S^*$ is closed

$I \subseteq I(L^*, S^*) = \{0\}$ so in both cases the proof is complete.

The assumptions made in the theorem suggest two interrelated open problems:

- a) Is the theorem true without the assumptions?
 - b) For what convex cones L and S , both assumptions, (i) and (ii), do not hold?
- Notice that if L and S are polyhedral then all the assumptions hold. We remark that, in general, assumptions (iia) and (iib) are independent. Let S and L be closed convex cones in R^3 such that

$$S^* = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad x \geq 0, \quad z \geq 0, \quad 2xz \geq y^2 \right\}$$

and L^* is the x -axis. Then obviously (iib) holds but, e.g. (2, p. 7), (iia) does

not. Conversely, let

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}; x > 0, y > 0 \right\} \cup \{0\}$$

and

$$L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}; x > 0, y < 0 \right\} \cup \{0\}$$

be (not closed) convex cones in \mathbb{R}^2 . Then (iia) holds but (iib) is false.

In conclusion, we point out some special cases.

The real version of the theorem with $S = \mathbb{R}_+^n$ is due to Epelman and Waksman (3).

Taking S to be polyhedral and L the null space of a matrix $A, L^* = L^\perp = R(A^H)$ and replacing $v \in R(A^H) \cap S^*$ by $A^H y \in S^*$ one gets the Key Theorem of Abrams and Ben-Israel (1). As shown in (1), the theorem of Tucker is the real special case where $S = \mathbb{R}_+^n$. Its complex extension, due to Levinson (8), is the special case where

$$S = T_\alpha = \{z \in \mathbb{C}^n; |\arg z_i| \leq \alpha_i\}$$

$$\alpha = (\alpha_i) \leq \frac{\pi}{2} e, \quad e\text{-vector of ones}$$

$$\text{and } S^* = T_{\frac{\pi}{2} e - \alpha}.$$

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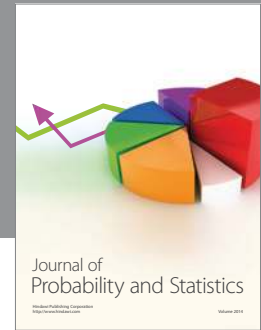
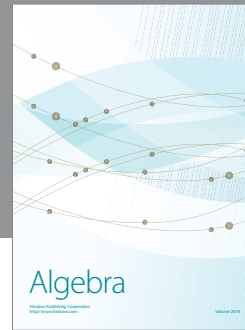
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