Anders Holmbom; Jeanette Silfver; Nils Svanstedt; Niklas Wellander
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Applications of Mathematics, Vol. 51 (2006), No. 3, 247–262

Persistent URL: http://dml.cz/dmlcz/134639

51 (2006) APPLICATIONS OF MATHEMATICS No. 3, 247–262

ON TWO-SCALE CONVERGENCE AND RELATED SEQUENTIAL COMPACTNESS TOPICS

Anders Holmbom, Östersund, Jeanette Silfver, Östersund,
Nils Svanstedt, Göteborg, and Niklas Wellander, Linköping

(Received August 19, 2004, in revised version November 11, 2004)

Abstract. A general concept of two-scale convergence is introduced and two-scale compactness theorems are stated and proved for some classes of sequences of bounded functions in $L^2(\Omega)$ involving no periodicity assumptions. Further, the relation to the classical notion of compensated compactness and the recent concepts of two-scale compensated compactness and unfolding is discussed and a defect measure for two-scale convergence is introduced.

Keywords: two-scale convergence, compensated compactness, two-scale transform, unfolding

MSC 2000: 40A30

1. Introduction

In 1989 Nguetseng [15] presented a new approach for the homogenization of partial differential equations, the so-called *two-scale convergence method*. The name two-scale convergence was introduced by Allaire in [1], where the method was applied to a variety of problems. Nguetseng's method has been widely used and developed in various ways. A careful treatment of the theoretical fundaments of the method is found in the recent survey [12] by Lukkassen et al. Let us also mention [4], where Amar proved two-scale compactness for a sequence of functions defined on BV. The extension to the almost periodic case is found in Casado-Diaz and Gayto [7], and in [6] Bourgeat et al develop a stochastic two-scale convergence (in the mean). Further, in [3] two-scale convergence is extended to the linear stationary multiscale case by Allaire and Briane, and in [11] by Lions et al to the monotone stationary multiscale case. In [10] a multiscale homogenization theorem is proved for parabolic problems. The two-scale convergence method relies on the sequential matching between a bounded sequence $\{u_h\}$ of functions in $L^2(\Omega)$ and a sequence $\{\mu_h\}$ of functions defined through $\mu_h(x) = v\bigl(x, x/\varepsilon_h\bigr)$, $v \in L^2(\Omega \times Y)$, where Y is the unit cube in $\mathbb{R}^N$ and Ω an open bounded set in $\mathbb{R}^N$. The original result by Nguetseng says that for v sufficiently smooth and Y-periodic in the second argument it holds up to a subsequence that

$$\lim_{h\to\infty} \int_\Omega u_h(x) v\Bigl(x, \frac{x}{\varepsilon_h}\Bigr)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for $\varepsilon_h \to 0$ when $h \to \infty$. The purpose of the present paper is to state and prove general compactness results that do not depend upon any periodicity assumptions on the test functions. We discuss it in the context of the classical compensated compactness by Murat and Tartar and the recent concept of two-scale compensated compactness by Birnir, Svanstedt and Wellander. In particular, we will discuss the fact that the additional scale appearing in the two-scales limit allows for a relaxation of the regularity needed in order to obtain results of compensated compactness type.

R e m a r k 1. In the sequel we denote spaces of functions periodic with respect to the unit cell Y as

$$F_\sharp(Y) = \{u \colon\ u \in F_{\mathrm{loc}}(\mathbb{R}^N),\ \ u \text{ is } Y\text{-periodic}\},$$

where F may be, for example, L^p, $W^{1,p}$, C or C^∞.

2. Weak convergence and general two-scale convergence

Let us first recall the usual weak compactness in $L^2(\Omega)$. For a bounded sequence $\{u_h\}$ in $L^2(\Omega)$ it is well-known that, up to a subsequence, $\{u_h\}$ converges weakly, i.e., for some $u \in L^2(\Omega)$ it holds that

$$\int_\Omega u_h(x)\mu(x)\,\mathrm{d}x \to \int_\Omega u(x)\mu(x)\,\mathrm{d}x$$

for all $\mu \in L^2(\Omega)$. Replacing μ with a bounded sequence $\{\mu_h\}$ in $L^2(\Omega)$ the situation gets less obvious. Depending on in which way the involved sequences of functions are chosen it may or may not hold that, still up to a subsequence,

$$\int_\Omega u_h(x)\mu_h(x)\,\mathrm{d}x \to \int_\Omega u(x)\mu(x)\,\mathrm{d}x, \tag{1}$$

