**Added in proof** (April 1991). I. Condition 2.2(2) implies convergence of the series in 2.2(1) and the summability assumption implies condition 2.2(2). However, a large part of the theory holds assuming only 2.2(1)–(2). The summability assumption is required for the density result (Theorem 2.10) and to commutate the interpolation functor and $L^p$ (Theorem 3.3).

II. Theorem 2.6 can be proved directly (without appeal to Lemmas 2.4 and 2.5) by means of decomposition and ideas similar to the ones already used in Theorem 2.3.

III. The interpolation result 2.7 can be improved to obtain the constant $2M_0\,\bar{\varrho}(M_1/M_0)$. One has to follow Gustavsson–Peetre's ideas and recall the symmetry properties of the Rademacher functions.

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# On two classes of Banach spaces with uniform normal structure

by

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**Abstract.** We give two classes of Banach spaces $X$ that have uniform normal structure. The first class is closed under duality, and contains the uniformly convex spaces as well as the uniformly smooth spaces. The second class is defined by $J(X) < 3/2$, where $J(X) = \sup\{\|x+y\| \wedge \|x-y\|: \|x\| = \|y\| = 1\}$. Both classes of spaces are uniformly nonsquare, their properties are being studied.

**§ 1. Introduction.** A Banach space $X$ is said to have *normal structure* [2, 8] if for each bounded closed convex subset $K$ in $X$ that contains more than one point, there exists a point $x \in K$ such that

$$\sup\{\|x-y\|: y \in K\} < \operatorname{diam} K.$$

$X$ is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any subset $K$ as above, there exists $x \in K$ such that

$$\sup\{\|x-y\|: y \in K\} < c \operatorname{diam} K.$$

It is well known that uniform convexity in every direction implies normal structure [8, 28], whereas uniform convexity and uniform smoothness imply uniform normal structure [8, 27]. Our main purpose in this paper is to give two new classes of Banach spaces with uniform normal structure and study their relevant properties.

Let $S(X) = \{x \in X: \|x\| = 1\}$ be the unit sphere of $X$. For $x \in X$, let $\nabla_x$ denote the set of norm 1 supporting functionals $f$ of $S(X)$ at $x$. In [16] Lau introduced the following notion to study the Chebyshev subset of $X$:

DEFINITION 1.1. A Banach space $X$ is called a *U-space* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(1.1)\qquad \forall x, y \in S(X),\quad \|(x+y)/2\| > 1-\delta \Rightarrow \langle f, y\rangle > 1-\varepsilon,\ \forall f \in \nabla_x.$$

Some of the properties of $U$-spaces in [16] are summarized in the following theorem.

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THEOREM 1.2. (i) *If $X$ is a U-space, then $X$ is uniformly nonsquare, in particular, $X$ is superreflexive* [8];

(ii) *$X$ is a U-space if and only if $X^*$ is a U-space;*

(iii) *Uniformly convex spaces and uniformly smooth spaces are U-spaces.*

One of the main results in this paper is

THEOREM 1.3. *If $X$ is a U-space, then $X$ has uniform normal structure.*

In an attempt to simplify Schäffer's notion of girth and perimeter [22], the authors studied in [12] the parameter

$$J(X) = \sup\{\|x+y\| \wedge \|x-y\|:\ x, y \in S(X)\}.$$

They showed that $\sqrt{2} \leqslant J(X) \leqslant 2$; $J(X) < 2$ if and only if $X$ is uniformly nonsquare; $J(l_2) = \sqrt{2}$, $J(l_1) = J(l_\infty) = 2$, and more general, $J(l_p) = J(L_p) = \max\{2^{1/p}, 2^{1/q}\}$, where $1/p+1/q = 1$, $p \geqslant 1$. Our second main theorem is

THEOREM 1.4. *Let $X$ be a Banach space with $J(X) < 3/2$. Then $X$ has uniform normal structure.*

Let $\delta(\varepsilon) = \inf\{1-\frac{1}{2}\|x+y\|:\ \|x-y\| \geqslant \varepsilon,\ x, y \in S(X)\}$, $0 \leqslant \varepsilon \leqslant 2$, be the *modulus of convexity* of $X$. The relation of $J(X)$ and $\delta(\varepsilon)$ is: $J(X) < \varepsilon$ if and only if $\delta(\varepsilon) > 1-\varepsilon/2$ (Corollary 5.5). As a consequence, we have

COROLLARY 1.5. *Let $X$ be a Banach space with $\delta(3/2) > 1/4$. Then $X$ has uniform normal structure.*

This result is closely related to a theorem of Goebel [13], namely: if $\delta(1) > 0$, then $X$ has uniform normal structure. It is worthwhile to mention that if a modulus of convexity $\delta$ satisfies the condition $\delta(3/2) > 1/4$ and the function $\delta$ is convex, then $\delta(1) > 0$ (since $\delta(1) = 0$ implies $\delta(2^-) = \lim_{t\to 2^-} \delta(t) \leqslant 1/2$). Therefore Corollary 1.5 is a new result for the spaces with nonconvex moduli of convexity. Prus [22] and we both independently found different examples of Banach spaces with $\delta(3/2) > 1/4$ and $\delta(1) = 0$.

In [12], it is shown that $J(X)$ can be estimated through isomorphism. We improve that result when $X$ is isomorphic to $l_p$ or $L_p$:

THEOREM 1.6. *For any isomorphism $T$ from $X$ to $l_p$ or $L_p$, $1 < p < \infty$, $J(X) \leqslant \|T\| \cdot \|T^{-1}\| \max\{2^{1/p}, 2^{1/q}\}$, where $1/p+1/q = 1$.*

As a consequence we can conclude that certain isomorphs of $l_p$ or $L_p$ are uniformly nonsquare (Corollary 6.5).

The paper is arranged as follows: In §2, we prove some preparation lemmas. The main result is Lemma 2.3. It amounts to saying that if $X$ does not have $w$-normal structure, then the unit ball contains a hexagon with a certain property. As a direct application, we give a simple proof of a result of Turett [27] concerning the modulus of smoothness and normal structure (Corollary 2.4). In §3, Lemma 2.3 is used to prove the normal structure of $U$-spaces. We also obtain some equivalent forms of $U$-spaces. The ultraproduct of $U$-spaces is taken up in §4, which leads to *uniform* normal structure for such spaces. In §5, we use Lemma 2.3 again to prove the uniform normal structure for $J(X) < 3/2$. We also establish the connection of $J(X)$ and $\delta(\varepsilon)$, the modulus of convexity of $X$. In §6, some isomorphism results for $J(X)$ are obtained. Finally in §7, we give some remarks in connection with the works of Bynum [5] and Pichugov [21] on the normal structure coefficient, and also pose some open questions.



**§2. Some lemmas.** Let $X$ be a Banach space. For $r > 0$, $x \in X$, let $B(x, r) = \{y:\ \|x-y\| \leqslant r\}$, and let $B(X) = B(0, 1)$ be the unit ball of $X$.

