On two Hamilton cycle problems in random graphs

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Abstract

We study two problems related to the existence of Hamilton cycles in random graphs. The first question relates to the number of edge disjoint Hamilton cycles that the random graph $G_{n,p}$ contains. $\delta(G)/2$ is an upper bound and we show that if $p \leq (1+o(1))\ln n/n$ then this upper bound is tight **whp**. The second question relates to how many edges can be *adversarially* removed from $G_{n,p}$ without destroying Hamiltonicity. We show that if $p \geq K \ln n/n$ then there exists a constant $\alpha > 0$ such that **whp** $G-H$ is Hamiltonian for all choices of H as an n-vertex graph with maximum degree $\Delta(H) \leq \alpha K \ln n$.

1 Introduction

In this paper, we give results on two problems related to Hamilton cycles in random graphs.

1.1 Edge Disjoint Hamilton Cycles

It was shown by Komlós and Szemerédi [8] that if $p = \frac{\ln n + \ln\ln n + c}{n}$ then,

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \text{ is Hamiltonian}) = \lim_{n\to\infty} \mathbf{Pr}(\delta(G_{n,p}) \geq 2).$$

Bollobás [3], Ajtai, Komlós and Szemerédi [1] proved a hitting time version of this statement, i.e., **whp**[1], as we add random edges $e_1, e_2, \ldots, e_m$ one by one to an empty graph, the graph $G_m =$

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[1]A sequence of events $\mathcal{E}_n$ is said to occur with high probability (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$

$([n], \{e_1, e_2, \ldots, e_m\})$ becomes Hamiltonian at exactly the point when the minimum degree reaches two.

Let us say that a graph G has property $\mathcal{H}$ if it contains $\lfloor \delta(G)/2 \rfloor$ edge disjoint Hamilton cycles plus a further edge disjoint (near) perfect matching in the case $\delta(G)$ is odd. (Here a (near) perfect matching is one of size $\lfloor n/2 \rfloor$). Bollobás and Frieze [5] showed that **whp** G_m has property $\mathcal{H}$ as long as the minimum degree is $O(1)$.

It is reasonable to conjecture that **whp** $G_{n,p}$ has property $\mathcal{H}$ for any $0 \leq p \leq 1$. Our first result is to show that this is true for $p \leq (1 + o(1)) \ln n/n$ which strengthens the non-hitting time version the result quoted from [5].

Theorem 1 *Let $p(n) \leq (1 + o(1)) \ln n/n$. Then* **whp** *$G_{n,p}$ has property $\mathcal{H}$.*

We remark that Frieze and Krivelevich [7] showed that if p is constant then **whp** $G_{n,p}$ *almost* satisfies $\mathcal{H}$ in the sense that it contains $(1 - o(1))\delta(G_{n,p})/2$ edge disjoint Hamilton cycles.

1.2 Robustness of Hamiltonicity

In recent times, there is increasing interest in graphs which are only partially random. For example, Bohman, Frieze and Martin [2] considered graphs of the form $G = H + R$ where H is arbitrary, but with high minimum degree and R is random. In this section we consider graphs of the form $G = R - H$ where R is random and H is an arbitrary subset of R, subject to some restrictions. In particular $R = G_{n,p}$

Sudakov and Vu [10] have recently shown that if $p > (\ln n)^4/n$ and if $G = G_{n,p}$ then **whp** $G - H$ is Hamiltonian for all choices of H as an n-vertex graph with maximum degree $\Delta(H) \leq (1/2 - \varepsilon)np$. Here $\varepsilon > 0$ is an arbitrarily small constant. Note that this bound on $\Delta(H)$ is essentially best possible, otherwise $R - H$ could be a bipartite graph with an uneven partition. In this note we reduce p to $O(\ln n/n)$ but unfortunately, we have to reduce the bound on $\Delta(H)$ as well.

Theorem 2 *Let $G = G_{n,p}$ where $p \geq K \ln n/n$ for some sufficiently large constant $K > 0$. There exists a constant $\alpha > 0$ such that* **whp** *$G - H$ is Hamiltonian for all choices of H as an n-vertex graph with maximum degree $\Delta(H) \leq \alpha K \ln n$.*

2 Proof of Theorem 1

2.1 Preliminaries

Observe first that the assumption on the edge probability in this theorem can be easily seen to be essentially equivalent to the assumption that the minimum degree $\delta(G)$ of $G_{n,p}$ almost surely satisfies: $\delta(G) = o(\log n)$.

