72. On Two Classes of Subalgebras of $L^1(G)$

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1. Introduction. Let G and $\hat{G}$ be two locally compact Abelian groups in Pontrjagin duality. The Fourier transform of a function $f \in L^1(G)$ will be denoted by $\hat{f}$. For $1 \leqq p < \infty$, define

$$A^p(G) = \{f \in L^1(G) : \hat{f} \in L^p(\hat{G})\}, \qquad B^p(G) = L^1(G) \cap L^p(G).$$

The space $A^p(G)$ is a Banach algebra with respect to the norm $\|\cdot\|_{A^p(G)}$ defined by $\|f\|_{A^p(G)} = \|f\|_1 + \|\hat{f}\|_p$ and the usual convolution product. The Banach algebra $A^p(G)$ have been studied by Larsen-Liu-Wang [8], Lai [5]–[7], Martin-Yap [9], and others. The space $B^p(G)$ is a Banach algebra with respect to the norm $\|\cdot\|_{B^p(G)}$ defined by $\|f\|_{B^p(G)} = \|f\|_1 + \|f\|_p$ and the usual convolution product. The Banach algebras $B^p(G)$ have been studied by Warner [12], Yap [15], and others. The purpose of this paper is to extend some of the results on $A^p(G)$ and $B^p(G)$ to the spaces

$$A(p, q)(G) = \{f \in L^1(G) : \hat{f} \in L(p, q)(\hat{G})\}$$

and

$$B(p, q)(G) = L^1(G) \cap L(p, q)(G)$$

respectively (see next section for the definition of $L(p, q)(G)$ and some relevant facts about these spaces). In Section 2 we identify the maximal ideal spaces of the algebras $A(p, q)(G)$ and $B(p, q)(G)$, show that they satisfy Ditkin's condition and that the Shilov-Wiener Tauberian theorem holds for these algebras. In Section 3 we prove non-factorization theorems for these algebras.

2. Tauberian theorem for $A(p, q)(G)$ and $B(p, q)(G)$. For the convenience of the reader, we now review briefly what we need from the theory of $L(p, q)$ spaces.

Definition 2.1. Let f be a measurable function defined on (G, λ), where λ is the Haar measure of G. For $y \geq 0$, we define

$$m(f, y) = \lambda\{x \in G : |f(x)| > y\}.$$

For $x \geq 0$, we define

$$\begin{aligned} f^*(x) &= \inf\{y : y > 0 \text{ and } m(f, y) \leqslant x\} \\ &= \sup\{y : y > 0 \text{ and } m(f, y) > x\}, \end{aligned}$$

with the conventions $\inf \phi = \infty$ and $\sup \phi = 0$. For $x > 0$, we define

$$f^{**}(x) = x^{-1} \int_0^x f^*(t)\,dt.$$

We also define

$$\|f\|^*_{(p,q)}=\left\{\int_0^\infty [x^{1/p}f^*(x)]^q\frac{dx}{x}\right\}^{1/q}, \qquad (0<p<\infty, 0<q<\infty)$$

$$\|f\|^*_{(p,\infty)}=\sup_{x>0} x^{1/p}f^*(x) \qquad (0<p<\infty)$$

$$L(p,q)(G)=\{f: \|f\|^*_{(p,q)}<\infty\}.$$

It is quite easy to see that we have

$$\int_0^\infty f^*(x)^p\,dx=\int_G |f(x)|^p\,d\lambda(x)$$

and hence $L^p(G)=L(p,p)(G)$, $A^p(G)=A(p,p)(G)$, $B^p(G)=B(p,p)(G)$.

If we replace $f^*(x)$ by $f^{**}(x)$ in the definition of $\|f\|^*_{(p,q)}$, the resulting number will be denoted by $\|f\|_{(p,q)}$. For $1<p<\infty$, $1\leqslant q\leqslant\infty$, it is known that

(i) $\|f\|^*_{(p,q)}\le\|f\|_{(p,q)}\le p/(p-1)\cdot\|f\|^*_{(p,q)}$ (see the proof of (3.2) in [13]),

(ii) $(L(p,q), \|\cdot\|_{(p,q)})$ is a Banach space. (see [4, (2.6)], [10, (2.1)].)

