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ON TWO PROBLEMS OF ELECTRICAL HEATING OF CONDUCTORS

By

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Abstract. The thermal effects of the currents induced in a massive conductor by an external slowly varying magnetic field are studied with regard to existence and uniqueness of solutions. In the first part a theorem of existence of solution is also given for the thermistor problem with a current limiting device.

1. Introduction. In this paper we study various problems related to the Joule heating of a conductor. In Sec. 2 a theorem of existence is given for a version of the "thermistor problem" first proposed in [3] and then studied for special boundary conditions in [1]. The boundary conditions here considered are quite general and the same methods apply even to more general situations. In Sec. 3 a model is proposed for describing the Joule heating due to the eddy currents in a very long cylinder when the external variable magnetic field is parallel to the axis of the cylinder. A theorem of existence is given for this problem.

2. The thermistor problem with a current limiting device. Let us consider a solid body conductor of heat and electricity represented by an open, bounded, and connected subset Ω of $\mathbb{R}^3$. The boundary $\partial\Omega \in C^2$ consists of three parts, S_i, $i = 0, 1, 2$, such that $S_0 \neq \varnothing$, $S_1 \neq \varnothing$, $S_i \cap S_j = \varnothing$ when $i \neq j$, and $\partial\Omega = S_0 \cup S_1 \cup S_2$. S_0 and S_1 represent the metallic electrodes to which a difference of potential is applied. S_2 is insulated both thermally and electrically. The conductor is connected in series with a current limiting device of total resistance R. Steady conditions are supposed. $\mathbf{J}$ is the current density, $\mathbf{E}$ the electric field, $\mathbf{q}$ the heat flow, and u the temperature. The electric and thermal conductivities are given functions of the temperature denoted by $\sigma(u)$ and $\kappa(u)$ respectively. From the constitutive relations

$$\mathbf{J} = \sigma\mathbf{E}, \qquad \mathbf{q} = -\kappa\nabla u, \tag{2.1}$$

and the usual conservation laws

$$\nabla \bullet \mathbf{J} = 0, \qquad \nabla \bullet \mathbf{q} = \mathbf{E} \bullet \mathbf{J}, \tag{2.2}$$

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we obtain, since $\mathbf{E} = -\nabla\varphi$,

$$\nabla \bullet (\sigma(u)\nabla\varphi) = 0, \tag{2.3}$$

$$-\nabla \bullet (\kappa(u)\nabla u) = \sigma(u)|\nabla\varphi|^2. \tag{2.4}$$

Denoting by $\mathbf{n}$ the exterior pointing unit normal vector to $\partial\Omega$, we have

$$i = -\int_{S_0} \mathbf{J} \bullet \mathbf{n}\, ds = \int_{S_0} \sigma(u)\frac{d\varphi}{dn}\, ds \tag{2.5}$$

for the total current i entering through S_0. The surfaces S_0 and S_1 are almost exactly equipotential, hence we have

$$\varphi = 0 \quad \text{on } S_0, \qquad \varphi = \varphi_1 \quad \text{on } S_1, \tag{2.6}$$

where φ_1 is an unknown constant satisfying the condition

$$V - R\int_{S_1} \sigma(u)\frac{d\varphi}{dn}\, ds - \varphi_1 = 0, \tag{2.7}$$

which follows from $V = \varphi_1 + Ri$. V is the total difference of applied potential, a given positive constant. Since S_2 is insulated, we have

$$\frac{d\varphi}{dn} = 0 \quad \text{on } S_2, \tag{2.8}$$

$$\frac{du}{dn} = 0 \quad \text{on } S_2. \tag{2.9}$$

On the remaining part of the boundary we assume for the temperature a Dirichlet boundary condition, i.e.,

$$u = u_0 \quad \text{on } S_0 \cup S_1, \tag{2.10}$$

where $u_0 \geq 0$ is the restriction to $S_0 \cup S_1$ of a $C^2(\overline{\Omega})$-function. We arrive to the following problem Pb_1:

Find $\varphi(x)$ and $u(x)$ such that (2.3) and (2.4) hold in Ω. Moreover (2.6), (2.7), (2.8), (2.9), and (2.10) must be satisfied on $\partial\Omega$.