Notation: For a graph $G = (V, E)$ and two disjoint vertex subsets U, W we denote:

$$\begin{aligned} N(U,W) &:= \{w \in W : \ w \text{ has a neighbor in } U\}\,; \\ N(U) &:= N(U, V \setminus U)\,; \\ E(U,W) &:= \{e \in E(G) : |e \cap U| = 1, |e \cap W| = 1\}\,; \\ e(U,W) &:= |E(U,W)|\,. \end{aligned}$$

Definition 1 *A graph $G = (V, E)$ is called a (k, c)-*expander *if $|N(U)| \geq c|U|$ for every subset $U \subseteq V(G)$ of cardinality $|U| \leq k$.*

Set

$$d_0 = d_0(n,p) = \min\left\{k : n\binom{n-1}{k}p^k(1-p)^{n-1-k} \geq 1\right\} \ .$$

One can prove that **whp** $\delta(G_{n,p})$ satisfies (say):

$$|\delta(G) - d_0| \leq \ln\ln n \ .$$

Indeed, $u_k = n\binom{n-1}{k}p^k(1-p)^{n-1-k}$ is the expected number of vertices of degree k and $u_{k+1}/u_k = \frac{(n-1-k)p}{(k+1)(1-p)}$. Since $d_0 = o(\ln n)$ we see that $u_{d_0 - \ln\ln n} = o(1)$. Furthermore, $u_{d_0+\ln\ln n} \to \infty$ and we can use the Chebyshev inequality to show that $u_{d_0+\ln\ln n} \neq 0$ **whp**.

Define

$$\rho = \frac{2001(d_0 + \ln\ln n)}{n \ln n}\,,$$

observe that $\rho = o(1/n)$. Define $p_0 = p_0(n)$ by

$$1 - p = (1-p_0)(1-\rho)\,, \tag{1}$$

observe that $p_0 = p - \rho(1 - o(1))$. We can thus decompose $G \sim G_{n,p}$ as $G = G_0 \cup R$, where $G_0 \sim G_{n,p_0}$, $R \sim G_{n,\rho}$.

Notation. $\delta_0 = \delta(G_0)$.

Claim 1 *For a fixed G_0, almost surely over the choice of $R \sim G_{n,\rho}$, $\delta(G_0) = \delta(G_0 \cup R)$.*

Proof Clearly, $\delta(G_0) \leq \delta(G_0 \cup R)$. In the opposite direction, take a vertex v of minimum degree in G_0. Recall that $\rho = o(1/n)$, and therefore the edges of R almost surely miss v, implying $\delta(G_0 \cup R) \leq d_{G_0 \cup R}(v) = d_{G_0}(v) = \delta(G_0)$. $\square$

It thus follows that in order to prove Theorem 1 it is enough to prove that almost surely $G_0 \cup R$ contains $\lfloor \delta_0/2 \rfloor$ disjoint Hamilton cycles, plus an edge disjoint (near) perfect matching if δ_0 is odd..

Of course we may (and will indeed) assume that $p(n) = (1 + o(1)) \ln n/n$, as otherwise **whp** $\delta_0 \leq 1$ and there is nothing new to prove.

2.2 Properties of $G_0 = G_{n,p_0}$

Define
$$SMALL = \{v \in V : d_{G_0}(v) \le 0.1 \ln n\} \ .$$

Lemma 3 *The random graph $G_0 = G_{n,p_0}$, with p_0 defined by (1), has* **whp** *the following properties:*

(P1) *G_0 does not contain a path of at most four distinct edges (with possibly identical endpoints), both of whose endpoints lie in $SMALL$.*

(P2) *Every vertex has at most one neighbor in $SMALL$.*

(P3) *Every set $U \subset V$ of size $|U| \le 100n/\ln n$ spans at most $|U|(\ln n)^{1/2}$ edges in G_0.*

(P4) *For every two disjoint subsets $U, W \subset V$ satisfying: $|U| \le 100n/\ln n$, $|W| \le 10^{-4}|U| \ln n$,*
$$e_{G_0}(U, W) < 0.09|U| \ln n \,.$$

(P5) *For every two disjoint subsets $U, W \subset V$ satisfying: $|U| \ge 100n/\ln n$, $|W| \ge n/4$,*
$$e_{G_0}(U, W) \ge 0.1|U| \ln n \,.$$

Proof The above are rather standard statements about random graphs, so we will be relatively brief in our arguments.

