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ON THE REMARKABLE DISTRIBUTIONS OF MAXIMA OF SOME FRAGMENTS OF THE STANDARD REFLECTING RANDOM WALK AND BROWNIAN MOTION

BY

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Abstract. In this paper, we consider some distributions of maxima of excursions and related variables for standard random walk and Brownian motion. We discuss the infinite divisibility properties of these distributions and calculate their Lévy measures. Lastly we discuss Chung's remark related with Riemann's zeta functional equation.

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1. INTRODUCTION

In a number of studies in probability theory, e.g. Brownian motion (written later as BM) and empirical process, the laws of the following random variables play an important role:

(a) If $(B_t, t \geqslant 0)$ denotes BM, then variables of that kind are

$$\sup_{t \leqslant 1} |B_t|, \quad \sup_{t \leqslant g_1} |B_t|, \quad \sup_{t \leqslant d_1} |B_t|,$$

where for $u \geqslant 0$, $g_u = \sup\{s < u : B_s = 0\}$, $d_u = \inf\{s > u : B_s = 0\}$, to name only a few of them.

(b) Likewise, if $(Z_t, t \in N = \{0, 1, 2, \ldots\})$ denotes a standard random walk (written in the sequel as RW), i.e. $Z_t = \xi_1 + \ldots + \xi_t$, where $\xi_1, \ldots, \xi_t$ are i.i.d. and $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$, then there are obvious counterparts of the previous sup-type Brownian variables.

The distributions of such variables are not particularly simple: for instance, in the Brownian case, their densities may be expanded in terms of theta functions. However, if instead of the fixed time 1, we consider these maxima at random time θ, an independent exponential random variable with parameter $\lambda^2/2$, and if in the standard RW case, θ denotes a geometric random variable with parameter $1-q$, then the expressions of the corresponding laws are fairly simple — thanks to the independence of θ and B, and scaling properties of BM; it is not so difficult to invert the Laplace transforms, hence to recover the theta function expansions.

This remark justifies that in the present paper we are focussing attention on a number of Brownian functionals considered at independent exponential time. We also consider the RW counterparts, taken at an independent geometric time.

We present our results in the form of two self-explanatory tables.

TABLE 1. A list of interesting maxima

BM	$P(\cdot \leqslant A)$	RW	$P(\cdot < A)$
$\sup_{u\leqslant\theta}\lvert B_u\rvert \sim \sqrt{\theta}\sup_{u\leqslant 1}\lvert B_u\rvert$	$1-\dfrac{1}{\cosh(\lambda A)}$	$\sup_{u\leqslant\theta}\lvert Z_u\rvert$	$1-\dfrac{2}{\alpha^{-A}+\alpha^{A}}$
$\sup_{u\leqslant g_\theta}\lvert B_u\rvert \sim \sqrt{g_\theta}\sup_{u\leqslant 1}\lvert b_u\rvert$	$\tanh(\lambda A)$	$\sup_{u\leqslant g_\theta}\lvert Z_u\rvert$	$\dfrac{\alpha^{-A}-\alpha^{A}}{\alpha^{-A}+\alpha^{A}}$
$\sup_{g_\theta\leqslant u\leqslant\theta}\lvert B_u\rvert \sim \sqrt{\theta-g_\theta}\sup_{u\leqslant 1}m_u$	$\tanh(\lambda A/2)$	$\sup_{g_\theta\leqslant u\leqslant\theta}\lvert Z_u\rvert$	$\dfrac{\alpha^{-A/2}-\alpha^{A/2}}{\alpha^{-A/2}+\alpha^{A/2}}$
$\sup_{u\leqslant d_\theta}\lvert B_u\rvert$	$1-\dfrac{\tanh(\lambda A)}{\lambda A}$	$\sup_{u\leqslant d_\theta}\lvert Z_u\rvert$	$1-\dfrac{1}{A}\dfrac{1+\alpha}{1-\alpha}\dfrac{\alpha^{-A}-\alpha^{A}}{\alpha^{-A}+\alpha^{A}}$
$\sup_{\theta\leqslant u\leqslant d_\theta}\lvert B_u\rvert$	$1-\dfrac{1-e^{-\lambda A}}{\lambda A}$	$\sup_{\theta\leqslant u\leqslant d_\theta}\lvert Z_u\rvert$	$1-\dfrac{2}{\alpha^{-1}-\alpha}\dfrac{1-\alpha^{A}}{A}$
$\sup_{g_\theta\leqslant u\leqslant d_\theta}\lvert B_u\rvert \sim \sqrt{d_\theta-g_\theta}\sup_{u\leqslant 1}e_u$	$\coth(\lambda A)-\dfrac{1}{\lambda A}$	$\sup_{g_\theta\leqslant u\leqslant d_\theta}\lvert Z_u\rvert$	$\dfrac{\alpha^{-A}+\alpha^{A}}{\alpha^{-A}-\alpha^{A}}-\dfrac{1}{A}\dfrac{2}{\alpha^{-1}-\alpha}$

NOTATION. For BM: b denotes Brownian bridge, m — Brownian meander, e — normalized excursion, and $\theta \sim \exp(\lambda^2/2)$, i.e., its density is

$$f_\theta(x) = 1_{(0,\infty)}(x)\frac{\lambda^2}{2}\exp\left(-\frac{\lambda^2 x}{2}\right).$$