LEMMA 2.1. *For $0 < \varepsilon < 1$, let $L = [x_1; x_2]$ be a line segment in $B(X)$ such that $\frac{1}{2}\|x_1+x_2\| > 1-\varepsilon$. Then for any $z \in L$, $\|z\| > 1-2\varepsilon$.*

Proof. Let $z = tx_1+(1-t)x_2$, $0 \leqslant t \leqslant 1$. For $0 \leqslant t \leqslant 1/2$,

$$z = (2-2t)\frac{x_1+x_2}{2} - (1-2t)x_1,$$

and hence $\|z\| > (2-2t)(1-\varepsilon)-(1-2t) \geqslant 1-2\varepsilon$. A similar proof holds for $1/2 \leqslant t \leqslant 1$.

A Banach space $X$ is said to have *w-normal structure* if for each weakly compact convex set $K$ in $X$ that contains more than one point, there exists an $x \in K$ such that $\sup\{\|x-y\|:\ y \in K\} < \operatorname{diam} K$. It is clear that if $X$ is reflexive, then normal structure (as defined in §1) and $w$-normal structure coincide. The following is a special case of a result of van Dulst [11].

LEMMA 2.2. *Let $X$ be a Banach space without w-normal structure. Then for any $0 < \varepsilon < 1$, there exists a sequence $\{z_n\} \subseteq S(X)$ with $z_n \xrightarrow{w} 0$ and*

$$1-\varepsilon < \|z_{n+1}-z\| < 1+\varepsilon,$$

*for sufficiently large $n$, and any $z \in \operatorname{co}\{z_k\}_{k=1}^n$.*

LEMMA 2.3. *Let $X$ be a Banach space without w-normal structure. Then for any $0 < \varepsilon < 1$, there exist $x_1$, $x_2$, $x_3$ in $S(X)$ satisfying*

(i) $x_2-x_3 = ax_1$ *with* $|a-1| < \varepsilon$;

(ii) $|\,\|x_1-x_2\|-1|$, $|\,\|x_3-(-x_1)\|-1| < \varepsilon$; *and*

(iii) $\frac{1}{2}\|x_1+x_2\|$, $\frac{1}{2}\|x_3+(-x_1)\| > 1-\varepsilon$.

The geometric meaning of the lemma can be succinctly described as: if $X$ does not have $w$-normal structure, then there exists an inscribed hexagon in $S(X)$ with length of each side arbitrarily close to 1 (by (i) and (ii)), and with at least four sides whose distances to $S(X)$ are arbitrarily small (by (iii)).

Proof. Let $\eta = \varepsilon/4$ and let $\{z_n\}$ be chosen as in Lemma 2.2 with $\varepsilon$ replaced by $\eta$. We claim that for large $n$,

$$1-\eta < \|z_n - z_1/2\| < 1+\eta. \tag{2.1}$$

In fact, since $z_n \xrightarrow{w} 0$, 0 is in the weak closed convex hull of $\{z_n\}$, which equals the norm closed convex hull $\overline{\mathrm{co}}\{z_n\}_{n=1}^{\infty}$, and there exists $n_0$ and $y \in \mathrm{co}\{z_n\}_{n=1}^{n_0}$ with $\|y\| < \eta$. We may assume $n_0$ also satisfies

$$1-\eta/2 < \|z_n - z\| < 1+\eta/2,$$

for $n \geqslant n_0$ and $z \in \mathrm{co}\{z_k\}_{k=1}^{n_0}$ as in Lemma 2.2. The claim follows by observing that for $n \geqslant n_0$,

$$\|z_n - z_1/2\| \geqslant \|z_n - (y+z_1)/2\| - \|y/2\| > (1-\eta/2) - \eta/2 = 1-\eta,$$

$$\|z_n - z_1/2\| \leqslant \|z_n - (y+z_1)/2\| + \|y/2\| < (1+\eta/2) + \eta/2 = 1+\eta.$$

Let $f_1$ be the supporting functional of $z_1 \in S(X)$, i.e. $\|f_1\| = \langle f_1, z_1\rangle = 1$. Since $z_n \xrightarrow{w} 0$, we can assume, in conjunction with (2.1), that $z_{n_0}$ satisfies

$$|\langle f_1, z_{n_0}\rangle| < \eta, \qquad 1-\eta < \|z_{n_0} - z_1\|, \ \|z_{n_0} - z_1/2\| < 1+\eta.$$

Let $w = (z_1 - z_{n_0})/\|z_1 - z_{n_0}\|$. Then $w$, $z_1$ and $z_{n_0}$ will play the role of $x_1$, $x_2$ and $x_3$ respectively: condition (i) is satisfied by the definition of $w$; for (ii) we need only observe that

$$\|z_1 - w\| = \big\|(1-\|z_1 - z_{n_0}\|)w - z_{n_0}\big\| \leqslant \frac{1}{\|z_1 - z_{n_0}\|}(|1-\|z_1 - z_{n_0}\|| + \|z_{n_0}\|)$$

$$\leqslant \frac{\eta+1}{1-\eta} < 1+4\eta = 1+\varepsilon,$$

$$\|z_1 - w\| \geqslant \frac{1}{\|z_1 - z_{n_0}\|}(\|z_{n_0}\| - |1-\|z_1 - z_{n_0}\||) \geqslant \frac{1-\eta}{1+\eta} > 1-4\eta = 1-\varepsilon.$$

Hence $|\,\|z_1 - w\| - 1| < \varepsilon$. Similarly we have $|\,\|w + z_{n_0}\| - 1| < \varepsilon$. Finally to prove (iii), we observe that

$$\|w + z_1\| \geqslant \langle f_1, w + z_1\rangle = 1 + \langle f_1, w\rangle$$

$$= 1 + \frac{\langle f_1, z_1\rangle - \langle f_1, z_{n_0}\rangle}{\|z_1 - z_{n_0}\|} > 1 + \frac{1-\eta}{1+\eta} > 2-4\eta,$$

$$\|z_{n_0} - w\| \geqslant \|z_{n_0} - (z_1 - z_{n_0})\| - \|(z_1 - z_{n_0}) - w\| \geqslant 2\|z_{n_0} - z_1/2\| - \eta > 2-4\eta.$$

Let $\varrho(\tau) = \sup\{(\|x+\tau y\| + \|x-\tau y\| - 2)/2\colon x, y \in S(X)\}$ be the *modulus of smoothness* of $X$. As a simple consequence of Lemma 2.3 we have

COROLLARY 2.4 (Baillon, Turett [27]). *If $X$ is a Banach space with $\lim_{\tau\downarrow 0} \varrho(\tau)/\tau < 1/2$, then $X$ has $w$-normal structure.*

Proof. Suppose $X$ does not have $w$-normal structure. For $\tau > 0$, let $\varepsilon = \tau^2$ and choose $x_1$, $x_2$, $x_3$ in $S(X)$ satisfying the conditions in Lemma 2.3. Let $x = x_1$, $y = (x_2 - x_1)/\|x_2 - x_1\|$. Then for $0 < \tau < 1$, $\|x+\tau y\| > 1-2\varepsilon$ (by Lemmas 2.1 and 2.3(iii)), and



$$\|x-\tau y\| \geqslant \|x_1 - \tau(x_2 - x_1)\| - \left\|\tau(x_2 - x_1) - \frac{\tau(x_2 - x_1)}{\|x_2 - x_1\|}\right\|$$

$$= \|(1+\tau)x_1 - \tau(ax_1 + x_3)\| - \tau|\,\|x_2 - x_1\| - 1|$$

$$\geqslant \|(1+\tau)x_1 - \tau(x_3 + x_1)\| - \tau|1-a| - \tau\varepsilon$$

$$\geqslant (1+\tau)\left\|\frac{\tau}{1+\tau}x_3 + \frac{1}{1+\tau}(-x_1)\right\| - 2\varepsilon\tau$$

$$\geqslant (1+\tau)(1-2\varepsilon) - 2\varepsilon\tau.$$

Hence

$$\frac{\varrho(\tau)}{\tau} \geqslant \frac{\tau - 4\varepsilon - 4\varepsilon\tau}{2\tau} = \frac{1}{2} - 2\tau(1+\tau).$$

We have $\lim_{\tau\downarrow 0} \varrho(\tau)/\tau \geqslant 1/2$, which contradicts the assumption on $\varrho$.