where u and μ are the weak limits for $\{u_h\}$ and $\{\mu_h\}$. The easiest way to make sure that (1) holds is of course to make the stronger assumption that $\{\mu_h\}$ converges strongly. However, the notion of two-scale convergence provides an alternative. For suitable choices of v we have already noticed that

$$\int_\Omega u_h(x)\mu_h(x)\,\mathrm{d}x \to \int_\Omega \int_Y u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x, \tag{2}$$

where $\mu_h(x) = v\big(x, x/\varepsilon_h\big)$ is bounded in $L^2(\Omega)$ but not necessarily strongly convergent to any limit in $L^2(\Omega)$. It seems like the extra scale supports the convergence in cases where neither $\{u_h\}$ nor $\{\mu_h\}$ are strongly convergent.

A natural question to ask is whether there are other ways to generate weakly convergent sequences $\{\mu_h\}$ such that (2) is true. To find out we investigate sequences of integral expressions of the type

$$\int_\Omega u_h(x)\mu_h(x)\,\mathrm{d}x = \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x,$$

where

$$\tau_h\colon\ X \to L^2(\Omega)$$

and $X \subset L^2(\Omega \times A)$, where $\Omega \subset \mathbb{R}^{\mathbb{N}}$ and $A \subset \mathbb{R}^M$ are open and bounded. We prove that results of the same kind as (2) hold under general assumptions on the operators τ_h and the admissible set X. One of the advantages with two-scale convergence is that all the properties one seeks for in the sequence $\{u_h\}$ are lifted out by a suitable choice of test functions. Therefore the characterization of admissibility of test functions is one of the key problems in two-scale convergence. We will prove two theorems, Theorem 3 and Theorem 6, where the admissible set of test functions belongs to two different subsets of $L^2(\Omega \times A)$. One is based on separability and the other is characterized by its geometrical cone properties (Hahn-Banach). First we define two-scale convergence in a general setting:

Definition 2. A sequence $\{u_h\}$ in $L^2(\Omega)$ is said to two-scale converge to $u_0 \in L^2(\Omega \times A)$ with respect to $\{\tau_h\}$ if $\tau_h\colon\ X \to L^2(\Omega)$ and

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x = \int_\Omega \int_A u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for all $v \in X$.

We have the following compactness result:

Theorem 3. *Let Ω and A be bounded open subsets of $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{M}$, respectively, and assume that $\{u_h\}$ is a bounded sequence in $L^2(\Omega)$. Further, assume that $X \subset L^2(\Omega \times A)$ is a separable Banach space and that*

$$\tau_h\colon\ X \to L^2(\Omega)$$

satisfies

$$\lim_{h\to\infty} \|\tau_h v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega\times A)} \tag{3}$$

and

$$\|\tau_h v\|_{L^2(\Omega)} \leqslant C\|v\|_X. \tag{4}$$

Then there exists $u_0 \in L^2(\Omega\times A)$ and a subsequence $h\to\infty$ such that

$$\int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x \to \int_\Omega\int_A u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for all $v \in X$.

P r o o f. By the Schwarz inequality and (4)

$$\left|\int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x\right| \leqslant \|u_h\|_{L^2(\Omega}\|\tau_h v\|_{L^2(\Omega)} \leqslant C\|v\|_X. \tag{5}$$

This means that we can identify u_h with an element F_h in the dual X' via

$$(F_h, v)_{X',X} = \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x.$$

The sequence $\{F_h\}$ is bounded in X'. By (5) we get

$$\|F_h\|_{X'} = \sup_{\|v\|_X=1} |(F_h, v)_{X',X}| \leqslant C. \tag{6}$$

Now X' is the dual of the separable Banach space X and then the bound (6) means that there exists a weakly$*$ convergent subsequence of $\{F_h\}$ in X' such that

$$(F_h, v)_{X',X} \to (F, v)_{X',X}$$

for all $v \in X$. Further, (3), (5), and a passage to the limit yields

$$|(F, v)_{X',X}| \leqslant C\|v\|_{L^2(\Omega\times A)}$$

and thus, by using the Hahn-Banach theorem, there exists a bounded linear functional $G \in (L^2(\Omega \times A))'$ such that

$$(F, v)_{X',X} = (G, v)_{(L^2(\Omega\times A))', L^2(\Omega\times A)}$$

for all $v \in X$. Finally, according to the Riesz representation theorem (L^2-duality), there exists a unique $u_0 \in L^2(\Omega \times A)$ such that

$$(G, v)_{(L^2(\Omega\times A))', L^2(\Omega\times A)} = \int_\Omega \int_A u_0(x, y) v(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

and the proof is complete. □

Below we prove that the second scale in the two-scale limit is lost if we assume that $\{u_h\}$ converges strongly in $L^2(\Omega)$.