We start with proving **P1**. Observe that for a vertex $v \in V(G_0)$, the degree of v is binomially distributed with parameters $n-1$ and p_0. Therefore,

$$\begin{aligned}
\mathbf{Pr}[v \in SMALL] &= \sum_{k \le 0.1 \ln n} \mathbf{Pr}[B(n-1, p_0) = k] \le 0.1 \ln n \binom{n-1}{0.1 \ln n} p^{0.1 \ln n} (1-p)^{n-1-0.1 \ln n} \\
&\le 0.1 \ln n \left(\frac{10enp}{\ln n}\right)^{0.1 \ln n} e^{-p(n-1-0.1 \ln n)} < 29^{0.1 \ln n} e^{-(1-o(1)) \ln n} \\
&< n^{-0.6} \ .
\end{aligned}$$

Also, for a fixed pair $u \ne v \in V(G_0)$ the probability that u and v are connected by a path of length ℓ in G_0 is at most $n^{\ell-1} p_0^\ell = ((1+o(1)) \ln n)^\ell n^{-1}$. Therefore, using the FKG inequality,

$$\begin{aligned}
\mathbf{Pr}[(u, v \in SMALL) \& (dist(u, v) \le 4)] &\le Pr[u \in SMALL]\, \mathbf{Pr}[v \in SMALL]\, Pr[dist(u, v) \le 4] \\
&\le 4 \cdot n^{-0.6} \cdot n^{-0.6} \cdot \frac{(1+o(1)) \ln^4 n}{n} \\
&< n^{-2.1} \ .
\end{aligned}$$

Applying the union bound over all possible pairs of distinct vertices u, v ($O(n^2)$ of them), we establish **P1**. The case where $u = v$ is treated similarly.

Property **P2** follows directly from **P1**. Properties **P3**, **P4** are straightforward first moment calculations which we thus omit.

We conclude with proving **P5**. Fix U, W. Then the number of edges between U and W is distributed binomially with parameters $|U||W|$ and p_0 and has thus expectation $|U||W|p_0 \geq (1+o(1))|U|\ln n/4$. Therefore by applying standard Chernoff-type bounds on the lower tail of the binomial distribution, it follows that

$$\begin{aligned} \mathbf{Pr}[e_{G_0}(U,W) \leq 0.1|U|\ln n] &\leq \exp\left\{-\frac{(0.25|U|\ln n - 0.1|U|\ln n)^2}{2 \cdot 0.25|U|\ln n}\right\} = \exp\{-2 \cdot 0.15^2|U|\ln n\} \\ &< \exp\{-4n\} . \end{aligned}$$

As the pair (U, W) can be chosen in at most 4^n ways, **P5** follows by applying the union bound. □

2.3 Pósa's Lemma and its consequences

Definition 2 *Let $G = (V, E)$ be a non-Hamiltonian graph with a longest path of length ℓ. A pair $(u, v) \notin E(G)$ is called a* hole *if adding (u, v) to G creates a graph G' which is Hamiltonian or contains a path longer than ℓ. In addition, if the maximum size of a matching in G is $m < \lfloor n/2 \rfloor$ then $(u, v) \notin E(G)$ is called a hole if adding (u, v) to G creates a graph G' which is contains a matching of size $m + 1$.*

Lemma 4 *Let G be a non-Hamiltonian connected $(k, 2)$-expander. Then G has a path of length at least $3k - 1$ and at least $k^2/2$ holes.*

Proof

Let $P = (v_0, \ldots, v_k)$ be a longest path in graph G. A *Pósa rotation* of P [9] with v_0 fixed gives another longest path $P' = (v_0, \ldots v_i v_k \ldots v_{i+1})$ created by adding edge (v_k, v_i) and deleting edge (v_i, v_{i+1}). Let $END_G(v_0, P)$ be the set of endpoints obtained by a sequence of Pósa rotations starting with P, keeping v_0 fixed and using an edge (v_k, v_i) of G.

Each vertex $v_j \in END_G(v_0, P)$ can then be used as the initial vertex of another set of longest paths whose endpoint set is $END_G(v_j, P)$, this time using v_j as the fixed vertex, but again only adding edges from G. Let $END_G(P) = \{v_0\} \cup END_G(v_0, P)$.