**§ 3. The $U$-spaces.** We begin by giving a useful equivalent criterion for $U$-spaces. For $0 < \delta < r < 1$, $x \in X$, we use $N_r(x, \delta) = B(X) \setminus B(-rx, 1+r-\delta)$ to denote the lune determined by the two balls.

THEOREM 3.1. *Let $X$ be a Banach space. Then $X$ is a $U$-space if and only if for any $\varepsilon > 0$, $1 > r > 0$, there exists $r > \delta > 0$ such that for any $x \in S(X)$, and for any $y_1, y_2, z \in N_r(x; \delta)$,*

(3.1) $$|\langle f, y_1 - y_2\rangle| < \varepsilon, \quad \textit{for all } f \in \nabla_z.$$

Proof. The necessity is given in [16, Lemma 3.1]. To prove the sufficiency, for $\varepsilon > 0$, $1 > r > 0$, let $0 < \delta < r$ be chosen to satisfy (3.1). For any $x, y \in S(X)$ with $\|(x+y)/2\| > 1-\delta/2$, we have

$$\|y + rx\| = \|(x+y) - (1-r)x\| \geqslant \|x+y\| - (1-r)$$

$$> (2-\delta) - (1-r) = 1+r-\delta.$$

This implies that $y \notin B(-rx, 1+r-\delta)$, and hence $y \in N_r(x, \delta)$. Now for any $f \in \nabla_x$, (3.1) implies $|\langle f, x-y\rangle| = 1 - \langle f, y\rangle < \varepsilon$, and therefore $\langle f, y\rangle > 1-\varepsilon$ for all $f \in \nabla_x$. $X$ is then a $U$-space.

THEOREM 3.2. *Suppose $X$ is a $U$-space. Then $X$ has normal structure.*

Proof. For $1/3 > \varepsilon > 0$, $1 > r > 0$, let $\delta$ be defined as in (3.1), and let $\varepsilon' = \min\{\varepsilon, \delta/(2(1+r))\}$. Note that $X$ is reflexive (Theorem 1.2(i)). Suppose $X$ does not have normal structure; then it does not have $w$-normal structure either. There exist $x_1, x_2, x_3 \in S(X)$ satisfying the conditions in Lemma 2.3 with respect to $\varepsilon'$. We claim that $x_2, -x_3 \notin B(-rx_1, 1+r-\delta)$. In fact, let $y = (x_2 + rx_1)/(1+r)$. Then $y \in [x_1; x_2]$, and by Lemma 2.1,

$$\|x_2 + rx_1\| = (1+r)\|y\| \geqslant (1+r)(1-2\varepsilon') \geqslant 1+r-\delta.$$

The same argument holds for $-x_3$. Let $f \in F_{x_1}$. Then by (3.1), we have

$$2 = 2\langle f, x_1\rangle \leqslant |\langle f, x_1 - x_2\rangle| + |\langle f, x_1 - (-x_3)\rangle| + |\langle f, x_2 - x_3\rangle|$$
$$\leqslant \varepsilon + \varepsilon + \|x_2 - x_3\|.$$

But by Lemma 2.3(i), $\|x_2 - x_3\| \leqslant 1+\varepsilon$, which implies that $2 < 1 + 3\varepsilon$ and hence a contradiction.

To show that the above $X$ actually has uniform normal structure, we need a more sophisticated argument. We will establish another equivalent condition of $U$-space which will be used in the following section.

For any $x \in S(X)$, $F_x$ is a $w^*$-compact convex subset of $X^*$, and for any smooth point $x$ of $S(X)$, $F_x$ is a singleton. Let $K \subseteq X^*$ be a bounded closed subset. Then $f \in K$ is called a *$w^*$-strongly exposed point* if there exists $x \in S(X)$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\forall g \in K, \quad \langle f, x\rangle < \langle g, x\rangle + \delta \Rightarrow \|g - f\| < \varepsilon.$$

A Banach space $X$ is called an *Asplund space* if every continuous convex function on $X$ is Fréchet differentiable on a dense $G_\delta$ subset. It is well known that $X$ is an Asplund space if and only if every $w^*$-compact convex subset $K$ in $X^*$ is the $w^*$-closed convex hull of its $w^*$-strongly exposed points [1, 10, 19]. Also in this case, every sequence in $K$ has a $w^*$-convergent subsequence [26].

LEMMA 3.3. *Let $X$ be an Asplund space, $x \in S(X)$, and let $f$ be a $w^*$-strongly exposed point of $F_x$. Then there exist sequences $\{x_n\} \subseteq S(X)$, $\{f_n\} \subseteq S(X^*)$ with $x_n \xrightarrow{\|\cdot\|} x$, $f_n \xrightarrow{w^*} f$, where the $x_n$'s are Fréchet differentiable points of $S(X)$, and $f_n \in F_{x_n}$ is the (unique) supporting functional of $S(X)$ at $x_n$.*

In order to prove the above lemma, we will need the following three results.

LEMMA 3.4. *Let $X$ be a Banach space, let $\{x_n\}$, $x$ be in $S(X)$, and let $f_n \in F_{x_n}$. Suppose $x_n \xrightarrow{\|\cdot\|} x$ and $f_n \xrightarrow{w^*} f$. Then $f \in F_x$.*

Proof. This follows from

$$1 \geqslant |\langle f, x\rangle| = \lim_{n\to\infty} |\langle f_n, x_n\rangle + \langle f_n, x - x_n\rangle| \geqslant 1 - \lim_{n\to\infty} \|x - x_n\| = 1.$$

LEMMA 3.5 [9, p. 22]. *Let $X$ be a Banach space. For any $x, y \in S(X)$, let $u = (x+\lambda y)/\|x+\lambda y\|$ with $\lambda \geqslant 0$. Then for $f_x \in F_x$, $f_u \in F_u$,*

$$\langle f_x, y\rangle \leqslant (\|x+\lambda y\| - 1)/\lambda \leqslant \langle f_u, y\rangle.$$