Thus we can endow $A(p,q)(G)$ and $B(p,q)(G)$ $(1<p<\infty, 1\le q\le\infty)$ with the norms

$$\|f\|_{A(p,q)}=\|f\|_1+\|\hat{f}\|_{(p,q)}, \qquad \|f\|_{B(p,q)}=\|f\|_1+\|f\|_{(p,q)}$$

respectively.

We now single out the following fact for easy reference.

Lemma 2.2. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$. Let $\{f_n\}$ be a sequence in $L(p,q)(G)$ and $\|f_n-f\|_{(p,q)}\to 0$, where $f\in L(p,q)(G)$. Then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to f.*

Proof. See the proof of (2.3) in Hunt [4, p. 258] and (2.1(i)) above.

We will prove the main result in this section via the concept of Segal algebra whose definition we now recall. A subalgebra $S(G)$ of $L^1(G)$ is called a *Segal algebra* if:

(S–1) $S(G)$ is dense in $L^1(G)$ in the L^1-norm topology and if $f\in S(G)$ then $f_a\in S(G)$, where $f_a(x)=f(a^{-1}x)$;

(S–2) $S(G)$ is a Banach algebra under some norm $\|\cdot\|_S$ which also satisfies $\|f\|_S=\|f_a\|_S$ for all $f\in S(G)$, $a\in G$ (multiplication in $S(G)$ is the usual convolution);

(S–3) if $f\in(G)$, then for any $\varepsilon>0$ there exists a neighborhood U of the identity element of G such that $\|f_y-f\|_S<\varepsilon$ for all $y\in U$.

Proposition 2.3. *For $1<p<\infty$ and $1\le q<\infty$, the space $A(p,q)$ is a Segal algebra with respect to the norm $\|\cdot\|_{A(p,q)}$.*

Proof. Clearly $A(p,q)$ is a subalgebra of L^1 and $f_a\in A(p,q)$ whenever $f\in A(p,q)$, $a\in G$. Since $D=\{f\in L^1: \hat{f}$ has compact support$\}$ is dense in L^1 (see [11, 2.6.6]) and $D\subset A(p,q)$, $A(p,q)$ is dense in L^1. Thus condition (S–1) is satisfied.

That $A(p,q)$ is a Banach algebra with respect to the norm $\|\cdot\|_{A(p,q)}$ can be proved as in [8, Theorems 1 and 3], using Lemma (2.2) above.

It is clear that $\|f\|_{A(p,q)}=\|f_a\|_{A(p,q)}$ for all $f\in A(p,q)$, $a\in G$. Thus condition (S–2) is fulfiled.

Next we check that $A(p,q)$ satisfies condition (S–3). Let $0\neq f\in A(p,q)$ and let $\varepsilon>0$. First we choose a neighborhood U of the identity element e of G such that $\|f_y-f\|_1<\varepsilon/2$ for all $y\in U$. Define $\varepsilon'=\varepsilon(p-1)/p$. Choose a continuous function ϕ on $\hat{G}$ having compact support such that $\|\phi-\hat{f}\|^*_{(p,q)}<\varepsilon'/8$ (see [13, (4.2)]). Let K denote the support of ϕ, and let $K'=\hat{G}\backslash K$. It follows that

(1) $$\|\hat{f}\chi_{K'}\|^*_{(p,q)}<\varepsilon'/8.$$

Now define

$$N(K,\varepsilon')=\{y\in G:|(y,\gamma)-1|<\varepsilon'/4\|\hat{f}\|^*_{(p,q)} \text{ for all } \gamma\in K\}.$$

Then $N(K,\varepsilon')$ is a neighborhood of e in G. We now choose a symmetric neighborhood W of e such that $W\subset u\cap N(K,\varepsilon')$. It follows that

(i) for $y\in W$ and $\gamma\in K$ we have
$|\hat{f}_y(\gamma)-\hat{f}(\gamma)|=|(y^{-1},\gamma)-1|\cdot|\hat{f}(\gamma)|<\varepsilon'(4\|\hat{f}\|^*_{(p,q)})^{-1}|\hat{f}(\gamma)|$,
and hence $\|(\hat{f}_y-\hat{f})\chi_K\|^*_{(p,q)}<\varepsilon'/4$;

(ii) for $y\in W$ and $\gamma\in K'$ we have $|\hat{f}_y(\gamma)-\hat{f}(\gamma)|\leqq 2|\hat{f}(\gamma)|$. It follows from (1) that $\|(\hat{f}_y-\hat{f})\chi_{K'}\|^*_{(p,q)}<\varepsilon'/4$.