Proposition 4. *Let $\{\tau_h\}$ be as in Theorem 3 with the additional condition that $\mu_h = \tau_h v$ converges weakly to $\mu(x) = \int_A v(x, y)\, \mathrm{d}y$ in $L^2(\Omega)$. Assume further that $\{u_h\}$ converges strongly to u in $L^2(\Omega)$. Then*

$$\int_\Omega u_h(x)\tau_h v(x)\, \mathrm{d}x \to \int_\Omega \int_Y u(x) v(x, y)\, \mathrm{d}y\, \mathrm{d}x = \int_\Omega u(x)\mu(x)\, \mathrm{d}x$$

for all admissible v.

Proof. The "weak-strong" convergence immediately yields

$$\int_\Omega u_h(x)\tau_h v(x)\, \mathrm{d}x \to \int_\Omega u(x) \Big[\int_A v(x, y)\, \mathrm{d}y\Big]\, \mathrm{d}x = \int_\Omega u(x)\mu(x)\, \mathrm{d}x$$

and the proof is complete. □

For the proof of the second compactness result (Theorem 6), we first recall the following version of the Hahn-Banach theorem.

Lemma 5 (Hahn-Banach). *Let X be a normed linear space and Y a subset to X. Further, assume that $f\colon Y \to \mathbb{R}$ is linear and that*

$$\Big|\sum_{i=1}^n c_i f(v_i)\Big| \leqslant C \Big\|\sum_{i=1}^n c_i v_i\Big\|_X$$

for some C and all $v_i \in Y, c_i \in \mathbb{R}$.

Then there exists a linear functional g with $\|g\|_{X'} \leqslant C$ that extends f from Y to X.

Proof. Put $p(v) = C\|v\|_X$ in Theorem 2.3.1 in Edwards [9]. □

We are now ready to state and prove a second two-scale compactness result which holds under somewhat different assumptions than in Theorem 3.

Theorem 6. *Let Ω and A be bounded open subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, and assume that $\{u_h\}$ is a bounded sequence in $L^2(\Omega)$, X a subset contained in $L^2(\Omega \times A)$ endowed with the norm of $L^2(\Omega \times A)$, and*

$$\tau_h\colon X \to L^2(\Omega)$$

a sequence of linear maps such that, for some C independent of h,

$$\left\|\sum_{i=1}^n c_i \tau_h v_i\right\|_{L^2(\Omega)} \leqslant C \left\|\sum_{i=1}^n c_i v_i\right\|_{L^2(\Omega\times A)} \tag{7}$$

for all $v_i \in X$, $c_i \in \mathbb{R}$. Then, for some $u_0 \in L^2(\Omega \times A)$ and up to a subsequence,

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x = \int_\Omega \int_A u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

holds for all $v \in X$.

Proof. We introduce

$$(F_h, v)_{X',X} = \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x.$$

Clearly, by (7) and the Hölder inequality,

$$\begin{aligned}\left|\sum_{i=1}^n c_i (F_h, v_i)_{X',X}\right| &= \left|\sum_{i=1}^n c_i \int_\Omega u_h(x)(\tau_h v_i)(x)\,\mathrm{d}x\right| \\ &\leqslant C\left\|\sum_{i=1}^n c_i \tau_h v_i\right\|_{L^2(\Omega)} \leqslant D\left\|\sum_{i=1}^n c_i v_i\right\|_{L^2(\Omega\times A)}.\end{aligned} \tag{8}$$

Equation (8) and Lemma 5 yield that there exists an extension G^h of F^h such that

$$(G_h, v)_{(L^2(\Omega\times A))', L^2(\Omega\times A)} \leqslant D\|v\|_{L^2(\Omega\times A)}. \tag{9}$$

The inequality (9) and the separability of $L^2(\Omega\times A)$ imply that there exists a weakly$*$ convergent subsequence of $\{G^h\}$ in $(L^2(\Omega \times A))'$ such that

$$(G_h, v)_{(L^2(\Omega\times A))', L^2(\Omega\times A)} \to (G, v)_{(L^2(\Omega\times A))', L^2(\Omega\times A)}$$

for all $v \in L^2(\Omega\times A)$. Finally, due to L^2-duality, there exists a unique $u_0 \in L^2(\Omega\times A)$ such that

$$(G, v)_{(L^2(\Omega\times A))', L^2(\Omega\times A)} = \int_\Omega \int_A u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

and therefore, for the restriction F of G to X,

$$(F, v)_{X',X} = \int_\Omega \int_A u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for any $v \in X$. □