For RW: $\theta \sim \mathrm{Geom}(1-q)$, i.e., $P(\theta = k) = (1-q)q^k$ $(k = 0, 1, 2, \ldots)$, $\alpha = (1-\sqrt{1-q^2})/q$, and

$$g_t = \inf\{u \leqslant t : Z_u = 0\}, \quad d_t = \inf\{u > t : Z_u = 0\}.$$

We now explain the presence of Table 2: while dealing with the variables in Table 1, we found that the random times, which are very naturally involved,

TABLE 2. A list of interesting times

BM	Distribution	RW	Generating function $E(t^X)$
g_θ	$\frac{N^2}{\lambda^2}$	g_θ	$\frac{(1-q^2)^{1/2}}{(1-q^2t^2)^{1/2}}$
d_θ	$\frac{1}{\lambda^2}\left(\frac{N^2}{U^2}+\hat{N}^2\right)$	d_θ	$1-\frac{(1-t^2)^{1/2}}{(1-q^2t^2)^{1/2}}$
$\theta-g_\theta$	$\frac{N^2}{\lambda^2}$	$\theta-g_\theta$	$\sqrt{\frac{1-q}{1+q}}\sqrt{\frac{1+qt}{1-qt}}$
$d_\theta-\theta$	$\frac{1}{\lambda^2}\frac{N^2}{\hat{N}^2}(2e)$	$d_\theta-\theta$	$\frac{(1-q)t(t+q)}{\sqrt{1-q^2}(\sqrt{1-q^2}+\sqrt{1-t^2})}$
$d_\theta-g_\theta$	$\frac{N^2}{\lambda^2U^2}$	$d_\theta-g_\theta$	$\frac{\sqrt{1-q^2t^2}-\sqrt{1-t^2}}{\sqrt{1-q^2}}$
θ	$\frac{2}{\lambda^2}e$	θ	$\frac{1-q}{1-qt}$

NOTATION. N, $\hat{N} \sim \mathcal{N}(0, 1)$, $U \sim \mathcal{U}(0, 1)$, $e \sim$ exponential with mean 1.

deserve some particular attention, hence our Table 2, which helps to identify the laws of the scaling factors in Table 1 for BM.

The rest of the paper is devoted to the discussion of all the entries in the two tables. Before we shift to these detailed computations, let us discuss a little about the nature of this paper: although a number of results found here are well known to, e.g., Brownian experts, we feel that the RW counterparts are somewhat original, and moreover we also discuss infinite divisibility properties related to lengths of BM and RW excursions. As the reader will notice, we shall skip the discussion of well-known properties, in particular about independence of certain fragments of the Brownian path. Nonetheless, in the Appendix, we give a word of explanation about the identities in law found in the first column of Table 1; they are justified by the scaling property of Brownian motion together with some deeper results about random Brownian scaling, depending on the random times being considered.

Finally, let us indicate some notation used throughout the paper: $X \sim Y$ means that the random variables X and Y have the same distribution; such an identity in law will often involve $X = AB$, and $Y = CD$, say, with the understanding that A and B, on one hand, and C and D on the other hand are independent. In any case, the context should help the reader understand precisely how the various variables are stochastically linked or independent.

2. COMPUTATIONS OF DISTRIBUTIONS FOR THE SIX MAXIMA

(1) $P(\sup_{u \leqslant \theta} |B_u| \leqslant A)$

$$P(\sup_{u \leqslant \theta} |B_u| \leqslant A) = 1 - P(\sup_{u \leqslant \theta} |B_u| \geqslant A) = 1 - P(\theta \geqslant T_A^*)$$
$$= 1 - E\left(\exp\left(-\frac{\lambda^2 T_A^*}{2}\right)\right) \quad (\text{where } T_A^* = \inf\{t : |B_t| = A\})$$
$$= 1 - \frac{1}{\cosh(\lambda A)}.$$

(1′) $P(\sup_{u \leqslant \theta} |Z_u| < A)$

$$P(\sup_{u \leqslant \theta} |Z_u| < A) = 1 - P(\sup_{u \leqslant \theta} |Z_u| \geqslant A) = 1 - P(\theta \geqslant T_A^*)$$
$$= 1 - E(q^{T_A^*}) \quad (\text{where } T_A^* = \inf\{t : |Z_t| = A\})$$
$$= 1 - E(q^{T_{-A} \wedge T_A}) \quad (\text{where } T_{\pm A} = \inf\{t : Z_t = \pm A\})$$
$$= 1 - \frac{2(\alpha^{-A} - \alpha^{A})}{\alpha^{-2A} - \alpha^{2A}} = 1 - \frac{2}{\alpha^{-A} + \alpha^{A}}.$$

(2) $P(\sup_{t \leqslant g_\theta} |B_t| \leqslant A)$

$$P(\sup_{t \leqslant g_\theta} |B_t| \leqslant A) = P(g_\theta \leqslant T_A^*) = P(\theta \leqslant d_{T_A^*}) = 1 - P(\theta \geqslant d_{T_A^*})$$
$$= 1 - E\left(\exp\left(-\frac{\lambda^2}{2} d_{T_A^*}\right)\right) = 1 - E\left(\exp\left(-\frac{\lambda^2}{2} T_A^*\right)\right)$$
$$\times E\left(\exp\left(-\frac{\lambda^2}{2} T_A\right)\right) \quad (\text{where } T_A = \inf\{t : B_t = A\})$$
$$= 1 - \frac{1}{\cosh(\lambda A)} \exp(-\lambda A) = \tanh(\lambda A).$$