The Pósa condition (see, e.g., [4], Ch.8.2)

$$|N(END_G(v, P))| \leq 2\,|END_G(v, P)| - 1$$

for $v \in END_G(P)$ together with the fact that G is a $(k, 2)$-expander implies that $|END_G(v, P)| > k$. The connectivity of G implies that closing a longest path to a cycle either creates a Hamilton cycle or creates a longer path. For every $v \in END_G(P)$ and for every $u \in END_G(v, P)$, a pair (u, v) is a hole. This shows that the number of holes is at least $k^2/2$ (each hole is counted at most twice for both its endpoints). As all neighbors in G of a subset $U \subseteq END_G(v, P)$ of size $|U| = k$ belong to P, due to the maximality of P, and G is a $(k, 2)$-expander, it follows that the length of P is at least $3k - 1$. $\square$

The following lemma is taken from [5].

Lemma 5 *Let G be a $(k, 1)$-expander which does not contain a matching of size $\lfloor n/2 \rfloor$. Then G has a matching of size at least k and at least $k^2/2$ holes.*

Proof Let $\mathcal{M}$ denote the set of maximum size matchings in G and let $M \in \mathcal{M}$. Fix v uncovered by M and now let S_0 be the set of vertices reachable from v by an even length alternating path with respect to M. Clearly, every vertex of S_0 is either v or is covered by M. Let $x \in N(S_0)$. Then x is covered by M, as otherwise we can get a larger matching by using an alternating path from v to $y \in S_0$, and then an edge (y, x).

Let y_1 satisfy $(x, y_1) \in M$. We show that $y_1 \in S_0$. Now there exists $y_2 \in S_0$ such that $(x, y_2) \in E(G)$. Let P be an even length alternating path from v terminating at y_2. If P contains (x, y_1) we can truncate it to terminate with (x, y_1), otherwise we can extend it using edges (y_2, x) and (x, y_1).

It follows that $|N(S_0)| < |S_0|$ (as $v \in S_0$, v is not covered by M). Recalling that G is a $(k, 1)$-expander, we derive that $|S_0| > k$. But then obviously the union $S_0 \cup N(S_0)$ has at least $2k$ vertices and thus has at least $2k - 1$ vertices from M. This implies: $|M| \geq k$.

Now we prove that G has at least $k^2/2$ holes. Fix v uncovered by M and now let $S \neq \emptyset$ be the other vertices uncovered by M. Let $S_1 \supseteq S$ be the set of vertices reachable from S by an even length alternating path with respect to M. As before we can prove that $|S_1| > k$. For every $u \in S_1$ there is an even length alternating path with respect to M ending at u. Replacing the edges along this path belonging to M with those outside of M gives a maximum matching $M' \in \mathcal{M}$ not covering u. Thus (u, v) is a hole. Repeating now the above argument with u, M' instead of v, M, respectively, gives at least k holes touching u. Since $|S_1| \geq k$, and each hole is counted at most twice, altogether we get at least $|S_1|k/2 \geq k^2/2$ holes, as required. $\square$

2.4 Proof idea

We split the random graph R into $\lceil \delta_0/2 \rceil$ identically distributed random graphs R_i. We then create $\lfloor \delta_0/2 \rfloor$ Hamilton cycles H_i (plus a matching if needed). We use the random edges of R_i to fill a hole. Once H_i is created its edges are deleted from the graph and we proceed to the next phase. At the i-th stage, by the definition of δ_0, the graph G_i has minimum degree at least 2, moreover, most vertices in it have degree around $\ln n$ (as each vertex loses at most $\delta_0 = o(\ln n)$ neighbors during the process), and therefore G_i is connected, is an $(n - cn/\ln n, 2)$-expander by properties **P1-P5** and has a path P_i of length at least $n - cn/\ln n$. We gradually augment P_i to a Hamilton path, and then to a Hamilton cycle. At each substage of augmenting P_i, the current graph has a quadratic number of holes, and therefore a constant number of random edges are expected to augment the current path to a longer one/close a Hamilton cycle. If δ_0 is odd, we need a final stage to create a (near) perfect matching.

2.5 Formal argument

We may assume that $\delta_0 \geq 2$ as otherwise there is nothing new to prove.

Define ρ_i by

$$1 - \rho = (1 - \rho_i)^{\lceil \delta_0/2 \rceil}$$

observe that

$$\rho_i \geq \frac{\rho}{\lceil \delta_0/2 \rceil} = \frac{2001(d_0 + \ln\ln n)}{\lceil \delta_0/2 \rceil n \ln n} \geq \frac{4000}{n \ln n} \ .$$

We then represent

$$R = \bigcup_{i=1}^{\lceil \delta_0/2 \rceil} R_i \, ,$$

where $R_i \sim G(n, \rho_i)$.