R e m a r k 7. All results in this chapter are easily generalized from L^2 to the L^p-case when $p > 1$. The case $p = 1$ has to be handled in a somewhat different manner. Since $L^1(\Omega)$ is not reflexive, we cannot apply weak sequential compactness. We can however argue as follows. Let $C_0(\Omega)$ denote the set of continuous functions with compact support in Ω. Then it is well known that its dual is $(C_0(\Omega))' = M(\Omega)$, i.e., the space of Radon measures on Ω. Let us now as usual identify $L^1(\Omega)$ with a subspace of $M(\Omega)$. It follows that, if $\{u_h\}$ is a sequence which is uniformly bounded in $L^1(\Omega)$ and if

$$\tau_h\colon\ C_0(\Omega \times A) \to C_0(\Omega)$$

is a sequence of maps such that

$$\lim_{h\to\infty} \|\tau_h v\|_{C_0(\Omega)} \leqslant C\|v\|_{C_0(\Omega\times A)}, \tag{10}$$

then there exists a Radon measure $\mu_0 \in M(\Omega \times A)$ and a subsequence, still denoted $\{u_h\}$, in $L^1(\Omega)$ such that

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x = \langle \mu_0(x,y), v(x,y)\rangle_{M(\Omega\times A),C_0(\Omega\times A)}.$$

If the limit element μ_0 belongs to $L^1(\Omega \times A)$, then the two-scale convergence is compact in L^1. Weak two-scale compactness in L^1 can also be proved using the usual Dunford-Pettis characterization. For a complete exposition of periodic two-scale convergence of Radon measures we refer to [4] by Amar; in [13] Mascarenhas and Toader introduce a concept called "scale-convergence" for Young measures.

R e m a r k 8 (Periodic case). Let $\tau_h v(x) = v(x, x/\varepsilon_h)$, where $v(x,y)$ is periodic (unit period for instance) and continuous in the second argument, and where $\{\varepsilon_h\}$ is a sequence of positive numbers tending to zero as h tends to $+\infty$. Then (3) becomes

$$\lim_{h\to\infty} \Big\|v\Big(x, \frac{x}{\varepsilon_h}\Big)\Big\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega\times Y)} \tag{11}$$

and, if $\{u_h\}$ is bounded in $L^2(\Omega)$, it holds up to a subsequence that

$$\lim_{h\to\infty} \int_\Omega u_h(x) v\Big(x, \frac{x}{\varepsilon_h}\Big)\,\mathrm{d}x = \int_\Omega\int_Y u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for some $u_0 \in L^2(\Omega \times Y)$, where $Y = (0,1)^N$. Typical examples of admissible test functions as above are those in $L^2(\Omega; C_\sharp(Y))$ and, for Ω bounded, $L^2_\sharp(Y; C(\overline{\Omega}))$. In fact, for these function spaces, (11) holds with equality and with $C = 1$.

R e m a r k 9 (Periodic multiscale case). Let $\tau_h v(x) = v(x, x/\varepsilon_h^1, \ldots, x/\varepsilon_h^q)$, where $v(x, y_1, \ldots, y_q)$ is periodic (unit period for instance) and continuous in $y_1, \ldots, y_q$, and where ε_h is a sequence of positive numbers tending to zero as h tends to $+\infty$. Then (3) becomes

$$\lim_{h\to\infty} \Big\| v\Big(x, \frac{x}{\varepsilon_h^1}, \ldots, \frac{x}{\varepsilon_h^q}\Big)\Big\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega\times Y_1\times\ldots\times Y_q)},$$

and, for $\{u_h\}$ bounded in $L^2(\Omega)$,

$$\lim_{h\to\infty} \int_\Omega u_h(x) v\Big(x, \frac{x}{\varepsilon_h^1}, \ldots, \frac{x}{\varepsilon_h^q}\Big)\,\mathrm{d}x$$
$$= \int_\Omega \int_{Y_1} \ldots \int_{Y_q} u_0(x, y_1, \ldots, y_q) v(x, y_1, \ldots, y_q)\,\mathrm{d}y_1 \ldots \mathrm{d}y_q\,\mathrm{d}x,$$

up to a subsequence, for some $u_0 \in L^p(\Omega \times Y_1 \times \ldots \times Y_q)$, where $Y_i = (0,1)^N$, $i = 1, \ldots, q$. See [3].