(2′) $P(\sup_{t \leqslant g_\theta} |Z_t| < A)$

$$P(\sup_{t \leqslant g_\theta} |Z_t| < A) = P(g_\theta < T_A^*) = P(\theta < d_{T_A^*}) = 1 - P(\theta \geqslant d_{T_A^*})$$
$$= 1 - E(q^{d_{T_A^*}}) = 1 - E(q^{T_A^*}) E(q^{T_A})$$
$$= 1 - \frac{2}{\alpha^{-A} + \alpha^{A}} \alpha^{A} = \frac{\alpha^{-A} - \alpha^{A}}{\alpha^{-A} + \alpha^{A}}.$$

(3) $P(\sup_{g_\theta \leqslant t \leqslant \theta} |B_t| \leqslant A)$

We start with

$$P(\sup_{t\leqslant g_\theta} |B_t| \leqslant A)\, P(\sup_{g_\theta \leqslant t \leqslant \theta} |B_t| \leqslant A) = P(\sup_{t\leqslant\theta} |B_t| \leqslant A),$$

since pre-g_θ events and post-g_θ events are independent (see Revuz and Yor [10], Chapter XII). Then we get

$$P(\sup_{g_\theta \leqslant t \leqslant \theta} |B_t| \leqslant A) = \frac{\tanh(\lambda A)}{1-1/\cosh(\lambda A)} = \tanh\left(\frac{\lambda A}{2}\right).$$

(3′) $P(\sup_{g_\theta \leqslant t \leqslant \theta} |Z_t| < A)$

For random walk, we do as with the preceding argument:

$$P(\sup_{t\leqslant g_\theta} |Z_t| < A)\, P(\sup_{g_\theta \leqslant t \leqslant \theta} |Z_t| < A) = P(\sup_{t\leqslant\theta} |Z_t| < A).$$

Then we get

$$P(\sup_{g_\theta \leqslant t \leqslant \theta} |Z_t| < A) = \frac{1-2/(\alpha^{-A}+\alpha^{A})}{(\alpha^{-A}-\alpha^{A})/(\alpha^{-A}+\alpha^{A})} = \frac{\alpha^{-A/2}-\alpha^{A/2}}{\alpha^{-A/2}+\alpha^{A/2}}.$$

(4) $P(\sup_{t\leqslant d_\theta} |B_t| \leqslant A)$

$$\begin{aligned} P(\sup_{t\leqslant d_\theta} |B_t| \leqslant A) &= P(d_\theta \leqslant T_A^*) = 1-P(\theta > g_{T_A^*}) \\ &= 1-E(\exp(-\lambda^2 g_{T_A^*}/2)) \\ &= 1-\frac{1/\cosh(\lambda A)}{\lambda A/\sinh(\lambda A)} = 1-\frac{\tanh(\lambda A)}{\lambda A}, \end{aligned}$$

since the following equality holds:

$$E\left(\exp\left(\frac{-\lambda^2 g_{T_A^*}}{2}\right)\right) E\left(\exp\left(\frac{-\lambda^2}{2}(T_A^* - g_{T_A^*})\right)\right) = E\left(\exp\left(\frac{-\lambda^2}{2} T_A^*\right)\right).$$

(4′) $P(\sup_{t\leqslant d_\theta} |Z_t| < A)$

$$\begin{aligned} P(\sup_{t\leqslant d_\theta} |Z_t| < A) &= P(d_\theta < T_A^*) = 1-P(\theta \geqslant g_{T_A^*}) \\ &= 1-E(q^{g_{T_A^*}}) = 1-\frac{2/(\alpha^{-A}+\alpha^{A})}{A(\alpha^{-1}-\alpha)/(\alpha^{-A}+\alpha^{A})} \\ &= 1-\frac{1}{A}\frac{2}{\alpha^{-1}-\alpha}\frac{\alpha^{-A}-\alpha^{A}}{\alpha^{-A}+\alpha^{A}}. \end{aligned}$$

(5) $P(\sup_{\theta\leqslant t\leqslant d_\theta}|B_t|\leqslant A)$

If $0\leqslant x\leqslant A$, we have

$$P(\sup_{\theta\leqslant t\leqslant d_\theta}|B_t|\leqslant A \mid |B_\theta|=x)=P(T_0<T_{A-x})=1-\frac{x}{A}.$$

Clearly, if $A\leqslant x$, we obtain $P(\sup_{\theta\leqslant t\leqslant d_\theta}|B_t|\leqslant A \mid |B_\theta|=x)=0$. Then we get

$$P(\sup_{\theta\leqslant t\leqslant d_\theta}|B_t|\leqslant A)=\int_0^A\left(1-\frac{x}{A}\right)f_{|B_\theta|}(x)\,dx$$

$$=\int_0^A\left(1-\frac{x}{A}\right)\lambda e^{-\lambda x}\,dx=1-\frac{1-e^{-\lambda A}}{\lambda A}.$$

(5′) $P(\sup_{\theta\leqslant t\leqslant d_\theta}|Z_t|<A)$

If $x\leqslant A$, we have

$$P(\sup_{\theta\leqslant t\leqslant d_\theta}|Z_t|<A \mid |Z_\theta|=x)=P(T_0<T_{A-x})=1-\frac{x}{A}.$$