LEMMA 3.6. *Let $X$ be a two-dimensional Banach space, and let $f, g \in F_x$, and $z \in S(X)$ such that $-\alpha = \langle g, z\rangle < \langle f, z\rangle = 0$. Let $A = \{y \in S(X): \langle g, y\rangle \leqslant \langle f, y\rangle\}$ denote the half sphere. Suppose $\{y_n\} \subseteq A$ with $y_n \xrightarrow{\|\cdot\|} x$, and suppose $\{f_n\} \subseteq S(X^*)$ satisfies*

(3.2) $$\langle f_n, y_n\rangle \geqslant 1 - \tfrac{1}{4}\alpha\|x - y_n\|, \quad \text{for every } n.$$

*Then $\langle f_n, z\rangle \geqslant 0$ for large $n$.*



Proof. Note that $x, z \in A$, and $y_n = (x+\lambda_n z)/\|x+\lambda_n z\|$, $\lambda_n \geqslant 0$. Since $\|y_n - x\| \to 0$, it is clear that $\lambda_n \to 0$ as $n \to \infty$. By Lemma 3.5,

$$\liminf_{n\to\infty} \langle g, x - y_n\rangle/\lambda_n = \liminf_{n\to\infty} [(\|x+\lambda_n z\| - 1) + \lambda_n\alpha]/\lambda_n \geqslant \langle f, z\rangle + \alpha = \alpha,$$

hence there exists $N_1$ such that for $n > N_1$,

(3.3) $$\langle g, y_n\rangle = 1 - \langle g, x - y_n\rangle < 1 - \lambda_n\alpha/2 \leqslant 1 - \tfrac{1}{4}\alpha\|x - y_n\|.$$

Let $u_n$ be the support point of $f_n$. We claim that $u_n \in A$. Suppose this is false. By passing to a subsequence, we can assume that $\{u_n\}$ is contained in the complementary half sphere $A' = S(X)\setminus A$. The hypothesis implies that $\langle f_n, x\rangle \to 1$, so that no subsequence of $\{u_n\}$ converges to $-x$; hence there exists $N_2$ and $\delta > 0$ such that if $N > N_2$, then $\|u_n - (-x)\| > \delta$. Also there exists $N_3$ such that for $n > N_3$, $\|y_n - x\| < \delta$. For $n > N = \max\{N_1, N_2, N_3\}$, $x$ can be represented as

$$x = \frac{u_n + \lambda_n y_n}{\|u_n + \lambda_n y_n\|}, \quad \lambda_n \geqslant 0.$$

Apply Lemma 3.5 again: we have $\langle f_n, y_n\rangle \leqslant \langle g, y_n\rangle$. This is impossible in view of (3.2) and (3.3), and the claim is proved.

Now for $n > N$, $u_n \in A$, either $u_n = (x+\lambda_n z)/\|x+\lambda_n z\|$ or $u_n = (-x+\lambda_n z)/\|-x+\lambda_n z\|$, $\lambda_n \geqslant 0$. In the first case, Lemma 3.5 implies that $\langle f_n, z\rangle \geqslant \langle f, z\rangle = 0$. In the second case, take $-f$ as a support functional at $-x$. Then Lemma 3.5 again implies that $\langle f_n, z\rangle \geqslant \langle -f, z\rangle = 0$. This proves the lemma.

Proof of Lemma 3.3. If $F_x$ is a singleton, the assertion is clear. Assume $F_x$ contains more than one point, and let $f$ be a $w^*$-strongly exposed point of $F_x$. Then there exists a $z \in S(X)$ such that

(3.4) $$\langle g, z\rangle < \langle f, z\rangle, \quad \text{for all } g \in F_x\setminus\{f\}.$$

Without loss of generality, assume $\langle f, z\rangle = 0$. Let $X_2$ be the two-dimensional subspace spanned by $x$ and $z$. Let

$$A' = \{y \in S(X_2): \langle g, y\rangle \leqslant \langle f, y\rangle \text{ for all } g \in F_x\setminus\{f\}\}.$$

Then $A'$ is a half sphere. Suppose $\{y_n\} \subseteq A'$ with $y_n \xrightarrow{\|\cdot\|} x$. Let $\{\bar{y}_n\} \subseteq S(X)$ be the Fréchet differentiable points such that $\|y_n - \bar{y}_n\| < \|x - y_n\|/n$, and let $f_n$ be the unique supporting functional of $S(X)$ at $\bar{y}_n$. We claim that $\langle f_n, z\rangle \geqslant 0$ for large $n$. In fact, let $\eta$ be small enough such that $K = \overline{\mathrm{co}}^{w^*}(F_x\setminus B(f, \eta)) \neq \varnothing$. Since $f$ is a $w^*$-strongly exposed point of $F_x$, $f \notin K$. By the $w^*$-compactness of $K$, we can find a $g_1 \in K$ such that

$$-\alpha = \langle g_1, z\rangle = \sup\{\langle g, z\rangle: g \in K\} < 0.$$

Let

$$A'' = \{y \in S(X_2): \langle g_1, y\rangle \leqslant \langle f, y\rangle\}.$$

It is obvious that $A'' = A'$. It follows from the choice of $f_n$ that

$$1 \geqslant \langle f_n, y_n\rangle = \langle f_n, \bar{y}_n\rangle - \langle f_n, \bar{y}_n - y_n\rangle \geqslant 1 - \|\bar{y}_n - y_n\| > 1 - \tfrac{1}{4}\alpha\|x - y_n\|,$$

for large $n$, and since $y_n \xrightarrow{\|\cdot\|} x$, $\langle f_n, x\rangle \to 1$. Let $\bar{f}_n$ be the normalization of the restriction of $f_n$ on $X_2$. Then from the above $\langle \bar{f}_n, y_n\rangle > 1 - \frac{1}{4}\alpha\|x - y_n\|$. Use Lemma 3.6: we have $\langle \bar{f}_n, z\rangle \geqslant 0$, and so $\langle f_n, z\rangle \geqslant 0$.

Note that since $X$ is an Asplund space, there exists a $w^*$-convergent subsequence of $\{f_n\}$ which converges weak* to some $h \in X^*$ [26]. Without loss of generality, let $\{f_n\}$ itself be this subsequence, i.e. $f_n \xrightarrow{w^*} h$. Then $h \in V_x$ (by Lemma 3.4) and $\langle h, z\rangle \geqslant 0$ (by the above proof); (3.4) hence implies that $h = f$.

Our goal is to obtain the following equivalence statement for $U$-spaces.

THEOREM 3.7. *Let $X$ be a Banach space. Then the following are equivalent:*

(i) *$X$ is a $U$-space;*

(ii) *There exists a function $\varphi$: $S(X) \to S(X^*)$ such that $\varphi(x) \in V_x$ and (1.1) is satisfied for $f = \varphi(x)$ (instead of $\forall f \in V_x$);*

(iii) *There exists a dense $G_\delta$ subset $D \subseteq S(X)$ and a function $\varphi$: $D \to S(X^*)$ such that $\varphi(x) \in V_x$ and (1.1) is satisfied for $x \in D$ and $f = \varphi(x)$.*

Proof. It is clear that (i) implies (ii), and (ii) implies (iii). To prove (iii) implies (i), we first claim that $X$ is uniformly nonsquare. Indeed, for $0 < \varepsilon < 1$, let $\delta$ be chosen to satisfy the conditions in (iii). If $X$ is not uniformly nonsquare, then for the above $\delta$, there exist $x, y \in D$ such that $\frac{1}{2}\|x+y\|, \frac{1}{2}\|x-y\| > 1-\delta$. Hence by (1.1), $\langle \varphi_x, y\rangle$ and $\langle \varphi_x, -y\rangle \geqslant 1-\varepsilon$; this is impossible. It follows that $X$ is reflexive, the set of Fréchet differentiable points $F$ of $S(X)$ is a dense $G_\delta$ set, and the $w^*$-strongly exposed points of bounded closed convex sets $K$ in $X^*$ coincide with the strongly exposed points of $K$. Without loss of generality, we assume $D = F$ and hence $\{\varphi(x)\} = V_x$ for $x \in D$.