3. Two-scale convergence and compensated compactness

In Theorems 3 and 6 we did not ask for anything more than boundedness in $L^2(\Omega)$ of $\{u_h\}$, while we made more specific assumptions on the sequence $\{\mu_h\}$ generated by functions $v = v(x,y)$. A famous result by Murat and Tartar, the *div-curl* lemma, addresses a similar situation under somewhat different assumptions. In their approach they impose certain differential constraints on sequences $\{u_h\}$ and $\{\mu_h\}$ in addition to boundedness in $L^2(\Omega)$. In [16] Tartar utilizes this study and proves general compactness results for quadratic forms $Q(u_h)$ under the name of *compensated compactness*. The div-curl lemma reads:

Theorem 10. *Let $\{u_h\}$ and $\{\mu_h\}$ be bounded sequences in $[L^2(\Omega)]^N$ and u and μ the weak limits of suitable subsequences. If, in addition, $\{\operatorname{div} u_h\}$ and $\{\operatorname{curl} \mu_h\}$ are compact in $W^{-1,2}(\Omega)$ and $[W^{-1,2}(\Omega)]^{N\times N}$, respectively, then*

$$\lim_{h\to\infty} \int_\Omega u_h(x)\cdot\mu_h(x)\varphi(x)\,\mathrm{d}x = \int_\Omega u(x)\cdot\mu(x)\varphi(x)\,\mathrm{d}x \tag{12}$$

for any $\varphi \in C_0^\infty(\Omega)$.

Recently a version of Tartar's compensated compactness result has been developed in connection to the method of two-scale convergence in [5] by Birnir, Svanstedt and Wellander. The two-scale version of the div-curl lemma reads:

Theorem 11. *Suppose that* $\{u_h\}$ *and* $\{\mu_h\}$ *are bounded sequences in* $[L^2(\Omega)]^N$ *and denote by* u_0 *and* v_0 *their weak two-scale limits, respectively. If, in addition,* $\{\operatorname{div} u_h\}$ *and* $\{\operatorname{curl} \mu_h\}$ *are bounded in* $L^2(\Omega)$ *and* $[L^2(\Omega)]^{N\times N}$*, respectively, then*

$$\begin{aligned}(13)\qquad &\lim_{h\to\infty}\int_\Omega u_h(x)\varphi\Big(x,\frac{x}{\varepsilon_h}\Big)\cdot\mu_h(x)\varphi\Big(x,\frac{x}{\varepsilon_h}\Big)\,\mathrm{d}x\\ &\qquad=\int_\Omega\int_Y u_0(x,y)\varphi(x,y)\cdot v_0(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

up to a subsequence for any $\varphi\in C_0^\infty(\Omega;C_\sharp^\infty(Y))$.

Remark 12. The relationship between the convergence (12) in Theorem 10 and the convergence (13) in Theorem 11 deserves some attention. One immediately observes that, by choosing $\varphi=\varphi(x)$, (13) leads to (12) and, consequently,

$$(14)\qquad \int_Y u_0(x,y)\cdot v_0(x,y)\,\mathrm{d}y=\int_Y u_0(x,y)\,\mathrm{d}y\cdot\int_Y v_0(x,y)\,\mathrm{d}y.$$

If, for example, $u_0(x,y)=u(x)$, the identity (14) follows immediately from the fact that the weak limit u is obtained from the two-scale limit u_0 through

$$u(x)=\int_Y u_0(x,y)\,\mathrm{d}y.$$

It is well known that the loss of the second scale occurs when $\{u_h\}$ is strongly convergent. However, for this case (12) holds trivially by elementary functional analysis. An important question in this connection is whether the second scale may vanish under some conditions not including strong convergence. We demonstrate such a situation below. By the assumption that $\{\operatorname{curl}\mu_h\}$ is bounded in $[L^2(\Omega)]^{N\times N}$ we conclude from the two-scale compactness that $\operatorname{curl}_y v_0=0$. Classical vector calculus arguments then say that there exists a function Ψ such that

$$v_0(x,y)=\mu(x)+\nabla_y\Psi(x,y).$$

Moreover, $\operatorname{div}_y u_0=0$ and this together with integration by parts applied to the second term yields

$$\begin{aligned}&\int_\Omega\Big(\int_Y u_0(x,y)\cdot(\mu(x)+\nabla_y\Psi(x,y))\,\mathrm{d}y\Big)\varphi^2(x)\,\mathrm{d}x\\ &\qquad=\int_\Omega\Big(\int_Y u_0(x,y)\,\mathrm{d}y\Big)\cdot\mu(x)\varphi^2(x)\,\mathrm{d}x.\end{aligned}$$

Thus, by letting

$$u(x)=\int_Y u_0(x,y)\,\mathrm{d}y,$$

we get the identity

$$\int_\Omega \left(\int_Y u_0(x,y) \cdot v_0(x,y)\,\mathrm{d}y \right) \varphi^2(x)\,\mathrm{d}x = \int_\Omega u(x) \cdot \mu(x) \varphi^2(x)\,\mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$.