Assume

$$\sigma(\zeta) \in C^1(\mathbb{R}^1_+), \qquad \kappa(\zeta) \in C^1(\mathbb{R}^1_+), \tag{2.11}$$

$$\sigma_1 \geq \sigma(\zeta) \geq \sigma_0 > 0 \quad \forall\zeta \geq 0, \tag{2.12}$$

$$\kappa(\zeta) > 0 \quad \forall\zeta \geq 0, \tag{2.13}$$

$$\int_0^\infty \frac{\kappa(\zeta)}{\sigma(\zeta)}\, d\zeta = \infty. \tag{2.14}$$

In the sequel we need the following

LEMMA 2.1. Given $a(x) \in C^0(\overline{\Omega})$, $b(x) \in C^0(S_1)$, and a constant $V > 0$ such that

$$a(x) \geq a_0 > 0, \qquad x \in \overline{\Omega}, \tag{2.15}$$

$$b(x) \geq b_0 > 0, \qquad x \in S_1, \tag{2.16}$$

the problem

$$\nabla \bullet (a(x)\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{2.17}$$

$$\varphi = 0 \quad \text{on } S_0, \tag{2.18}$$

$$\varphi = V - \int_{S_1} b\frac{d\varphi}{dn}\,ds \quad \text{on } S_1, \tag{2.19}$$

$$\frac{d\varphi}{dn} = 0 \quad \text{on } S_2 \tag{2.20}$$

has one and only one regular solution.

Proof. Uniqueness. Let φ and φ' be two solutions. Define $\psi = \varphi - \varphi'$. The function ψ satisfies

$$\nabla \bullet (a(x)\nabla\psi) = 0 \quad \text{in } \Omega, \tag{2.21}$$

$$\psi = 0 \quad \text{on } S_0, \tag{2.22}$$

$$\psi = -\int_{S_1} b\frac{d\psi}{dn}\,ds \quad \text{on } S_1, \tag{2.23}$$

$$\frac{d\psi}{dn} = 0 \quad \text{on } S_2. \tag{2.24}$$

Let ψ_1 be the constant value of ψ on S_1. We claim that $\psi_1 \geq 0$. Assume by contradiction $\psi_1 < 0$. By the maximum principle in Hopf's form [7] we have $\frac{d\psi}{dn} < 0$ on S_1, hence by (2.23) we obtain $\psi_1 > 0$. Similarly we have $\psi_1 \leq 0$. Therefore $\psi_1 = 0$. This implies $\psi = 0$ in $\overline{\Omega}$.

Existence. Let $\Gamma \in [0, V]$ and $\varphi(x; \Gamma)$ be the unique solution of the problem

$$\nabla \bullet (a(x)\nabla\varphi) = 0 \quad \text{in } \Omega,$$

$$\varphi = 0 \quad \text{on } S_0, \qquad \varphi = \Gamma \quad \text{on } S_1, \qquad \frac{d\varphi}{dn} = 0 \quad \text{on } S_2.$$

By standard results, $\varphi(x; \Gamma)$ is regular and depends continuously on Γ. Define

$$g(\Gamma) = V - \Gamma - \int_{S_1} b\frac{d\varphi}{dn}\,ds.$$

We have $g(\Gamma) \in C^0[0, V]$, $g(0) = V$, and $g(V) = -\int_{S_1} b\frac{d\varphi}{dn}\,ds < 0$, since $\frac{d\varphi}{dn} > 0$ on S_1. Hence there exists $\phi_0 \in (0, V)$ such that $g(\phi_0) = 0$. The solution of problem (2.17)–(2.20) is given by $\varphi(x; \phi_0)$. □

LEMMA 2.2. Let (φ, u) be any solution of problem Pb_1. We have

$$0 < \varphi < V \quad \text{in } \Omega, \tag{2.25}$$

$$\frac{d\varphi}{dn} > 0 \quad \text{on } S_1. \tag{2.26}$$

Proof. Let φ_1 be the constant value of φ on S_1. We claim that $\varphi_1 > 0$. By contradiction, let $\varphi_1 = 0$. This implies $\varphi = 0$ in Ω against (2.7) since $V > 0$. On the other hand, if $\varphi_1 < 0$ we have, by the maximum principle, $\frac{d\varphi}{dn} < 0$ on S_1

which contradicts (2.7). Hence $\varphi_1 > 0$. Therefore $\frac{d\varphi}{dn} > 0$ on S_1 and by (2.7) we have $\varphi_1 < V$. Again by the maximum principle we conclude that (2.25) and (2.26) hold. □