For $i = 1, \ldots, \lceil \delta_0/2 \rceil$, let G_i be a graph obtained from $G_0 \cup \bigcup_{j=1}^{i-1} R_j$ after having deleted the first $i-1$ Hamilton cycles (assuming that the previous $i-1$ stages were successful, of course). Each vertex v has its degree in G_0 reduced by at most $2(i-1)$ in G_i. Therefore if $i \leq \lfloor \delta_0/2 \rfloor$ then the minimum degree $\delta(G_i)$ satisfies $\delta(G_i) \geq \delta_0 - 2(i-1) \geq 2$. If δ_0 is odd, then $\delta(G_{\lceil \delta_0/2 \rceil}) \geq 1$.

We will now show that if $i \leq \lfloor \delta_0/2 \rfloor$ then G_i is a $(k, 2)$-expander for $k = n/3 - 100n/(3 \ln n)$. Let $X \subset V$ be a set of $|X| = t$ vertices.

Case 1: $t \leq 100n/\ln n$.

Denote $X_0 = X \cap SMALL$, $|X_0| = t_0$, $X_1 = X \setminus X_0$, $|X_1| = t_1$. Observe first that $|N_{G_i}(X_0, V \setminus X)| \geq 2t_0 - t_1$. Indeed, in G_i all edges touching X_0 have their second endpoint outside X_0, by Property **P1**. We currently have at least two edges per each vertex in X_0. By Property **P2** each vertex outside $SMALL$ has at most one neighbor in X_0 in the graph G_i. Thus the other endpoints of the edges from G_i touching X_0 are distinct, and at most t_1 of them land in X_1.

Now, X_1 spans at most $t_1(\ln n)^{1/2}$ edges in G_0, by Property **P3**. As the degrees in G_0 of all vertices in X_1 are at least $0.1 \ln n$, by the definition of $SMALL$, at least $0.09 t_1 \ln n$ edges leave X_1 in G_0. But then by Property **P4** $|N_{G_0}(X_1)| \geq 10^{-4} t_1 \ln n$. By Property **P1** at most t_1 of those neighbors fall into $X_0 \cup N_{G_0}(X_0)$, implying:

$$|N_{G_0}(X_1, V \setminus X) - N_{G_0}(X_0, V \setminus X)| \geq 10^{-4} t_1 \ln n - t_1 \ .$$

As in G_i every vertex lost at most δ_0 neighbors compared to G_0, we have

$$\begin{aligned} |N_{G_i}(X_1, V \setminus X) - N_{G_0}(X_0, V \setminus X)| &\geq 10^{-4} t_1 \ln n - t_1 - \delta_0 t_1 \\ &\geq 10^{-5} t_1 \ln n \ . \end{aligned}$$

Altogether,

$$|N_{G_i}(X)| \geq 2t_0 - t_1 + 10^{-5} t_1 \ln n \geq 2t \, ,$$

as claimed.

Case 2: $t \geq 100n/\ln n$.

Recall that $t \leq n/3 - 100n/(3\ln n)$. Assume to the contrary that $|N_{G_i}(X)| < 2|X|$. Then in G_i there is a vertex subset Y disjoint from X such that $|Y| = n - 3t$, and G_i has no edges between X and Y. But then there were at most $2\min\{\lfloor\delta_0/2\rfloor|X|, \lfloor\delta_0/2\rfloor|Y|\}$ edges between X and Y in G_0.

If $t \leq n/4$, then $n - 3t \geq n/4$, and we get a contradiction to Property **P5** with X, Y substituted for U, W, respectively. If $n/4 \leq t \leq n/3 - (100n)/(3\ln n)$, then $n - 3t \geq 100n/\ln n$, again contradicting Property **P5** with Y, X instead of U, W, respectively.

We have proved that given properties **P1-P5** of G_0, for each i the graph G_i is deterministically an $(n/3 - 100n/(3\ln n), 2)$-expander.

A similar argument, in the case where δ_0 is odd, shows that the graph $G_{\lfloor\delta_0/2\rfloor}$ is an $(n/2 - 100n/(2\ln n), 1)$-expander.