Now for any $\varepsilon > 0$, let $\delta_1 > 0$ such that for any $x \in D$, $y \in S(X)$ with $\|(x+y)/2\| > 1-\delta_1$, $\langle \varphi_x, y\rangle > 1-\varepsilon/2$. Let $\delta = \min\{\delta_1/2, \varepsilon/2\}$. Then for any $x \in S(X)$, and for any strongly exposed point $f \in V_x$, Lemma 3.3 implies that there exists a sequence of Fréchet differentiable points $\{x_n\} \subseteq D$ such that $x_n \xrightarrow{\|\cdot\|} x$, and $\varphi_{x_n} \xrightarrow{w} f$. Suppose $y \in S(X)$ and $\|(x+y)/2\| > 1-\delta$. Take $N$ such that $\|x_N - x\| < \delta$ and $|\langle \varphi_{x_N} - f, y\rangle| < \delta$. Then

$$\|(x_N+y)/2\| > \|(x+y)/2\| - \|(x_N-x)/2\| > (1-\delta) - \delta/2 > 1-\delta_1,$$

and hence

$$\langle f, y\rangle = \langle \varphi_{x_N}, y\rangle + \langle f - \varphi_{x_N}, y\rangle > 1 - \varepsilon/2 - \delta > 1-\varepsilon.$$

Note that $V_x$ is the closed convex hull of its strongly exposed point $f$, hence we have $\langle f, y\rangle \geqslant 1-\varepsilon$ for all $f \in V_x$ and (i) follows.

**§4. Ultraproducts and uniform normal structure of $U$-spaces.** Let $\mathscr{F}$ be a *filter* on an index set $I$, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space $X$. Then $\{x_i\}_{i\in I}$ is said to converge to $x$ with respect to $\mathscr{F}$, denoted by $\lim_{\mathscr{F}} x_i = x$, if for each neighborhood $U$ of $x$, $\{i \in I: x_i \in U\} \in \mathscr{F}$. A filter $\mathscr{U}$ on $I$ is called an *ultrafilter* if it is maximal with respect to the ordering of set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A: A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathscr{U}$ is an ultrafilter, then (i) for any $A \subseteq I$, either $A \in \mathscr{U}$ or $I \backslash A \in \mathscr{U}$; (ii) if $\{x_i\}_{i\in I}$ has a cluster point $x$, then $\lim_{\mathscr{U}} x_i$ exists and equals $x$.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I}\|x_i\| < \infty$.

DEFINITION 4.1 [7, 24]. Let $\mathscr{U}$ be an ultrafilter on $I$ and let

$$N_{\mathscr{U}} = \{(x_i) \in l_\infty(I, X_i): \lim_{\mathscr{U}}\|x_i\| = 0\}.$$

The *ultraproduct* of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathscr{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathscr{U}}$ to denote the element of the ultraproduct. It follows from property (ii) above and the definition of quotient norm that

$$\|(x_i)_{\mathscr{U}}\| = \lim_{\mathscr{U}}\|x_i\|. \tag{4.1}$$

In the following we will restrict our index set $I$ to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space $X$. For an ultrafilter $\mathscr{U}$ on $\mathbb{N}$, we use $X_{\mathscr{U}}$ to denote the ultraproduct. Note that if $\mathscr{U}$ is nontrivial, then $X$ can be embedded into $X_{\mathscr{U}}$ isometrically.

LEMMA 4.2 [24]. *Suppose $\mathscr{U}$ is an ultrafilter on $\mathbb{N}$ and $X$ is a Banach space. Then $(X^*)_{\mathscr{U}} \cong (X_{\mathscr{U}})^*$ if and only if $X$ is superreflexive; and in this case, the mapping $J$ defined by*

$$\langle J((f_i)_{\mathscr{U}}), (x_i)_{\mathscr{U}}\rangle = \lim_{\mathscr{U}}\langle f_i, x_i\rangle, \quad \text{for all } (x_i)_{\mathscr{U}} \in X_{\mathscr{U}},$$

*is the canonical isometric isomorphism from $(X^*)_{\mathscr{U}}$ onto $(X_{\mathscr{U}})^*$.*

THEOREM 4.3. *Suppose $X$ is a $U$-space. Then for any ultrafilter $\mathscr{U}$ on $\mathbb{N}$, $X_{\mathscr{U}}$ is also a $U$-space.*

Proof. Since a $U$-space is uniformly nonsquare, it is hence superreflexive [8], and by Lemma 4.2, $(X_{\mathscr{U}})^* = (X^*)_{\mathscr{U}}$. For any $\varepsilon > 0$, let $\delta$ be as in the definition of $U$-space, i.e.

$$\forall x, y \in S(X), \quad \|(x+y)/2\| > 1-\delta \Rightarrow \langle f_x, y\rangle > 1-\varepsilon, \quad \forall f_x \in V_x. \tag{4.2}$$

Let $(x_i)_{\mathscr{U}}, (y_i)_{\mathscr{U}} \in S(X_{\mathscr{U}})$ and $\|((x_i)_{\mathscr{U}} + (y_i)_{\mathscr{U}})/2\| > 1-\delta$. Without loss of generality we may assume $\|x_i\| = \|y_i\| = 1$ for all $i \in \mathbb{N}$. Then from (4.1), $I = \{i: \|(x_i+y_i)/2\| > 1-\delta\} \in \mathscr{U}$, and $I \neq \varnothing$. For each $i \in \mathbb{N}$, take an $f_{x_i} \in V_{x_i}$. Since $\langle (f_{x_i})_{\mathscr{U}}, (x_i)_{\mathscr{U}}\rangle = \lim_{\mathscr{U}}\langle f_{x_i}, x_i\rangle = 1$, $(f_{x_i})_{\mathscr{U}} \in V_{(x_i)_{\mathscr{U}}}$. From (4.2), we have $\langle f_{x_i}, y_i\rangle > 1-\varepsilon$, for all $i \in I$, so $\langle (f_{x_i})_{\mathscr{U}}, (y_i)_{\mathscr{U}}\rangle = \lim_{\mathscr{U}}\langle f_{x_i}, y_i\rangle \geqslant 1-\varepsilon$. By letting $\varphi$: $S(X_{\mathscr{U}}) \to S((X_{\mathscr{U}})^*)$, defined by $\varphi((x_i)_{\mathscr{U}}) = (f_{x_i})_{\mathscr{U}}$, $(x_i)_{\mathscr{U}} \in S(X_{\mathscr{U}})$, Theorem 3.7(ii) implies that $X_{\mathscr{U}}$ is a $U$-space.