Define $u_m = \min_{S_0 \cup S_1} u_0$ and $u_M = \max_{S_0 \cup S_1} u_0$. We introduce now the basic transformation of the problem. Define:

$$F(u) = \int_{u_m}^{u} \frac{\kappa(t)}{\sigma(t)}\, dt \tag{2.27}$$

and

$$\theta = \varphi^2/2 + F(u). \tag{2.28}$$

If $(\varphi(x), u(x))$ is a solution of problem Pb_1, it is easy to verify that $\theta(x)$ given by (2.28) satisfies

$$\nabla \bullet (\sigma(u)\nabla\theta) = 0 \quad \text{in } \Omega. \tag{2.29}$$

Moreover, evaluating $\theta(x)$ on $\partial\Omega$ we have $\theta = \theta_0(x)$ on S_0 with $\theta_0(x) = F(u_0(x)) \geq 0$, $\theta = \theta_1(x)$ on S_1, where $\theta_1(x) = F(u_0(x)) + \varphi_1^2/2$, $x \in S_1$, and $\frac{d\theta}{dn} = 0$ on S_2 by (2.8), (2.9). Define

$$\theta_M = V^2/2 + F(u_M). \tag{2.30}$$

We have

$$0 \leq \theta \leq \theta_M \quad \text{on } S_0 \cup S_1 \tag{2.31}$$

by (2.25) and, by the maximum principle,

$$0 \leq \theta \leq \theta_M \quad \text{in } \overline{\Omega}. \tag{2.32}$$

Since $\lim_{t\to\infty} F(t) = \infty$ and $F'(t) > 0$, we can consider

$$u = F^{-1}(\theta - \varphi^2/2). \tag{2.33}$$

Let us define

$$\hat{u} = \max_{(\varphi,\theta)\in[0,\theta_M]\times[0,V]} F^{-1}(\theta - \varphi^2/2).$$

By (2.25) and (2.32) we have

$$0 \leq u(x) \leq \hat{u} \quad \text{in } \overline{\Omega}. \tag{2.34}$$

We are now in a position to state the main result of this section.

THEOREM 2.1. There exists at least a regular solution of problem Pb_1.

Proof. Let $A = \{w(x) \in C^0(\overline{\Omega}),\, 0 \leq w(x) \leq \hat{u}\}$. Fix $w(x) \in A$ and consider the linear problem

$$\nabla \bullet (\sigma(w)\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{2.35}$$

$$\varphi = 0 \quad \text{on } S_0, \qquad \frac{d\varphi}{dn} = 0 \quad \text{on } S_2, \tag{2.36}$$

$$\varphi + R\int_{S_1} \sigma(w)\frac{d\varphi}{dn}\, ds = V \quad \text{on } S_1. \tag{2.37}$$

By Lemma 2.1 there exists one and only one solution $\varphi \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, of problem (2.35)–(2.37). Besides, we have $0 \leq \varphi \leq V$ in $\overline{\Omega}$ and, by standard results of regularity,

$$\|\varphi\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_1, \tag{2.38}$$

where the constant C_1 depends only on σ_0, σ_1, V, and Ω. Next we solve the problem

$$\nabla \bullet (\sigma(w)\nabla\theta) = 0 \quad \text{in } \Omega, \tag{2.39}$$

$$\theta = F(u_0) \quad \text{on } S_0, \qquad \theta = F(u_0) + \frac{\varphi_1^2}{2} \quad \text{on } S_1, \qquad \frac{d\theta}{dn} = 0 \quad \text{on } S_2, \tag{2.40}$$

where φ_1 is the constant value on S_1 of the solution $\varphi(x)$ of (2.35)–(2.37). As in (2.32) we have $0 \leq \theta \leq \theta_M$ in $\overline{\Omega}$ and $\theta \in C^{0,\alpha}(\overline{\Omega})$. Moreover, by elliptic estimates, we obtain

$$\|\varphi\|_{C^{0,\beta}(\overline{\Omega})} \leq C_2, \qquad \beta \in (0, 1), \tag{2.41}$$

where the constant C_2 depends only on the data and not on w. Define

$$u(x) = F^{-1}(\varphi^2(x)/2 - \theta(x)).$$

We get $0 \leq u \leq \hat{u}$ in $\overline{\Omega}$ and

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C_3, \qquad \gamma \in (0, 1). \tag{2.42}$$