Recall that a random graph R_i added at the i-th stage is distributed according to G_{n,ρ_i} with $\rho_i \geq \frac{4000}{n\ln n}$, so $\rho_i \geq \frac{120}{n^2} \cdot \frac{100n}{3\ln n}$ and $\rho_i > \frac{20}{n^2} \cdot \frac{100n}{2\ln n}$. Theorem 1 will thus follow from:

Lemma 6

(a) *Let $G = (V, E)$ be a $(n/3 - k, 2)$-expander on n vertices, where $k = o(n)$. Let R be a random graph $G_{n,p}$ with $p(n) = 120k/n^2$. Then*

$$Pr[G \cup R \text{ is not Hamiltonian}] < e^{-\Omega(k)} .$$

(b) *Let $G = (V, E)$ be a $(n/2 - k, 1)$-expander on n vertices, where $k = o(n)$. Let R be a random graph $G_{n,p}$ with $p(n) = 20k/n^2$. Then*

$$Pr[G \cup R \text{ does not contain a (near) perfect matching}] < e^{-\Omega(k)} .$$

Proof
(a) Observe that by Pósa's Lemma and its consequences (Lemma 4):

- G is connected
  (Due to expansion of G there is no room for two connected components);
- G has a path of length at least $n - 3k - 1$
  (due to Lemma 4);
- If a supergraph of G_i is non-Hamiltonian it has at least $(n/3 - k)^2/2 > n^2/20$ holes.

We split the random graph R into $6k$ independent identically distributed graphs

$$R = \bigcup_{i=1}^{6k} R_i ,$$

where $R_i \sim G_{n,p_i}$ and $p_i \geq p/(6k) = 20/n^2$. Set $G_0 = G$, and for each $i = 1, \ldots 6k$ define

$$G_i = G \cup \bigcup_{j=1}^{i} R_j \ .$$

At Stage i we add to G_{i-1} the next random graph R_i. A stage i is called *successful* if a longest path in G_{i+1} is longer than that of G_i, or if G_{i+1} is already Hamiltonian. Clearly, if at least $3k+1$ stages are successful then the final graph G_{6k} is Hamiltonian. Observe that for Stage i to be successful, if G_{i-1} is not yet Hamiltonian, it is enough for the random graph R_i to hit one of the holes of G_{i-1}. Thus, Stage i is unsuccessful with probability at most $(1-p_i)^{n^2/20} < 1/e$. Let X be the random variable counting the number of successful stages. Then X stochastically dominates $Bin(6k, 1-1/e)$. Hence by standard estimates on the tails of the binomial distribution,

$$Pr[G \cup R \text{ is not Hamiltonian}] \leq Pr[X \leq 3k] < e^{-\Omega(k)} ,$$

as claimed.

The proof of (b) is similar. $\square$

3 Proof of Theorem 2

We will prove the result for $G_{n,m}$, $m = \frac{1}{2}Kn\ln n$. This implies the result for the $G_{n,p}$ model.

This time we will use the coloring argument of Fenner and Frieze [6]. Consider the following properties:

(Q1) $K\ln n/2 \leq \delta(G) \leq \Delta(G) \leq 2K\ln n$.

(Q2) $|S| \leq \frac{n}{K^3(\ln n)^2}$ implies $|E(S)| \leq 2|S|$.

(Q3) $\frac{n}{K^3(\ln n)^2} \leq |S| \leq n/(K\ln n)$ implies $|N(S)| \geq (K\ln n/5)|S|$.

(Q4) If S, T are disjoint sets of vertices and $|S| \geq |T| \geq n/10$ then $e(S,T) \geq (K\ln n/20)|T|$.

Lemma 7 *If K is sufficiently large, $G = G_{n,m}$ satisfies* **Q1–Q4 whp**.

Proof We will prove that $G_{n,p}$ has these properties where $p = K\ln n/n$. Inflating error probabilities by $O(n^{1/2})$ will show them for $G_{n,m}$. **Q1, Q2** are simple first moment calculations. We will check **Q3, Q4**. The size of $N(S)$ is distributed as the binomial $B(n-s, 1-(1-p)^s)$. Now $1-(1-p)^s \geq sp/2$ if $sp \leq 1$. Applying a Chernoff bound we see that

$$\mathbf{Pr}(\exists S \text{ failing } \mathbf{Q3}) \leq \sum_{s=\frac{n}{K^3(\ln n)^2}}^{n/(K\ln n)} \binom{n}{s} e^{-(n-s)sp/32} = o(1).$$

Similarly,

$$\mathbf{Pr}(\exists S, T \text{ failing } \mathbf{Q4}) \le \sum_{s \ge n/10} \sum_{t \ge n/10} \binom{n}{s}\binom{n}{t} e^{-K \ln n|T|/80} = o(1).$$

$\square$

In the following we will asssume that K is sufficiently large and α is sufficiently small so that our claimed inequalities hold. We do not attempt to optimise, since we are far from getting α close to 1/2.