4. General two-scale convergence and defect measures

In this final section we exhibit an explicit example of a sequence $\{\tau_h\}$ of operators of the type introduced in Theorem 3 and investigate the relationship with the well-known cases. Outgoing from this we introduce an approach that can be seen as a type of unfolding (see [8]), where both scales are visible all the time. This special case indicates the relation between two-scale convergence and unfolding in a way easier to understand. We also make some observations concerning how to introduce a defect measure for general two-scale convergence. Below we will discuss the consequences for the compactness if the conditions on sequences $\{u_h\}$ and $\{\mu_h\}$ no longer are strong enough to guarantee, e.g., "weak-strong" or compensated compactness. We discuss properties of $\{u_h\}$ and $\{\mu_h\}$ beyond the point of breakdown for this type of convergence that still allow for some related kind of convergence and provide an example, where the switch between these two modes is visualized. For this purpose we construct a simple prototype for two-scale convergence and study the relationship between two-scale convergence in a general sense, some more traditional weak compactnesses and the recent concept of periodic unfolding, also known as the two-scale transform method, see [14].

Let us consider again the expression

$$\int_\Omega u_h(x)\mu_h(x)\,\mathrm{d}x.$$

Now let $\{\mu_h\}$ appear through a sequence of Hilbert-Schmidt operators $\{\tau_h\}$:

$$\mu_h(x) = \tau_h v(x) = \int_Y w_h(y) v(x,y)\,\mathrm{d}y,$$

where $\{w_h\}$ is weakly convergent in $L^2(\Omega)$ and v is regular enough. By the reflexivity of $L^2(\Omega)$ and the compactness of the Hilbert-Schmidt operator, it follows that $\{\mu_h\}$ converges strongly in' $L^2(\Omega)$ (see [2, 8.9]). Consequently,

$$\int_\Omega u_h(x)\mu_h(x)\,\mathrm{d}x \to \int_\Omega u(x)\mu(x)\,\mathrm{d}x.$$

For the corresponding operator with two scales given by

$$\mu_h(x) = \tau_h v(x) = \int_Y w_h(x,y)v(x,y)\,\mathrm{d}y,$$

where $\{w_h\}$ is bounded in $L^2(\Omega \times A)$, the strong convergence for $\{\mu_h\}$ does not hold in general. If we choose, e.g.,

$$w_h(x,y) = w_h^1(x)w^2(y)$$

with $\{w_h^1\}$ bounded in $L^2(\Omega)$ and $w^2 \in L^2(A)$, the convergence of $\{\mu_h\}$ in $L^2(\Omega)$ will not be stronger than that of $\{w_h^1\}$. However, by imposing conditions not much stronger than these, we will prove compactness for general two-scale convergence in the sense of (15) below.

For two-scale convergence in the traditional setting the operators τ_h are of the type $\tau_h v(x) = v(x, w_h(x))$ where $w_h(x) = x/\varepsilon_h$. Consequently, it is natural to require that the admissible test functions in the periodic two-scale convergence are of Carathéodory type. However, for other choices of operators τ_h it turns out that the Carathéodory continuity condition can be removed. We are now ready to show that, under assumptions including neither strong convergence in any Lebesgue space nor differentiability or continuity requirements on the functions involved, it holds up to a subsequence that

$$\lim_{h\to\infty} \int_\Omega u_h(x)\mu_h(x)\,\mathrm{d}x = \int_\Omega \int_A u_0(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x, \tag{15}$$

where $u_0 \in L^2(\Omega \times A)$ and

$$u_h \rightharpoonup u = \int_A u_0(\cdot, y)\,\mathrm{d}y \quad \text{weakly in } L^2(\Omega). \tag{16}$$