It follows that the mapping $u = Tw$ maps A into a compact subset of A. It is also easy to check that T is continuous. Hence by the fixed point theorem of J. Schauder, T has a fixed point. By the usual bootstrap argument, we conclude that the regularity of the solution of problem Pb_1 depends only on the assumptions of regularity made on σ, K, and $\partial\Omega$. □

REMARK 2.1. Uniqueness is not to be expected for problem Pb_1 without further hypotheses on σ and κ. An example of nonuniqueness is given in [1].

3. Heating in massive conductor caused by the eddy currents. In certain industrial processes small pieces of metal are brought to the melting point with the intense Joule-heating generated by the Foucault currents induced in the specimen by an external variable magnetic field. Now the Wiedemann-Franz law relates the electrical and thermal conductivities with the temperature in metals and reads $\sigma = Ck/u$, where C is a universal constant. This implies a substantial variation for σ in the range of temperature practically encountered. In this section we propose a model, valid for a geometrically simple situation, for describing the heating due to the eddy currents. A theorem of existence of solution is also given.

Let us consider a long, conducting, and homogeneous cylinder of magnetic permeability μ and orthogonal cross-section Ω, an open and bounded subset of $\mathbb{R}^2$ with a regular boundary $\partial\Omega$. An insulating medium of permeability μ_0 fills the space utside the cylinder. Moreover, a time-periodic magnetic field is given at infinity by

$$\mathbf{H}_\infty = \overline{H}(\tau)\mathbf{i}_3, \tag{3.1}$$

where $\overline{H}(\tau)$ is a T-periodic regular function and $\mathbf{i}_3$ is the unit vector parallel to the axis of the cylinder. We want to find the magnetic field $\mathbf{H}$ everywhere, the current density $\mathbf{J}$ and the temperature U in the cylinder corresponding to the Joule heating due to the eddy currents. Given the geometry of the situation, we assume

$$\mathbf{H} = H(X, \tau)\mathbf{i}_3, \quad X = (X_1, X_2). \tag{3.2}$$

By (3.2) the equation

$$\nabla \bullet \mathbf{H} = 0 \tag{3.3}$$

is satisfied.

We assume we operate with quasi-stationary fields, thus we neglect, the displacement's current in the Maxwell equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{1}{C^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{3.4}$$

Outside the cylinder, where $\mathbf{J} = 0$, we have

$$\nabla \times \mathbf{H} = 0, \tag{3.5}$$

i.e., by (3.2), $\nabla H = 0$. This implies the constancy in space of $\mathbf{H}$ in the insulating medium. Inside the cylinder we get, by Ohm's law and (3.4),

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E}. \tag{3.6}$$

Hence

$$\mathbf{E} = \rho \nabla H \times \mathbf{i}_3, \tag{3.7}$$

where $\rho = 1/\sigma$. Recalling that

$$\nabla \times \mathbf{E} = -\mu \mathbf{H}_\tau \tag{3.8}$$

we obtain from (3.7)

$$\nabla \bullet (\rho \nabla H) = \mu H_\tau. \tag{3.9}$$

The energy equation reads in the present case

$$\varepsilon c U_\tau - \nabla \bullet (\kappa \nabla U) = \rho |\nabla H|^2, \tag{3.10}$$

since the expression for the Joule heating is $\mathbf{E} \bullet \mathbf{J} = \rho |\nabla H|^2$. In (3.10) ε is the mass density, c the specific heat, and κ the thermal conductivity. By the continuity of H across the surface of the cylinder we obtain

$$H = \overline{H}(\tau) \quad \text{on } \partial\Omega \times [0, \infty). \tag{3.11}$$

In addition, we assume

$$U(X, 0) = 0, \tag{3.12}$$

$$U = 0 \quad \text{on } \partial\Omega \times [0, \infty). \tag{3.13}$$

The thermal and electrical conductivities are supposed to be given functions of th temperature.