Thus for $y\in W$ we have $\|\hat{f}_y-\hat{f}\|^*_{(p,q)}<\varepsilon'/2$, and hence $\|f_y-f\|_{A(p,q)}<\varepsilon$ for all $y\in W$.

Proposition 2.4. *For $1<p<\infty$ and $1\leqq q<\infty$, the space $B(p,q)$ is a Segal algebra with respect to some norm which is equivalent to the norm $\|\cdot\|_{B(p,q)}$.*

Proof. Blozinski [1, (2.9)] shows that if $f\in L^1$ and $g\in L(p,q)$ then $\|f*g\|_{(p,q)}\leqq C(p,q)\|f\|_1\cdot\|g\|_{(p,q)}$, where $C(p,q)$ is a constant depending only on p,q. We assume with no loss of generality that $C(p,q)\geq 1$. It follows that if $f,g\in B(p,q)$ then $\|f*g\|_{B(p,q)}\leqq C(p,q)\|f\|_{B(p,q)}\cdot\|g\|_{B(p,q)}$. Thus $|||f|||_{B(p,q)}=C(p,q)\|f\|_{B(p,q)}$ defines a norm in $B(p,q)$ under which $B(p,q)$ is a Banach algebra. Since $B(p,q)$ contains all the continuous functions with compact supports, $B(p,q)$ is dense in L^1. Thus conditions (S–1) and (S–2) are satisfied.

We now prove that $B(p,q)$ satisfies condition (S–3). Let $0\neq f\in B(p,q)$ and let $\varepsilon>0$. First choose a continuous function ϕ with compact support such that $\|\phi-f\|^*_{(p,q)}<\varepsilon'/4$, where $\varepsilon'=\varepsilon(p-1)/pC(p,q)$. Let $K=$support of ϕ. By the uniform continuity of ϕ, there is a neighborhood V of the identity element e in G such that

$$\|\phi-\phi_x\|_\infty<\frac{\varepsilon'}{4}(q/p)^{1/q}(2\lambda(K))^{-1/p}$$

for all $x\in V$. It follows that $\|\phi-\phi_x\|^*_{(p,q)}<\varepsilon'/4$ for all $x\in V$. Next choose a neighborhood W of e such that $W\subset V$ and $\|f-f_x\|_1<\varepsilon/4C(p,q)$ for all $x\in W$. Thus for $x\in W$ we have

$$\begin{aligned}|||f-f_x|||_{B(p,q)} &= C(p,q)\cdot\|f-f_x\|_1+C(p,q)\cdot\|f-f_x\|_{(p,q)}\\ &<\varepsilon/4+C(p,q)[\|f-\phi\|_{(p,q)}+\|\phi-\phi_x\|_{(p,q)}+\|\phi_x-f_x\|_{(p,q)}]\\ &<\varepsilon/4+C(p,q)\frac{p}{p-1}(\varepsilon'/4+\varepsilon'/4+\varepsilon'/4)=\varepsilon.\end{aligned}$$

Theorem 2.5. *Let $S(G)=A(p,q)(G)$ or $B(p,q)(G)$. Then*

(i) *the maximal ideal space of $S(G)$ can be identified with the dual group $\hat{G}$ of G;*

(ii) *the algebra $S(G)$ satisfies Ditkin's condition;*

(iii) *the Shilov-Wiener Tauberian theorem holds in $S(G)$.*

Proof. Immediate from Propositions (2.3) and (2.4) and the fact that every Segal algebra has properties (i)–(iii) (Yap [16]).

3. Non-factorization in $A(p,q)(G)$ and $B(p,q)(G)$. We recall that an algebra A is said to have the *factorization property* if $A=A\cdot A$, where $A\cdot A=\{xy : x,y\in A\}$. We use A^2 to denote the ideal in A generated by $A\cdot A$. The group algebra $L^1(G)$ is known to have the factorization property (Cohen [2]), but in general $A^p(G)$ and $B^p(G)$ do not satisfy this property (Martin-Yap [9] and Yap [15]). In this section we extend these non-factorization theorems to the algebras $A(p,q)(G)$ and $B(p,q)(G)$.

Lemma 3.1. $A(p,q)^2\subset A(p/2,q/2)$.