Now let H be a graph with $\Delta(H) \le \alpha K \ln n$ and let X be any βm subset of $E(G - H)$ satisfying $\Delta(X) \le 2\beta K \ln n$. Here we will be assuming $1 \gg \beta \gg \alpha$. Let $\Gamma = G - H - X$.

Lemma 8 *If* **Q1–Q4** *hold then*

(a) Γ *is an* $(n/30, 2)$*-expander.*

(b) Γ *is connected.*

Proof
(a)
(i) $|S| \le \frac{n}{3K^3(\ln n)^2}$.
By construction, we have $\delta(\Gamma) \ge (1/2 - \alpha - 2\beta)K \ln n$. So if $|N_\Gamma(S)| < 2|S|$ we find that $N_\Gamma(S) \cup S$ contains at least $((1/2 - \alpha - 2\beta)K \ln n)|S|/2$ edges, contradicting **Q2**.

(ii) $\frac{n}{3K^3(\ln n)^2} \le |S| \le n/(K \ln n)$.
It follows from **Q3** that

$$|N_\Gamma(S)| \ge ((1/5 - \alpha - 2\beta)K \ln n)|S| \ge 2|S|.$$

(iii) $n/(K \ln n) \le |S| \le n/30$.
Choose $S' \subseteq S$ of size $n/(K \ln n)$. Then

$$|N_\Gamma(S)| \ge |N_\Gamma(S')| - |S| \ge (1/5 - \alpha - 2\beta)n - |S| \ge 2|S|.$$

(b) It follows from (a) that if Γ is not connected then each component is of size at least $n/10$. But then **Q4** implies that there are at least $(1/20 - \alpha - 2\beta)K|T| \ln n$ edges between each component in Γ, contradiction. $\square$

We now resort to our coloring argument. Let $G_1, G_2, \ldots, G_M$, $M = \binom{\binom{n}{2}}{m}$ be an enumeration of graphs with vertex set $[n]$ and m edges.

For each i let H_i be a *fixed* sub-graph of G_i with $\Delta(H_i) \le \alpha K \ln n$ such that $G_i - H_i$ is non-Hamiltonian, if one exists. Otherwise H_i is an arbitrary sub-graph of G_i with the same restrictions on the maximum degree. If graph G is non-Hamiltonian, let $\lambda(G)$ denote the length of the longest path in G and let $\lambda(G) = n$ if G is Hamiltonian. Now for a graph G_i, let $X_{i,1}, X_{i,2}, \ldots,$ be an enumeration of all βm-subsets of $E(G_i - H_i)$. Let $\Gamma_{i,j} = G_i - H_i - X_{i,j}$. Then let

$$a_{i,j} = \begin{cases} 1 & \begin{cases} (a)\ G_i \text{ satisfies } \mathbf{Q1–Q4} \\ (b)\ \lambda(G_i - H_i) = \lambda(G_i - H_i - X_{i,j}) \\ (c)\ G_i - H_i \text{ is not Hamiltonian} \\ (d)\ \Delta(X) \leq 2\beta K \ln n \end{cases} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The notation $A_n \gtrapprox B_n$ stands for $A_n \geq (1 - o(1))B_n$.

Lemma 9 *If G_i satisfies (a) and (c) of* (2) *then $\sum_j a_{i,j} \gtrapprox \binom{(1-\alpha)m-n}{\beta m}$.*

Proof H_i has at most $\frac{1}{2}\alpha K n \ln n = \alpha m$ edges and to ensure (b) all we have to do is avoid some fixed longest path of $\Gamma_{i,j}$. Furthermore, almost all choices of βm edges will induce a sub-graph with maximum degree at most $2\beta K \ln n$. $\square$

Lemma 10 *Let $N = \binom{n}{2}$. Then,*

$$\sum_{i,j} a_{i,j} \leq m \binom{N}{m} \binom{m}{\beta m} \binom{(1-\beta)m}{\alpha m} (899/900)^{\beta m}.$$

Proof Let $K_{i,j} = G_i - X_{i,j}$ and for a fixed graph K with $(1-\beta)m$ edges let us estimate the number of (i,j) with $K_{i,j} = K$ and $a_{i,j} = 1$.