If L, H_0, U_0, ρ_0, and κ_0 are respectively characteristic constants for length, magnetic field, temperature, resistivity, and thermal conductivity, we can write the basic equations in nondimensional form defining

$$x = X/L, \quad t = \tau/T, \quad h(x, t) = H(Lx, Tt)/H_0,$$
$$\theta(x, t) = U(Lx, Tt)/U_0, \quad r = \rho/\rho_0, \quad k = \kappa/\kappa_0.$$

Equations (3.9) and (3.10) become

$$a_1 h_t - \nabla \bullet (r(\theta)\nabla h) = 0, \tag{3.14}$$
$$a_2\theta_t - \nabla \cdot (k(\theta)\nabla\theta) = a_3 r(\theta)|\nabla h|^2, \tag{3.15}$$

where a_1, a_2, and a_3 are nondimensional parameters given by

$$a_1 = \frac{\mu L^2}{\rho_0 T}, \quad a_2 = \frac{\varepsilon c L^2}{\kappa_0 T}, \quad a_3 = \frac{\rho_0 H_0^2}{U_0 \kappa_0}. \tag{3.16}$$

We want to treat the case in which the ratio a_2/a_1 is much smaller than 1. This is not an unrealistic assumption for certain highly ferromagnetic substances like Si-Fe crystals for which a_2/a_1, as computed from the data for μ given in [4, p. 374], is of order 10^{-3}. For this reason we neglect, in a rather heuristic way, the time derivative in (3.15). This implies of course the impossibility of satisfying the initial condition $\theta(x, 0) = 0$. In this way we study the quasi-stationary situation which occurs after the body is fully heated up. Let us assume

$$r_1 \geq r(\theta) \geq r_0 > 0, \tag{3.17}$$
$$k_1 \geq k(\theta) \geq k_0 > 0. \tag{3.18}$$

Putting

$$F(\theta) = \int_0^\theta k(\zeta)\, d\zeta, \tag{3.19}$$

we can define a new scale of temperature $u = F(\theta)$. By (3.18), F maps $[0, \infty)$ on $[0, \infty)$ one-to-one. Let $d(u) = r[F^{-1}(u)]$. We obtain the following problem:

$$h_t - \nabla \bullet (d(u)\nabla h) = 0, \tag{3.20}$$
$$h = h_0(t) \quad \text{on } \partial\Omega \times [0, \infty), \tag{3.21}$$
$$h(x, t) = h(x, t+1), \tag{3.22}$$
$$-\Delta u = d(u)|\nabla h|^2, \tag{3.23}$$
$$u = 0 \quad \text{on } \partial\Omega \times [0, \infty), \tag{3.24}$$

where we assume $h_0(t) = h_0(t+1)$. We discuss first of all the nonphysical, but elementary case in which

$$h_0(t) = e^{-i2\pi t}. \tag{3.25}$$

We look for solutions of problem (3.20)–(3.25) of the form $h(x, t) = e^{-i2\pi t}[v(x) + iw(x)]$. Substituting in (3.20)–(3.23) we arrive at the following nonlinear elliptic system:

$$\nabla \bullet (d(u)\nabla v) - w = 0, \qquad v = 1 \text{ on } \partial\Omega, \tag{3.26}$$

$$\nabla \bullet (d(u)\nabla w) + v = 0, \qquad w = 0 \text{ on } \partial\Omega, \tag{3.27}$$

$$-\Delta u = d(u)[|\nabla v|^2 + |\nabla w|^2], \qquad u = 0 \text{ on } \partial\Omega. \tag{3.28}$$

In view of (3.26), (3.27), and (3.28) we obtain, with a simple calculation,

$$\nabla \bullet [d(u)\nabla\gamma] = 0, \tag{3.29}$$

where

$$\gamma = \frac{v^2}{2} + \frac{w^2}{2} + G(u), \qquad G(u) = \int_0^u \frac{dt}{d(t)}.$$