Clearly, if $A\leqslant x$, we obtain $P(\sup_{\theta\leqslant t\leqslant d_\theta}|Z_t|\leqslant A \mid |Z_\theta|=x)=0$. Then we get

$$P(\sup_{\theta\leqslant t\leqslant d_\theta}|Z_t|<A)=\sum_{k=0}^{A}\left(1-\frac{k}{A}\right)P(|Z_\theta|=k)=1-\frac{2}{\alpha^{-1}-\alpha}\frac{1-\alpha^A}{A},$$

where we used the following facts:

$$E(t^{Z_\theta})=\sum_{k=0}^{\infty}\left(\frac{t+t^{-1}}{2}\right)^k(1-q)\,q^k=\frac{2(1-q)}{q(\alpha^{-1}-\alpha)}\left(\sum_{k=0}^{\infty}\left(\frac{\alpha}{t}\right)^{k+1}+\sum_{k=0}^{\infty}\alpha^k t^k\right).$$

Then

$$P(Z_\theta=k)=\frac{1-\alpha}{1+\alpha}\alpha^k,\qquad k\in\boldsymbol{Z},$$

and we see that

$$P(|Z_\theta|=k)=\begin{cases}\dfrac{1-\alpha}{1+\alpha} & \text{for } k=0,\\[2mm] \dfrac{2(1-\alpha)}{1+\alpha}\alpha^k & \text{for } k\geqslant 1.\end{cases}$$

(6) $P(\sup_{g_\theta\leqslant t\leqslant d_\theta}|B_t|\leqslant A)$

$$P(\sup_{g_\theta\leqslant t\leqslant d_\theta}|B_t|\leqslant A)=\frac{P(\sup_{t\leqslant d_\theta}|B_t|\leqslant A)}{P(\sup_{t\leqslant g_\theta}|B_t|\leqslant A)}$$

$$=\frac{1-(\tanh(\lambda A))/\lambda A}{\tanh(\lambda A)}=\coth(\lambda A)-\frac{1}{\lambda A}.$$

(6′) $P(\sup_{g_\theta \leqslant t \leqslant d_\theta} |Z_t| < A)$

$$P(\sup_{g_\theta \leqslant t \leqslant d_\theta} |Z_t| < A) = \frac{P(\sup_{t \leqslant d_\theta} |Z_t| < A)}{P(\sup_{t \leqslant g_\theta} |Z_t| < A)}$$

$$= \frac{1 - \dfrac{1}{A} \dfrac{2}{\alpha^{-1} - \alpha} \dfrac{\alpha^{-A} - \alpha^{A}}{\alpha^{-A} + \alpha^{A}}}{\dfrac{\alpha^{-A} - \alpha^{A}}{\alpha^{-A} + \alpha^{A}}} = \frac{\alpha^{-A} + \alpha^{A}}{\alpha^{-A} + \alpha^{A}} - \frac{1}{A} \frac{2}{\alpha^{-1} - \alpha}.$$

3. COMPUTATIONS OF DISTRIBUTIONS FOR THE SIX TIMES

In this section, we slightly change our presentation, by first discussing the Brownian case, then the RW case.

We use the notation $\beta_{a,b}$ and γ_a for some beta and gamma distributed variables. Recall that

$$P(\beta_{a,b} \in dt) = \frac{t^{a-1}(1-t)^{b-1}\, dt}{B(a, b)} \qquad (0 < t < 1),$$

$$P(\gamma_a \in dt) = t^{a-1} e^{-t} \frac{dt}{\Gamma(a)} \qquad (t > 0).$$

We also use e_λ, or $Exp(\lambda)$, to denote an exponentially distributed variable with parameter λ. Finally, U is uniform on $(0, 1)$.

3.1. The BM case

(a) It is well known that g_t and $t - g_t$ are arcsine distributed. So we get

$$g_\theta \sim \theta - g_\theta \sim \frac{N^2}{\lambda^2}.$$

By scaling,

$$d_\theta - \theta \sim \theta(d_1 - 1) \sim \theta B_1^2 T_1 \sim \frac{2}{\lambda^2} \frac{N^2}{\hat{N}^2} e,$$

since

$$d_1 - 1 = \inf\{u \mid B_{1+u} = 0\} = \inf\{u \mid B_{1+u} - B_1 = -B_1\},$$

$$\frac{d_1 - g_1}{1 - g_1} = 1 + \frac{d_1 - 1}{1 - g_1} \sim 1 + \left(\frac{B_1^2}{1 - g_1}\right) \frac{1}{N^2}$$

$$\sim 1 + \frac{m_1^2}{N^2} \sim 1 + \frac{2e}{2\gamma_{1/2}} \sim \frac{1}{\beta_{1/2,1}} \sim \frac{1}{U^2},$$

where we used the fundamental fact about the meander

$$\left(m_u = \frac{|B_{g_1+u(1-g_1)}|}{\sqrt{1-g_1}},\ u \leqslant 1\right)$$

that $m_1^2 \sim 2e$ (see Biane and Yor [5]). Then we get

$$d_\theta - g_\theta \sim \frac{\theta - g_\theta}{U^2} \sim \frac{N^2}{\lambda^2 U^2}.$$

(b) We remark that we get similar results for any Bessel process with dimension $\delta = 2(1-\alpha)$ $(0 < \delta < 2)$, which, in fact, has been the starting point of the paper by Bertoin et al. [1], where four properties of $(d_\theta - g_\theta)$ are discussed.