For each sub-graph $H \subseteq K$ with $\Delta(H) \leq \alpha K \ln n$, we let $\theta(K, H)$ denote the number of choices of βm edges X such that (i) $K + X$ satisfies **Q1–Q4** and (ii) $\lambda(K - H) = \lambda(K + X - H)$. Then

$$\sum_{i,j} a_{i,j} \leq \sum_{K,H} \theta(K, H). \tag{3}$$

This is because for each (i,j) with $a_{i,j} = 1$ there is a corresponding $K_{i,j} = G_i - X_{i,j}$ such that $G_i = K_{i,j} + X_{i,j}$ satisfies **Q1–Q4** and an H_i such that $\lambda(K_{i,j} - H_i) = \lambda(K_{i,j} + X_{i,j} - H_i)$.

Now if $K + X$ satisfies **Q1–Q4** then from Lemmas 4 and 8 we see that to ensure $\lambda(K - H) = \lambda(K + X - H)$, X must avoid at least $(n/30)^2/2$ edges i.e.

$$\theta(K, H) \leq \binom{N - (1-\beta)m}{\beta m} (899/900)^{\beta m}.$$

Consequently,

$$\begin{aligned} \sum_{K,H} \theta(K, H) &\leq \sum_{t=0}^{\alpha m} \binom{N}{(1-\beta)m} \binom{(1-\beta)m}{t} \binom{N - (1-\beta)m}{\beta m} (899/900)^{\beta m} \\ &\leq m \binom{N}{(1-\beta)m} \binom{(1-\beta)m}{\alpha m} \binom{N - (1-\beta)m}{\beta m} (899/900)^{\beta m} \\ &= m \binom{N}{m} \binom{m}{\beta m} \binom{(1-\beta)m}{\alpha m} (899/900)^{\beta m}. \end{aligned}$$

$\square$

Let ν_H denote the number of i such that G_i satisfies **Q1–Q4** and yet $G_i - H_i$ non-Hamiltonian and let $M = \binom{N}{m}$. We must show that $\nu_H = o(M)$.

It follows from Lemma 9 that

$$\sum_{i,j} a_{i,j} \gtrapprox \nu_H \binom{(1-\alpha)m - n}{\beta m}.$$

On the other hand, Lemma 10 implies

$$\begin{aligned}
\frac{\nu_H}{\binom{N}{m}} &\lessapprox \frac{m\binom{m}{\beta m}\binom{(1-\beta)m}{\alpha m}(899/900)^{\beta m}}{\binom{(1-\alpha)m-n}{\beta m}} \\
&\leq m\left(\frac{me}{(1-\alpha)m - n - \beta m}\right)^{\beta m}\left(\frac{(1-\beta)e}{\alpha}\right)^{\alpha m}(899/900)^{\beta m} \\
&= o(1),
\end{aligned}$$

and Theorem 2 follows. $\square$

References

[1] M. Ajtai, J. Komlós and E. Szemerédi, The first occurrence of Hamilton cycles in random graphs, *Annals of Discrete Mathematics* 27 (1985) 173-178.

[2] T. Bohman, A.M. Frieze and R. Martin, How many random edges make a dense graph Hamiltonian?, *Random Structures and Algorithms* 22 (2003) 33-42.

[3] B. Bollobás, The evolution of sparse graphs, Graph Theory and Combinatorics. (Proc. Cambridge Combinatorics Conference in Honour of Paul Erdős (B. Bollobás; Ed.)) Academic Press (1984) 35-57.

[4] B. Bollobás, Random graphs, 2nd Ed., Cambridge Univ. Press, Cambridge, 2001.

[5] B. Bollobás and A.M. Frieze, On matchings and hamiltonian cycles in random graphs *Annals of Discrete Mathematics* 28 (1985) 23-46.

[6] T.I. Fenner and A.M. Frieze, On the existence of hamiltonian cycles in a class of random graphs, *Discrete Mathematics* 45 (1983) 301-305.

[7] A.M. Frieze and M. Krivelevich, On packing Hamilton Cycles in ε-regular Graphs, *Journal of Combinatorial Theory* B, 94 (2005) 159-172.

[8] J. Komlós and E. Szemerédi, Limit distributions for the existence of Hamilton circuits in a random graph, *Discrete Mathematics* 43 (1983) 55-63.

[9] L. Pósa, Hamiltonian circuits in random graphs, *Discrete Mathematics* 14 (1976) 359-364.

[10] B. Sudakov and V. Vu, Resilience of graphs, submitted.