Proof. It suffices to show that if $f,g\in A(p,q)$ then $f*g\in A(p/2, q/2)$. First we define $\alpha=2(p+q)/q$. Thus $|\hat{f}|^{p/\alpha}, |\hat{g}|^{p/\alpha}\in L(\alpha,\alpha q/p)$ and by O'Neil [10, 3.4] we see that $|\hat{f}\hat{g}|^{p/\alpha}\in L(r,s)$, where

$$1/r=1/\alpha+1/\alpha, \qquad 1/s=p/\alpha q+p/\alpha q.$$

It follows that $\widehat{f*g}=\hat{f}\hat{g}\in L(p/2,q/2)$, and hence $f*g\in A(p/2,q/2)$.

Theorem 3.2. If G is non-discrete, $1<p<\infty$, $1\leqq q<\infty$, then $A(p,q)(G)^2\neq A(p,q)(G)$.

Proof. Suppose $A(p,q)^2=A(p,q)$, then by Lemma (3.1) we would have $A(p,q)\subset A(p/2^n, q/2^n)$ for $n=1,2,3,\cdots$. We will show that this leads to a contradiction. Since G is non-discrete, $\hat{G}$ is non-compact, and we may choose a symmetric neighborhood U of the identity in $\hat{G}$ whose closure $\bar{U}$ is compact, and a sequence $\gamma_1,\gamma_2,\gamma_3,\cdots$ in $\hat{G}$ such that

$$\gamma_i U^2\cap\gamma_j U^2=\emptyset \qquad (i\neq j)$$

Now let N be a positive integer such that $p<2^N$. Define

$$\alpha=2^N/p,\ a_n=n^{-\alpha} \qquad (n=1,2,3,\cdots)$$

$$g=\chi_U,\ h=\sum_{k=1}^{\infty}a_k\chi_{\gamma_k U^2}.$$

Thus $g,h\in L^2(\hat{G})$ and so by Rudin [11, Theorem 1.6.3] there is a function $f\in L^1(G)$ such that $\hat{f}=g*h$. It follows that $\hat{f}(\gamma)=g*h(\gamma)=a_k\rho(U)$ for $\gamma\in\gamma_k U$, where ρ denotes the Harr measure of $\hat{G}$. Direct computations (similar to those in [14, p. 138]) show that $\hat{f}\in L(p,q)$, but $\hat{f}\notin L(p/2^N, q/2^N)$. Hence $f\in A(p,q)$, but $f\notin A(p/2^N, q/2^N)$.

Lemma 3.3. *If $f\in L(p_1,s)\cap L(p_2,s)$, then $f\in L(r,s)$ for all r such* *that* $p_1<r<p_2$.

Proof. Define $\beta=(1/r-1/p_2)(1/p_1-1/p_2)^{-1}$, and note that

$$\|f\|_{(r,s)}^{*s}=\int_0^\infty f^*(x)^s x^{s/r-1}dx$$
$$=\int_0^\infty [f^*(x)^{\beta s}x^{\beta(s/p_1-1)}]\cdot[f^*(x)^{(1-\beta)s}x^{(1-\beta)(s/p_2-1)}]dx$$
$$\leq \|f\|_{(p_1,s)}^{*s\beta}\cdot\|f\|_{(p_2,s)}^{*s(1-\beta)} \quad \text{(by Hölder's inequality).}$$

Theorem 3.4. *If G is non-discrete and $1<p<\infty$, $1\leqq q<\infty$, then* $B(p,q)(G)^2\neq B(p,q)(G)$.

Proof. Let $f, g\in B(p,q)$. Since $L^1=L(1,1)$, and $L(1,1)\subset L(1,q)$ (by [13, (3.3)]), it follows that $f, g\in L(1,q)$. Define $r=2p/(1+p)$. Clearly $1<r<p$, and so $f, g\in L(r,q)$ by Lemma (3.3). By [13, (3.5)] we have $f*g\in L(p,q/2)$. Thus $B(p,q)^2\subset B(p,q/2)$. But $B(p,q/2)$ is a *proper* subset of $B(p,q)$ (see the proof of Case I of Theorem (2.7) in Yap [14]).

Remark 3.5. Theorem (3.4) is valid for all (non-discrete) locally compact unimodular groups and the proof is the same.

Conjecture. For a Segal algebra $S(G)$, $S(G)^2\neq S(G)$ if $S(G)\neq L^1(G)$.

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