Let R_t be the Bessel process with dimension $\delta = 2(1-\alpha)$ $(0 < \delta < 2)$.

We define $g_u = \sup\{s < u \colon R_s = 0\}$, $d_u = \inf\{s > u \colon R_s = 0\}$ for $u \geqslant 0$. Then we get

$$d_1 - 1 = \inf\{u \colon R_{1+u} = 0\} \sim \frac{R_1^2}{2\gamma_\alpha}.$$

Consequently,

$$\frac{d_1 - 1}{1 - g_1} \sim \frac{R_1^2/(1-g_1)}{2\gamma_\alpha} \sim \frac{2e}{2\gamma_\alpha},$$

where we used the fact that the distribution of the meander process at time 1 is the same for every δ (see Yor [12]). Then we get

$$\frac{d_1 - g_1}{1 - g_1} \sim \frac{e}{\gamma_\alpha} + 1 \sim \frac{1}{\beta_{\alpha,1}} \sim \frac{1}{U^{1/\alpha}}.$$

Hence

$$d_\theta - g_\theta \sim \theta(d_1 - g_1) \sim \frac{\theta - g_\theta}{U^{1/\alpha}} \sim \frac{\gamma_{1-\alpha}}{U^{1/\alpha}},$$

which implies

$$E\big(\exp(-\lambda(d_\theta - g_\theta))\big) = (1+\lambda)^\alpha - \lambda^\alpha, \qquad \text{where } \theta \sim Exp(1/2).$$

Again this is the starting point of the paper by Bertoin et al. [1].

3.2. The RW case. The identity

$$P(g_{2n} = 2k) = \binom{2k}{k}\binom{2(n-k)}{n-k} 2^{-2n} \qquad (k = 0, 1, \dots, n)$$

is well known (see Feller [7]). Then

$$P(g_\theta = 2k) = \sum_{l=k}^{\infty} P(g_{2l} = 2k \cap \theta = 2l) + \sum_{l=k}^{\infty} P(g_{2l+1} = 2k \cap \theta = 2l)$$

$$= (1-q^2)\binom{2k}{k} \sum_{l=0}^{\infty} \binom{2l}{l}\left(\frac{q^2}{4}\right)^{l+k} = (1-q^2)^{1/2}\binom{2k}{k}\left(\frac{q^2}{4}\right)^k.$$

Consequently,

$$(*)\qquad E(t^{g_\theta}) = \sum_{k=0}^{\infty} P(g_\theta = 2k)\, t^{2k} = \left(\frac{1-q^2}{1-q^2 t^2}\right)^{1/2} = \sqrt{\frac{1+q}{1+qt}}\sqrt{\frac{1-q}{1-qt}}.$$

Similarly we get

$$P(\theta - g_\theta = 2k) = \sqrt{\frac{1-q}{1+q}}\binom{2k}{k}\left(\frac{q}{2}\right)^{2k} \qquad (k = 0, 1, 2, \ldots),$$

$$P(\theta - g_\theta = 2k+1) = \sqrt{\frac{1-q}{1+q}}\binom{2k}{k}\left(\frac{q}{2}\right)^{2k+1} \qquad (k = 0, 1, 2, \ldots).$$

Then

$$(**)\qquad E(t^{\theta - g_\theta}) = \sqrt{\frac{1-q}{1+q}}\sqrt{\frac{1+qt}{1-qt}} = \sqrt{\frac{1+qt}{1+q}}\sqrt{\frac{1-q}{1-qt}}.$$

Comparing formulae $(*)$ and $(**)$, we see that the distributions of g_θ and $\theta - g_\theta$ are not the same, but they are close and in the Brownian limit they coincide. We remark that

$$E(s^\theta t^{g_\theta}) = \frac{1-q}{1-sq} E(t^{g_{\theta'}}) = \frac{1-q}{1-sq}\sqrt{\frac{1-s^2 q^2}{1-s^2 q^2 t^2}},$$

where $\theta' \sim Geom(1-sq)$. By this formula we recover

$$E(t^\theta t^{-g_\theta}) = \sqrt{\frac{1+qt}{1+q}}\sqrt{\frac{1-q}{1-qt}}.$$

Next we investigate $E(t^{d_\theta})$:

$$E(t^{d_\theta}) = P(\theta' \geqslant d_\theta) = 1 - P(\theta' < d_\theta) = 1 - P(g_{\theta'} \leqslant \theta) = 1 - E(q^{g_{\theta'}})$$
$$= 1 - \sqrt{\frac{1-t^2}{1-q^2 t^2}}$$

where $\theta' \sim Geom(1-t)$. Hence, similarly we get

$$E(s^\theta t^{d_\theta}) = \frac{1-q}{1-sq}\left(1 - \sqrt{\frac{1-t^2}{1-s^2 q^2 t^2}}\right).$$

Then

$$E(t^{d_\theta - \theta}) = \frac{1-q}{1-qt}\left(1 - \sqrt{\frac{1-t^2}{1-q^2}}\right).$$

By the independence of $d_\theta - g_\theta$ and g_θ, we obtain

$$E(t^{d_\theta - g_\theta}) = \frac{E(t^{d_\theta})}{E(t^{g_\theta})} = \frac{\sqrt{1-q^2 t^2} - \sqrt{1-t^2}}{\sqrt{1-q^2}}.$$