This is true if $\{u_h\}$ is bounded in $L^2(\Omega)$,

$$\mu_h(x) = \tau_h v(x) = \int_A w_h(x,y)v(x,y)\,\mathrm{d}y$$

for some $v \in L^4(\Omega \times A)$, and $\{w_h\}$ is a bounded sequence in $L^4(\Omega \times A)$ such that, up to a subsequence,

$$w_h \rightharpoonup w \quad \text{weakly in } L^4(\Omega \times A)$$

and

$$w_h^2 \rightharpoonup W \quad \text{weakly in } L^2(\Omega \times A),$$

where $W \in L^\infty(\Omega \times A)$. We also normalize the sequence $\{w_h\}$, i.e. we let

$$\int_A w_h(x,y)\,\mathrm{d}y = 1.$$

The reason for this last normalization condition is that the average over A for the two-scale limit u_0 must coincide with the corresponding weak limit u. If we can show that

$$\lim_{h\to\infty} \|\tau_h v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega\times A)} \tag{17}$$

and

$$\|\tau_h v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^4(\Omega\times A)}, \tag{18}$$

then (15) will follow by Theorem 3.

By using the weak convergence of w_h^2 in $L^2(\Omega \times A)$ and the Jensen inequality we obtain (17):

$$\begin{aligned}
\|\tau_h v\|^2_{L^2(\Omega)} &= \left\| \int_A w_h(x,y)v(x,y)\,\mathrm{d}y \right\|^2_{L^2(\Omega)} \\
&= \int_\Omega \int_A (w_h(x,y)v(x,y)\,\mathrm{d}y)^2\,\mathrm{d}x \\
&\leqslant C_0 \int_\Omega \int_A w_h^2(x,y)v^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \to C_0 \int_\Omega \int_A W(x,y)v^2(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant C_1\|v\|^2_{L^2(\Omega\times A)}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\|\tau_h v\|^2_{L^2(\Omega)} &\leqslant C_0 \int_\Omega \int_A w_h^2(x,y)v^2(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant C_0\|w_h\|^2_{L^4(\Omega\times A)}\|v\|^2_{L^4(\Omega\times A)} \leqslant C_2\|v\|^2_{L^4(\Omega\times A)}
\end{aligned}$$

and hence also (18) is proven. For any $v = v(x) \in L^2(\Omega)$ we obtain

$$\begin{aligned}
\lim_{h\to\infty} \int_\Omega u_h(x)\mu_h(x)\,\mathrm{d}x &= \lim_{h\to\infty} \int_\Omega u_h(x) \int_A w_h(x,y)v(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lim_{h\to\infty} \int_\Omega u_h(x)\mu(x)\,\mathrm{d}x \\
&= \int_\Omega \int_A u_0(x,y)v(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_\Omega \left(\int_A u_0(x,y)\,\mathrm{d}y \right) v(x)\,\mathrm{d}x
\end{aligned}$$

and thus (16) is proven. Moreover, if we assume that $w \equiv 1$, it is easy to show that

$$\mu_h \rightharpoonup \int_A v(\cdot, y)\,\mathrm{d}y \quad \text{weakly in } L^2(\Omega).$$

R e m a r k 13. Changing the order of integration it is now possible to reformulate the left-hand side of (15) into weak convergence in $L^{\frac{4}{3}}(\Omega \times A)$:

$$\langle u_h, \tau_h v \rangle_{L^2(\Omega)} = \int_\Omega u_h(x) \tau_h v(x)\,\mathrm{d}x = \int_\Omega \int_A u_h(x) w_h(x,y) v(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

where $v \in L^4(\Omega \times A)$ and $\{u_h w_h\}$ is bounded in $L^{\frac{4}{3}}(\Omega \times A)$. We can look upon this as if we had "unfolded" u_h by means of the adjoint operator

$$\tau_h^* u_h(x,y) = u_h(x) w_h(x,y).$$

It is now possible to perform the two-scale convergence process in the equivalent form:

$$\begin{aligned}\langle \tau_h^* u_h, v \rangle_{L^2(\Omega \times Y)} &= \int_\Omega \int_A \tau_h^* u_h(x,y) v(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega \int_A u_h(x) w_h(x,y) v(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\to \int_\Omega \int_A u_0(x,y) v(x,y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Note that the limit $u_0 \in L^2(\Omega \times A)$ even though it is the weak limit of a sequence that is just bounded in $L^{\frac{4}{3}}(\Omega \times A)$

The remark below provides us with a link between the type of two-scale convergence introduced in this section and the traditional periodic two-scale convergence.