THEOREM 4.4. *If $X$ is a $U$-space, then $X$ has uniform normal structure.*

Proof. Suppose $X$ does not have uniform normal structure. We can find a sequence $\{C_n\}$ of bounded closed convex subsets of $X$ such that for each $n$,

$$0 \in C_n, \quad \operatorname{diam} C_n = 1, \tag{4.3}$$

$$\operatorname{rad} C_n = \inf_{x \in C_n} \sup_{y \in C_n} \|x-y\| > 1-1/n. \tag{4.4}$$

Let $\mathcal{U}$ be any nontrivial ultrafilter on $\mathbf{N}$, and let

$$C = \{(x_n)_{\mathcal{U}}\colon\ x_n \in C_n,\ n \in \mathbf{N}\}.$$

Then $C$ is a nonempty bounded closed convex subset of $X_{\mathcal{U}}$. It follows from (4.1), (4.3) and (4.4) that $\operatorname{diam} C = \operatorname{rad} C = 1$, so $X_{\mathcal{U}}$ does not have normal structure. On the other hand, from Theorem 4.3, $X_{\mathcal{U}}$ is a $U$-space. This contradicts Theorem 3.2, and $X$ must have uniform normal structure.

Theorems 1.2(ii) and 4.4 yield

COROLLARY 4.5. *Both uniformly convex spaces and uniformly smooth spaces have uniform normal structure.*

**§5. $J(X)$ and uniform normal structure.** It was proved in [12] that if $X$ is a two-dimensional Banach space whose unit sphere is defined by a right hexagon, then $J(X) = 3/2$. Hexagon plays an important role in normal structure as shown in Lemma 2.3. In the following we will pursue this connection further.

LEMMA 5.1. *Let $X$ be a Banach space, and let $0 < \varepsilon < 1$. Suppose there exist $x_1$, $x_2$ and $x_3$ in $S(X)$ satisfying the conditions in Lemma 2.3. Then $J(X) > 3/2-4\varepsilon$.*

Proof. Let $y = \frac{1}{2}(x_2+x_3)$. Then

$$\|(y-x_1)-x_3\| = \|(x_2-x_3)/2-x_1\| = 1-a/2,$$

and $1-a/2$ is bounded by $\frac{1}{2}(1\pm\varepsilon)$.

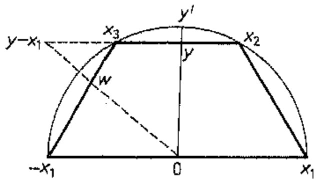


Fig. 1

Let $w = \lambda(y-x_1)$ be on the line segment $[-x_1; x_3]$. A comparison of similar triangles yields

$$\frac{\|(y-x_1)-w\|}{\|w\|} = \frac{1-a/2}{1}.$$



It follows that $(1-\lambda)/\lambda = 1-a/2$ and hence $1/\lambda \geqslant 3/2-\varepsilon$. Also hypothesis (iii) of Lemma 2.3 and Lemma 2.1 imply $\|w\| > 1-2\varepsilon$, and therefore we have

$$\|y-x_1\| = \frac{1}{\lambda}\|w\| > \frac{3}{2}-4\varepsilon.$$

Similarly $\|y+x_1\| > 3/2-4\varepsilon$. If we let $y' = y/\|y\| \in S(X)$, it is clear that

$$\|y'+x_1\| \geqslant \|y+x_1\|, \qquad \|y'-x_1\| \geqslant \|y-x_1\|$$

(for if $z_1$, $z_2$ are the intersections of a straight line and the unit sphere $S(X)$, and if $z(t) = z_2+(z_2-z_1)t$, $t \geqslant 0$, is one of the half lines outside the sphere, then $\|z(t)\|$ is an increasing function of $t$). This implies $J(X) \geqslant 3/2-4\varepsilon$.

We now proceed as in §4 to obtain the uniform normal structure for $J(X) < 3/2$. We first prove a result for ultraproduct.

THEOREM 5.2. *For any Banach space $X$, and for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbf{N}$, $J(X_{\mathcal{U}}) = J(X)$.*

Proof. For any $\varepsilon > 0$, choose $x, y$ so that $\|x\|, \|y\| \leqslant 1$ and $\|x\pm y\| \geqslant J(X)-\varepsilon$. Let $x_i = x$, $y_i = y$ for $i \in \mathbf{N}$. Then $\|(x_i)_{\mathcal{U}}\|, \|(y_i)_{\mathcal{U}}\| \leqslant 1$ and $\|(x_i)_{\mathcal{U}} \pm (y_i)_{\mathcal{U}}\| \geqslant J(X)-\varepsilon$. This implies that $J(X_{\mathcal{U}}) \geqslant J(X)-\varepsilon$ for any $\varepsilon > 0$.

To prove the reverse inequality, we can choose

$$\|(x_i)_{\mathcal{U}}\|,\ \|(y_i)_{\mathcal{U}}\| \leqslant 1, \qquad \|(x_i)_{\mathcal{U}} \pm (y_i)_{\mathcal{U}}\| > J(X_{\mathcal{U}})-\varepsilon.$$

From property (i) of ultrafilters (§4), and by (4.1), we know that the subsets $\{i \in \mathbf{N}\colon \|x_i\| < 1\}$, $\{i \in \mathbf{N}\colon \|y_i\| < 1\}$ and $\{i \in \mathbf{N}\colon \|x_i \pm y_i\| > J(X_{\mathcal{U}})-\varepsilon\}$ are all in $\mathcal{U}$. Hence their intersection $M$ is nonvoid. Let $i \in M$. Then

$$\|x_i-y_i\| \wedge \|x_i+y_i\| > J(X_{\mathcal{U}})-\varepsilon.$$

This implies that $J(X) > J(X_{\mathcal{U}})-\varepsilon$ for any $\varepsilon > 0$.

THEOREM 5.3. *Let $X$ be a Banach space with $J(X) < 3/2$. Then $X$ has uniform normal structure.*

Proof. Note that $J(X) < 2$ if and only if $X$ is uniformly nonsquare [12]. Now since $J(X) < 3/2$, $X$ is reflexive [15]. Lemmas 5.1 and 2.3 imply that $X$ has $w$-normal structure, and hence normal structure. By using the same proof as for Theorem 4.4 and making use of Theorem 5.2, we conclude that $X$ has uniform normal structure.