We note also that

$$E(t^{d_\theta} s^{g_\theta} u^{\theta}) = \frac{\sqrt{1-u^2q^2t^2}-\sqrt{1-t^2}}{\sqrt{1-t^2s^2u^2q^2}} \frac{1-q}{1-uq}.$$

To obtain the BM result, we put $t = e^{-\alpha\Delta t}$, $s = e^{-\beta\Delta t}$, $u = e^{-\gamma\Delta t}$, $q = 1-\mu\Delta t$, and letting Δt tend to 0, we get

$$E(e^{-\alpha d_\theta} e^{-\beta g_\theta} e^{-\gamma\theta}) = \frac{\mu}{\sqrt{\alpha+\beta+\gamma+\mu}\,(\sqrt{\alpha+\mu+\gamma}+\sqrt{\alpha})},$$

where $\theta \sim Exp(\mu)$. This allows to recover the BM result in (3.1).

4. SOME INFINITE DIVISIBILITY PROPERTIES OF LENGTHS OF EXCURSIONS IN BM AND RW

First we consider the infinite divisibility properties of random times associated with RW.

Formulae found in the next theorem express the Lévy–Khintchine representation of the infinitely divisible variable at hand.

THEOREM 4.1. *The generating functions of* θ, g_θ, $\theta-g_\theta$, $d_\theta-g_\theta$ *and* d_θ *are given by*:

(1) $$E(t^{\theta}) = \frac{1-q}{1-qt} = \exp\Big(-\sum_{n=1}^{\infty}(1-t^n)\frac{q^n}{n}\Big);$$

(2) $$E(t^{g_\theta}) = \left(\frac{1-q^2}{1-q^2t^2}\right)^{1/2} = \exp\big(-\sum_{n=1}^{\infty}(1-t^n)\mu_1\{n\}\big),$$

where $\mu_1\{2k\} = q^{2k}/2k$ *for* $k \geqslant 1$, $\mu_1\{2k+1\} = 0$ *for* $k \geqslant 0$;

(3) $$E(t^{\theta-g_\theta}) = \left(\frac{(1-q)(1+qt)}{(1+q)(1-qt)}\right)^{1/2} = \exp\big(-\sum_{n=1}^{\infty}(1-t^n)\mu_2\{n\}\big),$$

where $\mu_2\{2k+1\} = q^{2k+1}/(2k+1)$ *for* $k \geqslant 0$, $\mu_2\{2k\} = 0$ *for* $k \geqslant 1$;

(4) $$E(t^{d_\theta-g_\theta-2}) = \frac{\sqrt{1-q^2t^2}-\sqrt{1-t^2}}{t^2\sqrt{1-q^2}} = \exp\big(-\sum_{n=1}^{\infty}(1-t^n)\mu_3\{n\}\big),$$

where

$$\mu_3\{2l\} = \frac{1}{2l2^{2l}}\sum_{m=0}^{l}\binom{2(l-m)}{l-m}\binom{2m}{m}q^{2m}, \quad \mu_3\{2l+1\} = 0 \quad \textit{for } l \geqslant 0;$$

(5) $$E(t^{d_\theta-2}) = E(t^{g_\theta})E(t^{d_\theta-g_\theta-2}) = \exp\big(-\sum_{n=1}^{\infty}(1-t^n)(\mu_1+\mu_3\{n\}\big).$$

Proof. Since (1), (2), (3) are easy, we only prove (4). We have

$$\sum_{l=0}^{\infty} t^{2l} \frac{1}{2^{2l}} \sum_{m=0}^{l} \binom{2(l-m)}{l-m}\binom{2m}{m} q^{2m} = \sum_{m=0}^{\infty} \binom{2m}{m} \sum_{l=m} \binom{2(l-m)}{l-m}\left(\frac{t^{2l}}{4^l}\right)$$
$$= (1-t^2)^{-1/2}(1-q^2t^2)^{-1/2}.$$

On the other hand,

$$\frac{d}{dt}\log\frac{\sqrt{1-q^2t^2}-\sqrt{1-t^2}}{t^2\sqrt{1-q^2}} = \frac{1}{t\sqrt{1-t^2}\sqrt{1-q^2t^2}} - \frac{1}{t}.$$

Comparing these two last expressions, we obtain

$$\mu_3\{2l\} = \frac{1}{2l2^{2l}} \sum_{m=0}^{l} \binom{2(l-m)}{l-m}\binom{2m}{m} q^{2m} \quad \text{for } l \geqslant 1$$

and

$$\mu_3\{2l+1\} = 0 \quad \text{for } l \geqslant 0. \blacksquare$$

This result suggested us to find the analogous result in the BM case, which has been further extended in Bertoin et al. [1].