R e m a r k 14. Let $v \in C^\infty(\Omega; C^\infty_\sharp(Y))$, where $Y = (0,1)^N$, i.e. the set of smooth functions with (unit) period in the second argument. We can construct the usual set of periodic oscillating test functions $v(x, x/\varepsilon_h)$ by setting

$$v_h(x) = \tau_h v(x) = \int_{\mathbb{R}^n} \delta_{x/\varepsilon_h}(y) v(x,y)\,\mathrm{d}y = v(x, x/\varepsilon_h),$$

where δ_x is the usual delta distribution. We can think of this like when the dual space of functions is expanded to contain distributions which are not represented by functions in any ordinary sense then the space of admissible functions shrinks, which usually means stronger regularity assumptions.

Let us close this section by proposing a way to characterize a defect measure for the two-scale convergence. A general complication with two-scale convergence is that the sequence to be analyzed and the corresponding two-scale limit live in completely different spaces. The operator τ_h^* helps us to overcome this problem. Among other it makes it possible to introduce a defect measure ς_D for two-scale convergence. Below we suggest a defect measure and show how it can be simplified when the two-scale limit u_0 belongs to the admissible space X. We compare $\tau_h^* u_h$ with the two-scale limit in the norm topology of $L^2(\Omega \times A)$ to define a defect measure as follows:

$$\begin{aligned}
\varsigma_D(\{u_h\}, u_0) &= \lim_{h\to\infty} \|\tau_h^* u_h - u_0\|^2_{L^2(\Omega\times A)} \\
&= \lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2 - 2\tau_h^* u_h(x,y) u_0(x,y) + u_0^2(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2 + u_0^2(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - \lim_{h\to\infty} \int_\Omega 2u_h(x)\tau_h u_0(x)\,\mathrm{d}x \\
&= \lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_A u_0^2(x,y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

if $u_0 \in X$ and all the involved limits exist. In this case we have

$$\lim_{h\to\infty} \int_\Omega \int_A (\tau_h^* u_h(x,y))^2\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_A u_0^2(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

as a measure of what is missing to obtain strong convergence.

We close this section by presenting an explicit example of unfolding operators in the periodic setting.

Example. This example is studied independently by Cioranescu et al. in [8] and by Nechvátal in [14]. In [14] Nechvátal compares usual the two-scale convergence and the unfolding and proves a compactness result for the sequence defined by unfolding (or the two-scale transform) which shows that the limit coincides for the two sequences. Let $Y = (0,1)^N$ and let $\{\varepsilon_h\}$ be a sequence of positive real numbers tending to zero as $h \to \infty$. For any $x \in \mathbb{R}^N$ we write

$$x = \varepsilon_h\Big(\Big[\frac{x}{\varepsilon_h}\Big]_Y + \Big\{\frac{x}{\varepsilon_h}\Big\}_Y\Big),$$

where $[x]_Y = k$ denotes the vector of the greatest integers k_i less than or equal to x_i and $\{x\}_Y = x -$ $[x]_Y$ denotes the remainder. We have used the fact that for any $x \in \mathbb{R}^N$ and $h \in \mathbb{N}$ there exists a unique number $k \in \mathbb{Z}^N$ such that

$$\frac{x}{\varepsilon_h} = k + y$$

for some $y \in Y$. If we define $\tau_h v(x) = v(x, x/\varepsilon_h)$ (cf. Remark 14), then for any $u \in L^2(\Omega)$ we now define the corresponding unfolding operators in the periodic setting

$$\tau_h^*\colon\ L^2(\Omega) \to L^2(\Omega \times Y)$$

as

$$\tau_h^* u(x,y) = u\Big(\varepsilon_h \Big[\frac{x}{\varepsilon_h}\Big]_Y + \varepsilon_h y\Big).$$

Periodic two-scale convergence can now be viewed as weak convergence of the sequence $\{\tau_h^* u_h\}$ as in Remark 13.

Acknowledgement. The authors wish to thank Prof. François Murat for inspiring discussions particularly concerning the relationship between classical and two-scale compensated compactness.

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Authors' addresses: *A. Holmbom*, Department of Mathematics, Mid-Sweden University, SE-831 25 Östersund, Sweden, e-mail: `anders.holmbom@mh.se`; *J. Silfver*, Department of Mathematics, Mid-Sweden University, SE-831 25 Östersund, Sweden, e-mail: `jeanette.silfver@mh.se`; *N. Svanstedt*, Department of Computational Mathematics, Chalmers University, SE-412 96 Göteborg, Sweden, e-mail: `nilss@math.chalmers.se`; *N. Wellander*, Swedish Defence Research Agency, FOI, P.O. Box 1165, SE-581 11 Linköping, Sweden, e-mail: `niklas@foi.se`.