Since $\gamma = \frac{1}{2}$ on $\partial\Omega$ and $d_1 \geq d(u) \geq d_0 > 0$, we conclude

$$\frac{v^2}{2} + \frac{w^2}{2} + G(u) = \frac{1}{2}. \tag{3.30}$$

On the other hand, by the maximum principle we have $u \geq 0$ in $\overline{\Omega}$. Hence $|v|$, $|w|$, and u are pointwise bounded by an absolute constant. Moreover, since G maps $[0, \infty)$ on $[0, \infty)$ one-to-one, we can write

$$u = G^{-1}\left[\frac{1}{2} - \frac{v^2}{2} - \frac{w^2}{2}\right],$$

and problem (3.26)–(3.28) can be restated as follows:

$$\nabla \bullet [A(v, w)\nabla v] - w = 0, \qquad v = 1 \text{ on } \partial\Omega, \tag{3.31}$$

$$\nabla \bullet [A(v, w)\nabla w] + v = 0, \qquad w = 0 \text{ on } \partial\Omega, \tag{3.32}$$

where $A(v, w) = d[G^{-1}(\frac{1}{2} - v^2/2 - w^2/2)]$.

The Schauder's fixed point theorem can be applied quite easily to problem (3.31)–(3.32) and the existence of at least one classical solution can in this way be proved.

4. Existence for the eddy currents problem. In this section we study the following problem Pb_2:

$$h_t - \nabla \bullet (d(u)\nabla h) = 0, \tag{3.20}$$

$$h = h_0(t) \quad \text{on } \partial\Omega \times [0, \infty), \tag{3.21}$$

$$h(x, t) = h(x, t + 1), \tag{3.22}$$

where $h_0(t)$ is now an arbitrary 1-periodic, regular function. Integrating equation (3.23) over one time period we obtain

$$-\Delta u = d(u) \int_0^1 |\nabla h|^2 \, dt, \qquad u = 0 \text{ on } \partial\Omega. \tag{4.1}$$

We want to prove that problem (3.20), (3.21), (3.22), (4.1) has at least one weak solution. Let $Q = \Omega \times (0, 1)$, $N = \Omega \times \mathbb{R}^1$, and $S = \partial\Omega \times \mathbb{R}^1$. Let $C_0^\infty(Q)$ be the set of all functions $\eta(x, t)$ of class $C^\infty(\overline{N})$, periodic in t with period 1 and vanishing near S. Define $\overset{\circ}{W}{}_2^{1,0}(Q)$ as the closure of $C_0^\infty(Q)$ with respect to the norm

$$\|\eta\|_{\overset{\circ}{W}{}_2^{1,0}(Q)} = \left[\int_Q v^2\,dx\,dt + \int_Q |\nabla v|^2\,dx\,dt\right]^{1/2},$$

where $\nabla\eta = (\eta_{x_1}, \eta_{x_2})$. Let $\overset{\circ}{W}{}_2^{1,1}(Q)$ be the closure of $C_0^\infty(Q)$ with respect to the norm

$$\|\eta\|_{\overset{\circ}{W}{}_2^{1,1}(Q)} = \left[\int_Q v^2\,dx\,dt + \int_Q |\nabla v|^2\,dx\,dt + \int_Q v_t^2\,dx\,dt\right]^{1/2}.$$

As a weak formulation of problem Pb_2 we take

$$h - h_0 \in \overset{\circ}{W}{}_2^{1,0}(Q), \qquad \int_Q h\eta_t\,dx\,dt - \int_Q d(u)\nabla h \bullet \nabla\eta\,dx\,dt = 0 \tag{4.2}$$

for all $\eta \in \overset{\circ}{W}{}_2^{1,1}(Q)$, and

$$u \in H_0^1(\Omega), \qquad \int_\Omega \nabla u \bullet \nabla v\,dx = \int_Q d(u)|\nabla h|^2 v\,dx\,dt \tag{4.3}$$

for all $v \in \overset{\circ}{H}{}^1(\Omega) \cap L^\infty(\Omega)$, where $\overset{\circ}{H}{}^1(\Omega)$ is the usual Sobolev space obtained as completion of the function $v(x) \in C^\infty(\overline{\Omega})$ vanishing near $\partial\Omega$ with respect to the norm

$$\|v\|_{\overset{\circ}{H}{}^1(\Omega)} = \left[\int_\Omega |\nabla v|^2\,dx\right]^{1/2}.$$