THEOREM 4.2. *For BM, taking $\theta \sim e/m \sim Exp(m)$, we obtain*

$$E\big(\exp(-\mu(d_\theta - g_\theta))\big) = \frac{\sqrt{m}}{\sqrt{\mu+m}+\sqrt{\mu}} = \exp\Big(-\int_0^\infty (1-e^{-\mu x})\, n(dx)\Big),$$

where

$$n(dx) = \frac{dx}{x}\int_0^x \frac{1}{2\pi}\frac{1}{\sqrt{y(x-y)}} e^{-my}\, dy.$$

Proof. We have

$$\int_0^\infty (1-e^{-\mu x})\frac{dx}{x}\int_0^x \frac{e^{-my}\,dy}{2\pi\sqrt{y(x-y)}} = \int_0^\infty \Big(\int_0^\mu e^{-xu}\,du\Big)dx \int_0^x \frac{e^{-my}\,dy}{2\pi\sqrt{y(x-y)}}$$
$$= \frac{1}{2\pi}\int_0^\mu du \int_0^\infty \frac{e^{-my}}{\sqrt{y}}\,dy \int_y^\infty \frac{e^{-xu}}{\sqrt{x-y}}\,dx$$
$$= \frac{1}{2}\int_0^\mu \frac{1}{\sqrt{u(m+u)}}\,du = \big[\log(s+\sqrt{m+s^2})\big]_0^{\sqrt{\mu}}$$
$$= -\log\frac{\sqrt{m}}{\sqrt{m+\mu}+\sqrt{\mu}}. \blacksquare$$

We generalize this result to Bessel processes and investigate further results in Bertoin et al. [1].

5. RELATION TO RIEMANN'S ZETA FUNCTIONAL EQUATION

It is known that the maximum of BM excursion is related to Riemann's zeta function; see Biane [2], Biane et al. [3], Biane and Yor [4], Chung [6], Yor [13], Williams [11]. In this section we note that the distribution of the maximum of the Brownian excursion stradding an independent exponential time gives Riemann's zeta functional equation via another time randomization than the exponential one. This represents a slight variation of some of the arguments used by the previous authors.

By Table 1 (last row) and Table 2 (row before last), taking $\lambda = 1$, we have

$$P\left(A \geqslant \frac{|N|\, M_{\text{exc}}}{U}\right) = \coth(A) - \frac{1}{A}, \quad \text{where } M_{\text{exc}} = \sup_{u \leqslant 1} e_u.$$

Putting $A = \sqrt{e}/(\lambda^2/2)$, we get

$$E\left(\exp\left(\frac{-\lambda^2}{2}\frac{N^2 (M_{\text{exc}})^2}{U^2}\right)\right) = \int_0^\infty \frac{\lambda^2}{2} \exp\left(\frac{-\lambda^2 u}{2}\right)\left(\coth(\sqrt{u}) - \frac{1}{\sqrt{u}}\right) du.$$

Putting $\hat{M}_{\text{exc}} = M_{\text{exc}}/U$, and using integration by parts, we obtain

$$2\int_0^\infty E\left(\exp\left(\frac{-\lambda^2}{2} x^2 (\hat{M}_{\text{exc}})^2\right)\right)\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) dx$$

$$= \int_0^\infty \exp\left(\frac{-\lambda^2 v^2}{2}\right)\left(\frac{1}{v^2} - \frac{1}{\sinh^2 v}\right) dv.$$

Then we have

$$\frac{1}{v^2} - \frac{1}{\sinh^2 v} = \sqrt{\frac{\pi}{2}}\, E\left(\frac{1}{\hat{M}_{\text{exc}}}\exp\left(\frac{-v^2}{2(\hat{M}_{\text{exc}})^2}\right)\right).$$

Then noting

$$\frac{1}{v^2} - \frac{1}{\sinh^2 v} = \frac{1}{v^2}\left(1 - E\left(\exp\left(\frac{-v^2 T}{2}\right)\right)\right) = E\left(\int_0^T \exp\left(\frac{-v^2 x}{2}\right) dx\right),$$

where $T = T_1^{(3)} + \hat{T}_1^{(3)}$, i.e. T is the independent sum of two first hitting times to 1 by 3-dimensional Bessel processes, we get

$$P(T > x) = \sqrt{\frac{2}{\pi}}\, E\left(M_{\text{exc}}, \frac{1}{M_{\text{exc}}^2} > x\right).$$

Using Chung's remark $(M_{\text{exc}})^2 \sim T_{\pi/2}^{(3)} + \hat{T}_{\pi/2}^{(3)}$, we get

$$P(T > x) = \sqrt{\frac{2}{\pi}}\, E\left(\frac{\pi}{2}\sqrt{T}, \frac{1}{T} > \frac{\pi^2}{4} x\right).$$

This shows that

$$(*) \qquad E(\mathscr{T}^s) = E(\mathscr{T}^{1/2-s}), \quad \text{where } \mathscr{T} = \frac{\pi}{2} T.$$

We now explain the connection between (*) and the functional equation for Riemann's zeta function ζ. First, ζ is defined classically as

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} \quad \text{for } \operatorname{Re}(s) > 1$$

and extended meromorphically to the complex plane. Its functional equation is usually written as follows:

$$\xi(s) = \xi(1-s), \quad \text{where } \xi(s) = \frac{s(s-1)}{2} \Gamma(s/2) \pi^{-s/2} \zeta(s).$$

This is a translation of (*), since it is not difficult to show that $2\xi(2s) = E(\mathscr{T}^s)$ (see Biane [2], Biane et al. [3], Biane and Yor [4], Chung [6], Yor [13], Williams [11]).