Let $\delta(\varepsilon)$ be the modulus of convexity of $X$. The relationship of $J(X)$ and $\delta(\varepsilon)$ is as follows:

THEOREM 5.4. *Let $X$ be a Banach space. Then $J(X) = \sup\{\varepsilon\colon\ \delta(\varepsilon) \leqslant 1-\varepsilon/2\}$.*

Proof. Let $\varepsilon_0 = \sup\{\varepsilon\colon\ \delta(\varepsilon) \leqslant 1-\varepsilon/2\}$. We first show that $J(X) \leqslant \varepsilon_0$. Since $J(X) \leqslant 2$ [12, Theorem 2.5], the inequality is obvious if $\varepsilon_0$ equals 2. We can hence assume that $\varepsilon_0 < 2$. For any $\varepsilon > \varepsilon_0$, and for any $x$, $y$ in $S(X)$, either

$\|x-y\| \leqslant \varepsilon$ or $\|x-y\| > \varepsilon$. The latter case implies, by the definition of $\varepsilon_0$ and the choice of $\varepsilon$, that $\delta(\varepsilon) > 1-\varepsilon/2$. It follows that

$$1-\|(x+y)/2\| > 1-\varepsilon/2,$$

i.e. $\|x+y\| < \varepsilon$. We have in either case

$$\|x+y\| \wedge \|x-y\| \leqslant \varepsilon.$$

This implies $J(X) \leqslant \varepsilon$, and since $\varepsilon > \varepsilon_0$ is arbitrary, $J(X) \leqslant \varepsilon_0$.

To prove the reverse inequality, we let $0 < \eta < \varepsilon_0/3$, and let $\varepsilon = \varepsilon_0 - \eta$. There exist $x$, $y$ in $S(X)$ such that $\|x-y\| > \varepsilon$ and

$$1-\|(x+y)/2\| < \delta(\varepsilon)+\eta,$$

i.e. $\|x+y\| > 2-2\delta(\varepsilon)-2\eta$. This implies that

$$J(X) \geqslant \|x+y\| \wedge \|x-y\| \geqslant \min\{2(1-\delta(\varepsilon)-\eta), \varepsilon\} \geqslant \min\{\varepsilon-2\eta, \varepsilon\} \geqslant \varepsilon_0 - 3\eta.$$

Since $\eta > 0$ is arbitrary, $J(X) \geqslant \varepsilon_0$, and the proof is complete.

COROLLARY 5.5. *Let $X$ be a Banach space* ($\dim X \geqslant 2$). *Then*

(i) $J(X) \geqslant \sqrt{2}$; *and*
(ii) *for* $0 < \varepsilon \leqslant 2$, $\delta(\varepsilon) > 1-\varepsilon/2$ *if and only if* $J(X) < \varepsilon$.

Proof. (i) follows directly from Theorem 5.4 and a result of Nördlander [20]: for any Banach space $X$ with $\dim X \geqslant 2$, and for $0 < \varepsilon < 2$, $\delta(\varepsilon) \leqslant 1-(1-\varepsilon^2/4)^{1/2}$. (ii) is an easy consequence of Theorem 5.4.

As a direct corollary of Theorem 5.3 and Corollary 5.5(ii), we have

COROLLARY 5.6. *Let $X$ be a Banach space with $\delta(3/2) > 1/4$. Then $X$ has uniform normal structure.*

COROLLARY 5.7. *Let $X$ be a Banach space and suppose there exists $0 < \varepsilon \leqslant 3/2$ such that $\delta(\varepsilon) \geqslant \frac{1}{6}\varepsilon$. Then $X$ has uniform normal structure.*

Proof. If $X$ does not have uniform normal structure, then $\delta(3/2) \leqslant 1/4$. It is known that $\delta(\varepsilon)/\varepsilon$ is an increasing function for $0 < \varepsilon < 2$ [18]. We have $\delta(\varepsilon)/\varepsilon \leqslant \delta(3/2)/(3/2) \leqslant 1/6$ for all $0 < \varepsilon \leqslant 3/2$. This contradicts the assumption.

**§ 6. Isomorphism and $J(X)$.** Let $X$ be a given Banach space and let $\mathscr{X}$ be the class of Banach spaces isomorphic to $X$. Let $\Delta$ be the semimetric on $\mathscr{X}$ defined by

$$\Delta(Y, Z) = \inf\{\ln\|T\|\cdot\|T^{-1}\|:\ T\colon Y \to Z \text{ is an isomorphism}\}.$$

Let $X$, $Y$ be Banach spaces and let $T\colon X \to Y$ be an isomorphism. In [12], we proved that

$$(\|T\|\cdot\|T^{-1}\|)^{-1} \leqslant \frac{J(X)+2}{J(Y)+2} \leqslant \|T\|\cdot\|T^{-1}\|. \tag{6.1}$$



THEOREM 6.1. *If $J(Y) < 3/2$, and if*

$$\Delta(X, Y) < \ln\frac{7}{2(J(Y)+2)},$$

*then $X$ has uniform normal structure.*

Proof. By (6.1)

$$J(X) \leqslant (\exp\Delta(X, Y))\cdot(J(Y)+2)-2 < 7/2-2 = 3/2.$$

Hence Theorem 5.3 implies that $X$ has uniform normal structure.

In this section we will improve (6.1) with $Y = l_p$ or $L_p$.

LEMMA 6.2 [3, 23]. *Let $X$ be a two-dimensional Banach space, and let $K_1$, $K_2$ be closed convex subsets of $X$ with nonvoid interiors. If $K_1 \subseteq K_2$, then $r(K_1) \leqslant r(K_2)$, where $r(K_i)$ denotes the length of the circumference of $K_i$, $i = 1, 2$.*

LEMMA 6.3. *Let $X$ be a Banach space, and let $u, v \in X$. Then for any $a, b \geqslant 1$,*

$$\|u+v\|+\|u-v\| \leqslant \|au+bv\|+\|au-bv\|.$$

Proof. We can assume that $a \geqslant b \geqslant 1$. Then the triangle with vertices $bu$, $bv$ and $-bv$ is contained in the triangle with vertices $au$, $bv$ and $-bv$, so from Lemma 6.2, we have

$$\|u+v\|+\|u-v\| \leqslant \|bu+bv\|+\|bu-bv\| \leqslant \|au+bv\|+\|au-bv\|.$$

THEOREM 6.4. *For any isomorphism $T$ from $X$ to $l_p$, $1 < p < \infty$,*

$$J(X) \leqslant \|T\|\cdot\|T^{-1}\|\max\{2^{1/p}, 2^{1/q}\}, \quad \text{where } 1/p+1/q = 1.$$

Proof. Suppose $p \geqslant 2$. By applying the method of Lagrange multipliers to the function $F(s, t) = s+t$ subject to the constraint $s^p+t^p \leqslant 2^p$ for $s, t \geqslant 0$, $F(s, t)$ assumes its maximum value $2\cdot 2^{1/q}$ at the point $s = t = 2^{1/q}$ where $1/p+1/q = 1$. For $p \geqslant 2$, the Clarkson inequality [6] implies

$$\|x+y\|^p+\|x-y\|^p \leqslant 2^p, \quad \forall\, x, y \in S(l_p).$$

For $x, y \in S(l_p)$, let $u = Tx/\|Tx\|$, $v = Ty/\|Ty\|$, $a = \|T^{-1}\|\cdot\|Tx\|$, and $b = \|T^{-1}\|\cdot\|Ty\|$. Then $a, b \geqslant 1$, and by Lemma 6.3, we have

$$\begin{aligned}2(\|u+v\| \wedge \|u-v\|) &\leqslant \|u+v\|+\|u-v\| \leqslant \|au+bv\|+\|au-bv\| \\ &= \|T^{-1}\|\cdot\|Tx+Ty\|+\|T^{-1}\|\cdot\|Tx-Ty\| \\ &\leqslant \|T^{-1}\|\cdot\|T\|(\|x+y\|+\|x-y\|) \leqslant \|T^{-1}\|\cdot\|T\|\cdot 2\cdot 2^{1/q}.\end{aligned}$$

Since the above $u$, $v$ cover all the elements of $S(X)$, as $T$ is an isomorphism, we have $J(X) \leqslant \|T\|\cdot\|T^{-1}\|\cdot 2^{1/q}$.