The main difficulty in treating problem Pb_2 lies in the fact that the left-hand side of Eq. (4.1) belongs "a priori" only to $L^1(\Omega)$. For this reason we consider the following sequence of approximating problems Pb_n:

$$\frac{\partial h_n}{\partial t} - \nabla \bullet (d(u_n)\nabla h_n) = 0, \tag{4.4}$$

$$h_n = h_0 \quad \text{on } S, \qquad h_n(x, t) = h_n(x, t+1), \tag{4.5}$$

$$\frac{1}{n}\Delta\Delta u_n - \Delta u_n = d(u_n)\int_0^1 |\nabla h_n|^2\,dt, \tag{4.6}$$

$$u_n = 0, \quad \Delta u_n = 0 \quad \text{on } \partial\Omega, \tag{4.7}$$

and the corresponding weak formulation: $h_n - h_0 \in \overset{\circ}{W}{}_2^{1,0}(Q)$,

$$\int_Q h_n\eta_t\,dx\,dt - \int_Q d(u_n)\nabla h_n \bullet \nabla\eta\,dx\,dt = 0 \tag{4.8}$$

for all $\eta \in \overset{\circ}{W}{}_2^{1,1}(Q)$,

$$\begin{gathered} u_n \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega), \\ \frac{1}{n}\int_\Omega \Delta u_n \Delta v\,dx + \int_\Omega \nabla u_n \cdot \nabla v\,dx = \int_Q d(u_n)|\nabla h_n|^2 v\,dx\,dt, \end{gathered} \tag{4.9}$$

for all $v \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$.

First of all we derive various "a priori" estimates for the solutions of problem Pb_n. By the parabolic maximum principle we have in Q:

$$\min_{[0,1]} h_0(t) \le h_n(x, t) \le \max_{[0,1]} h_0(t). \tag{4.10}$$

Moreover if we proceed as in Lemma 3.1 of [2], where a similar situation occurs, we can show that

$$u_n \ge 0 \quad \text{in } \overline{\Omega}. \tag{4.11}$$

Since $u_n \in H^2(\Omega)$, we have $h_n - h_0 \in \overset{\circ}{W}{}_2^{1,1}(Q)$. Choosing $\eta = h_n - h$ in (4.8) we obtain

$$\int_Q d(u_n)|\nabla h_n|^2\,dx\,dt \le C_1$$

and, by (3.17),

$$\int_Q |\nabla h_n|^2\,dx\,dt \le C_2, \tag{4.12}$$

where the constants C_1 and C_2 do not depend on n. Let us multiply (4.4) by $h_n u_n$. Since $h_n u_n = 0$ on S, integrating by part over Ω we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_n^2 u_n\,dx + \int_\Omega u_n d(u_n)|\nabla h_n|^2\,dx + \int_\Omega h_n d(u_n)\nabla h_n \cdot \nabla u_n\,dx = 0.$$

By (4.10) and (4.12) we infer

$$\int_Q u_n d(u_n)|\nabla h_n|^2\,dx\,dt \le C_3\left[\int_Q |\nabla u_n|^2\,dx\,dt\right]^{1/2}. \tag{4.13}$$

Selecting $v = u_n$ in (4.9), we have by (4.13)

$$\frac{1}{n}\int_\Omega |\Delta u_n|^2\,dx + \int_\Omega |\nabla u_n|^2\,dx \le C_3\left[\int_\Omega |\nabla u_n|^2\,dx\right]^{1/2}.$$

It follows that

$$\int_\Omega |\nabla u_n|^2\,dx \le C, \tag{4.14}$$

$$\frac{1}{n}\int_\Omega |\Delta u_n|^2\,dx \le C, \tag{4.15}$$

where again C is a constant not depending on n.

The following lemma is a straightforward application of the Schauder fixed point theorem.

LEMMA 4.1. For every fixed $n \in \mathbb{N}$, there exists at least one solution of problem Pb_n.

Sketch of Proof. We omit the index n. Let $w \in \overset{\circ}{H}{}^1(\Omega)$ and solve, using standard results of linear theory, the problem

$$h - h_0 \in \overset{\circ}{W}{}_2^{1,0}(Q), \qquad \int_Q h\eta_t\,dx\,dt - \int_Q d(w)\nabla h \cdot \nabla\eta\,dx\,dt = 0$$

for all $\eta \in \overset{\circ}{W}{}_2^{1,1}(Q)$. Then we solve

$$u \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega),$$
$$\int_\Omega \Delta u \Delta v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx = \int_Q d(u)|\nabla h|^2 v \, dx \, dt$$

for all $v \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$.