6. APPENDIX: A CASE STUDY OF AN INDEPENDENCE PROPERTY IN RANDOM BROWNIAN SCALING

Consider $(B_u, u \geqslant 0)$, a standard Brownian motion, starting at 0. Let $0 \leqslant a < b$ denote two random times, and

$$B_u^{[a,b]} \equiv \frac{1}{\sqrt{b-a}} B_{a+u(b-a)}, \quad u \leqslant 1.$$

Thus $(B_u^{[a,b]}, u \leqslant 1)$ is the process $(B_v, a \leqslant v \leqslant b)$, once it has been *Brownian scaled* so that the interval $[a, b]$ becomes $[0, 1]$.

In this Appendix, we look at a number of examples where $B^{[a,b]}$ is independent of the length $(b-a)$, and also at a number of opposite cases where $(b-a)$ is *not* independent of $B^{[a,b]}$. This is motivated by the property that in the 1st column of Table 1, for 4 rows out of 6, we find that

$$\sup_{a \leqslant v \leqslant b} |B_v| \sim \sqrt{b-a} \sup_{u \leqslant 1} |B_u^{[a,b]}|,$$

where independence is now meant on the right-hand side(!) and it is not true for the remaining two rows.

6.1. Three cases of independence

Case 1. $a = 0$, $b = g_t \equiv \sup\{u \leqslant t : B_u = 0\}$.

Then $B^{[0,g_t]}$ is a standard Brownian bridge independent of g_t, and in fact of $\sigma\{g_t; B_{g_t+u}, u \geqslant 0\}$.

Case 2. $a=g_t,\ b=d_t\equiv\inf\{u>t: B_u=0\}$.

Then $|B^{[g_t,d_t]}|$ is a standard BES (3) bridge, independent of

$$\sigma\{B_u;\ u\leqslant g_t\}\vee\sigma\{\mathrm{sgn}(B_t)\}\vee\sigma\{d_t;\ B_{d_t+u},\ u\geqslant 0\}.$$

In particular, $B^{[g_t,d_t]}$ is independent of the pair (g_t,d_t), hence of (d_t-g_t).

Case 3. $a=g_t,\ b=t$.

Then $|B^{[g_t,t]}|$ is a standard Brownian meander, independent of $\mathscr{F}_{g_t}\vee\sigma\{\mathrm{sgn}(B_t)\}$. As a consequence, $B^{[g_t,t]}$ is independent of g_t.

6.2. Two cases of non-independence. The next two cases do not enjoy the independence property.

Case 4. $a=t,\ b=d_t$.

Then $B^{[t,d_t]}$ is not independent of d_t (or d_t-t). Indeed, the variable

$$B_0^{[t,d_t]}\equiv\frac{B_t}{\sqrt{d_t-t}}$$

is not independent of (d_t-t), since

$$(B_t,\sqrt{d_t-t})\sim\left(B_t,\frac{|B_t|}{|N|}\right).$$

Thus

$$\left(\frac{B_t}{\sqrt{d_t-t}},\sqrt{d_t-t}\right)\sim\left(N,\frac{|B_t|}{|N|}\right),$$

which shows very clearly that (d_t-t) is not independent of $B_t/\sqrt{d_t-t}$. A more interesting question, which we leave to the reader, is the following: what is the law of (d_t-t) given $B^{[t,d_t]}$?

Case 5. $a=0,\ b=d_t$.

Then $B^{[0,d_t]}$ is not independent of d_t.

We give two different arguments, which are also different from the argument in Case 4.

5a. First, we look at the local times of $B^{[0,d_t]}$, i.e. for any test function $f: \boldsymbol{R}\to\boldsymbol{R}_+$,

$$\int_0^1 f\left(\frac{B_{ud_t}}{\sqrt{d_t}}\right)du\equiv\int_{-\infty}^{\infty}f(x)L_1^x(B^{[0,d_t]})\,dx.$$

The left-hand side is

$$\frac{1}{d_t}\int_0^{d_t}f\left(\frac{B_v}{\sqrt{d_t}}\right)dv=\frac{1}{d_t}\int_{-\infty}^{\infty}f\left(\frac{x}{\sqrt{d_t}}\right)L_{d_t}^x(B)\,dx=\frac{1}{\sqrt{d_t}}\int_{-\infty}^{\infty}f(y)L_{d_t}^{y\sqrt{d_t}}(B)\,dy.$$

Hence, we have

$$L_1^y(B^{[0,d_t]})\equiv\frac{1}{\sqrt{d_t}}L_{d_t}^{y\sqrt{d_t}}(B)\qquad\text{for every } y\in\boldsymbol{R},$$

and so

$$L_1^0(B^{[0,d_t]}) = \frac{1}{\sqrt{d_t}} L_t^0(B).$$

Now, we can infer from this identity that d_t is not independent of $B^{[0,d_t]}$. Assume the contrary; then

$$E(L_t^0(B)) = E(\sqrt{d_t})\, E(L_1^0(B^{[0,d_t]})).$$

But, the left-hand side is finite, whereas $E(\sqrt{d_t}) = \infty$.

5b. Another argument consists in noting that

$$g_t/d_t = \sup\{u < 1 : B^{[0,d_t]} = 0\}.$$

So g_t/d_t is measurable with respect to $B^{[0,d_t]}$, but is not independent of d_t. Indeed, if it were independent of d_t, we would have

$$E(g_t) = E(g_t/d_t)\, E(d_t)$$

but the left-hand side is bounded by t, and the right-hand side is infinite.

A more detailed study of this Case 5 is made in Fujita and Yor [8].

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