For $1 \leqslant p \leqslant 2$ the Clarkson inequality [6] becomes

$$\|x+y\|^p+\|x-y\|^p \leqslant 2^2, \qquad \forall x, y \in S(l_p).$$

By the same proof, we have

$$\|u+v\|+\|u-v\| \leqslant \|T\|\cdot\|T^{-1}\|\cdot 2\cdot 2^{1-1/q} = 2(\|T\|\cdot\|T^{-1}\|\cdot 2^{1/p}),$$

and hence the same conclusion holds.

COROLLARY 6.5. *For $1 < p < \infty$, $1/p+1/q = 1$, if $\Delta(X, l_p) \leqslant \min(1/p, 1/q) \times \ln 2$, then $X$ is uniformly nonsquare.*

Proof. Recall that $X$ is uniformly nonsquare if and only if $J(X) < 2$ [12, Theorem 3.4]. It follows from Theorem 6.4 that

$$J(X) \leqslant (\exp \Delta(X, l_p))\max\{2^{1/p}, 2^{1/q}\} < \min\{2^{1/p}, 2^{1/q}\}\cdot\max\{2^{1/p}, 2^{1/q}\} = 2.$$

Thus $J(X)$ is uniformly nonsquare.

THEOREM 6.6. *For $1 < p < \infty$, Theorem 6.4 and Corollary 6.5 also hold if $l_p$ is replaced by $L_p$.*

**§7. Some remarks and open questions.** Let $K$ be a bounded closed convex subset of $X$, and let $D(K) = \sup\{\|x-y\|: x, y \in K\}$ be the diameter of $K$. For each $x \in K$, let $r(x, K) = \sup\{\|x-y\|: y \in K\}$, and let $R(K) = \inf\{r(x, K): x \in K\}$, the *Chebyshev radius* of $K$ [14, p. 178]. In [5], Bynum introduced the following *normal structure coefficient* of $X$:

$$N(X) = \inf\{D(K)/R(K): K \text{ bounded closed convex subset of } X\}.$$

He showed that if $X$, $Y$ are isomorphic, then $N(Y) \leqslant (\exp \Delta(X, Y))N(X)$. It is known that $N(L_p) = N(l_p) = \min\{2^{1/p}, 2^{1/q}\}$ [21]. It will be interesting to know any direct connection of $J(X)$ and $N(X)$, for any Banach space $X$.

It is known that uniformly nonsquare does not imply normal structure, e.g. let $X = (l_2, \|\cdot\|)$ where $\|\cdot\|$ is an equivalent norm of $l_2$ defined by

$$\|x\| = \max\{\|x^+\|_2, \|x^-\|_2\}$$

[4, 25]. Hence there exists $X$ such that $J(X) < 2$ and $X$ does not have normal structure.

QUESTION 7.1. Is $J(X) < 3/2$ a sharp condition for (uniform) normal structure? In other words, is $3/2$ the largest such constant?

Note that the above example still has the fixed point property by a result of Lin [17]. We pose a more restricted form of the well known open problem concerning reflexive spaces, or superreflexive spaces.

QUESTION 7.2. Does $J(X) < 2$ (equivalently, $X$ is uniformly nonsquare) imply the fixed point property?



In all the classical spaces, e.g. $l_p$, $L_p$ spaces, we have $J(X) = J(X^*)$. Recently, Prus has given an example of a Banach space $X$ such that $J(X) \neq J(X^*)$. We ask

QUESTION 7.3. What is the relation of $J(X)$ and $J(X^*)$? What is the dual parameter corresponding to $J(X)$?

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# Complex interpolation and $L^p$ spaces

by

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**Abstract.** A variant of the first Calderón interpolation method is defined in connection with an analytic functional, for complex interpolation families in the sense of Coifman, Cwikel, Rochberg, Sagher and Weiss. Some of its interpolation, duality and reiteration properties are described and applied to the identification of the interpolated space for a family of $L^p$ spaces.

**§1. Introduction.** In this paper we consider interpolation spaces for interpolation families of Banach spaces in the sense of [4]. Our method is a natural extension of that of Schechter [7] and Lions [5].

Throughout the paper we shall use the notation $A \equiv B$ to indicate a two-sided inequality, that is, there exist two constants $C$ and $C'$ such that $CA \leqslant B \leqslant C'A$.

Let $D$ denote the disc $\{|z| < 1\}$ and $\Gamma$ its boundary, and let $\{B(\gamma);\ \gamma \in \Gamma\}$ be a complex interpolation family (c.i.f.) on $\Gamma$ with $\mathscr{V}$ as the containing Banach space and $\mathscr{B}$ as the log-intersection space, in the sense of [4]. That is:

(a) The complex Banach spaces $B(\gamma)$ are continuously embedded in $\mathscr{V}$ ($\|\cdot\|_\gamma$ will be the norm on $B(\gamma)$ and $\|\cdot\|_{\mathscr{V}}$ the norm on $\mathscr{V}$),

(b) for every $b \in \bigcap_{\gamma\in\Gamma} B(\gamma)$, $\gamma \in \Gamma \to \|b\|_\gamma$ is a measurable function on $\Gamma$,

(c) $\mathscr{B} = \{b \in \bigcap_{\gamma\in\Gamma} b(\gamma);\ \int_\Gamma \log^+ \|b\|_\gamma \, d\gamma < \infty\}$, and there exists a measurable function $K(\gamma)$ on $\Gamma$ such that

$$\int_\Gamma \log^+ K(\gamma)\, d\gamma < \infty \quad \text{and} \quad \|b\|_{\mathscr{V}} \leqslant K(\gamma)\|b\|_\gamma \quad \text{a.e. } \gamma \ (b \in \mathscr{B}).$$

In [4], for every $z \in D$, the Banach space $B[z] = \{f(z);\ f \in \mathscr{F}\}$ is defined with the norm $\|b\|_z = \inf\{\|f\|_{\mathscr{F}};\ f(z) = b\}$, where $\mathscr{F} = \mathscr{F}(B(\cdot), \Gamma)$ is a Banach space of $\mathscr{V}$-valued analytic functions $f$ on $D$ with a.e. nontangential boundary values $f(\gamma) = \mathscr{V}\text{-}\lim_{\xi\to\gamma} f(\xi)$, that can be described as the completion of the space

$$\mathscr{G} = \{g = \sum_{j=1}^{N} \varphi_j(\cdot) b_j;\ b_j \in \mathscr{B},\ \varphi_j \in N^+(D),\ \operatorname*{ess\,sup}_{\gamma\in\Gamma} \|g(\gamma)\|_\gamma < \infty\},$$

with the norm $\|f\|_{\mathscr{F}} = \operatorname{ess\,sup}_{\gamma\in\Gamma} \|f(\gamma)\|_\gamma$.

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