By the estimates (4.12), (4.14), and (4.15) there exists $R > 0$ such that the map $u = Tw$ maps the ball B_R of center zero and radius R of $\overset{\circ}{H}{}^1(\Omega)$ into itself. By (4.15) the set $T(B_R)$ is compact in $\overset{\circ}{H}{}^1(\Omega)$. Besides, T is continuous, hence T has a fixed point and problem Pb_n a solution. □

By (4.10), (4.12), and (4.14) we can extract from $\{h_n, u_n\}$ a subsequence $\{h_m, u_m\}$ such that

$$h_m \to h \quad \text{strongly in } L^2(Q), \tag{4.16}$$
$$\nabla h_m \to \nabla h \quad \text{weakly in } L^2(Q), \tag{4.17}$$
$$u_m \to u \quad \text{strongly in } L^2(Q), \tag{4.18}$$
$$\nabla u_m \to \nabla u \quad \text{weakly in } L^2(Q), \tag{4.19}$$
$$d(u_m) \to d(u) \quad \text{strongly in } L^p(Q),\ 1 \le p < \infty, \tag{4.20}$$
$$d(u_m) \to d(u) \quad \text{a.e. in } \Omega. \tag{4.21}$$

By (4.16), (4.17), and (4.20) we can pass to the limit in (4.8). We get

$$h - h_0 \in \overset{\circ}{W}{}_2^{1,0}(Q), \qquad \int_Q h\eta_t \, dx \, dt - \int_Q d(u) \nabla h \cdot \nabla \eta \, dx \, dt = 0 \tag{4.22}$$

for all $\eta \in \overset{\circ}{W}{}_2^{1,1}(Q)$.

From (4.8) and (4.22) we obtain

$$\begin{aligned} &\int_Q d(u_m)\nabla(h_m - h) \bullet \nabla\eta \, dx \, dt - \int_Q (h_m - h)\eta_t \, dx \, dt \\ &\quad = \int_Q (d(u) - d(u_m))\nabla h \bullet \nabla \eta \, dx \, dt, \end{aligned} \tag{4.23}$$

for all $\eta \in \overset{\circ}{W}{}_2^{1,1}(Q)$.

By (4.20) and (4.23) we conclude that

$$\int_Q d(u_m)|\nabla(h_m - h)|^2 \, dx \, dt \to 0, \tag{4.24}$$
$$\nabla h_m \to \nabla h \quad \text{strongly in } L^2(Q). \tag{4.25}$$

On the other hand,

$$\begin{aligned} d(u_m)|\nabla h_m|^2 &= d(u_m)|\nabla(h_m - h)|^2 \\ &\quad + 2d(u_m)\nabla h \bullet \nabla h_m - d(u_m)|\nabla h|^2, \end{aligned}$$

hence we obtain, by (4.24), (4.25), and (4.20),

$$\int_Q d(u_m)|\nabla h_m|^2\,dx\,dt \to \int_Q d(u)|\nabla h|^2\,dx\,dt. \tag{4.26}$$

From (4.20), (4.25), and (4.26) we have, using Lebesgue's dominated convergence theorem,

$$d(u_m)|\nabla h_m|^2 \to d(u)|\nabla h|^2 \quad \text{strongly in } L^1(Q). \tag{4.27}$$

We are now in a position to pass to the limit in (4.9) as $m \to \infty$. By (4.15) we infer

$$\frac{1}{m}\int_\Omega \Delta u_m \Delta v\,dx \to 0,$$

and by (4.19)

$$\int_\Omega \nabla u_m \bullet \nabla v\,dx \to \int_\Omega \nabla u \bullet \nabla v\,dx.$$

Recalling (4.27) we obtain

$$\int_Q d(u_m)|\nabla h_m|^2 v\,dx\,dt \to \int_Q d(u)|\nabla h|^2 v\,dx\,dt.$$

Whence we have, for all $v \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$,

$$u \in \overset{\circ}{H}{}^1(\Omega), \qquad \int_\Omega \nabla u \bullet \nabla v\,dx = \int_Q d(u)|\nabla h|^2\,dx\,dt. \tag{4.28}$$

By density, (4.28) holds true also for all $v \in \overset{\circ}{H}{}^1(\Omega) \cap L^\infty(\Omega)$. Hence (4.3) follows. This proves that problem Pb_2 has at least one weak solution.

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