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Research Article

On Two Systems of Difference Equations

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We give very short and elegant proofs of the main results in the work of Yalcinkaya et al. (2008).

1. Introduction and a Proof of Some Resent Results

Motivated by our paper [1], the authors of [2] studied the following two systems of difference equations:

$$x_{n+1}^{(i)} = \frac{x_n^{(i+1(\text{mod } k))}}{x_n^{(i+1(\text{mod } k))} - 1}, \quad i = 1, \ldots, k, \ n \in \mathbb{N}_0, \tag{1.1}$$

$$x_{n+1}^{(i)} = \frac{x_n^{(i-1(\text{mod } k))}}{x_n^{(i-1(\text{mod } k))} - 1}, \quad i = 1, \ldots, k, \ n \in \mathbb{N}_0, \tag{1.2}$$

where we regard that $0\,(\text{mod } k) = k\,(\text{mod } k) = k$.

Following line by line the proofs of the main results in [1] they proved the following result (see Theorems 2.1 and 2.4 in [2])

Theorem A. *Assume $k \in \mathbb{N}$, then the following statements are true.*

(a) *If $k = 0\,(\text{mod } 2)$, then every (well-defined) solution of systems* (1.1) *and* (1.2) *is periodic with period k.*

(b) *If $k = 1\,(\text{mod } 2)$, then every (well-defined) solution of systems* (1.1) *and* (1.2) *is periodic with period $2k$.*

Here we give a very short and elegant proof of Theorem A.

Proof of Theorem A. By using the change $y_n^{(i)} = x_n^{(i)} - 1$, $i = 1, \ldots, k$, system (1.1) becomes

$$y_n^{(i)} = \left(y_{n-1}^{(i+1(\bmod k))}\right)^{-1}, \quad i = 1, \ldots, k, \ n \in \mathbb{N}, \tag{1.3}$$

while system (1.2) becomes

$$y_n^{(i)} = \left(y_{n-1}^{(i-1(\bmod k))}\right)^{-1}, \quad i = 1, \ldots, k, \ n \in \mathbb{N}. \tag{1.4}$$

From (1.3) and (1.4), for each $i \in \{1, \ldots, k\}$, and $n \geq k$, we obtain correspondingly that

$$\begin{aligned} y_n^{(i)} &= \left(y_{n-k}^{(i+1+k-1(\bmod k)}\right)^{(-1)^k} = \left(y_{n-k}^{(i)}\right)^{(-1)^k}, \\ y_n^{(i)} &= \left(y_{n-k}^{(i-1-(k-1)(\bmod k)}\right)^{(-1)^k} = \left(y_{n-k}^{(i)}\right)^{(-1)^k}. \end{aligned} \tag{1.5}$$

From (1.5), with $k = 0 \ (\bmod 2)$, it follows that

$$y_n^{(i)} = y_{n-k}^{(i)}, \quad i = 1, \ldots, k, \tag{1.6}$$

from which the statement in (a) easily follows.

If $k = 1(\bmod 2)$, we have that

$$y_n^{(i)} = \left(y_{n-k}^{(i)}\right)^{-1}, \quad i = 1, \ldots, k, \tag{1.7}$$

from which it follows that

$$y_n^{(i)} = y_{n-2k}^{(i)}, \quad i = 1, \ldots, k, \tag{1.8}$$

$n \geq 2k$, implying the statement in (b), as desired. □

2. An Extension on Theorem A

Here we extend Theorem A in a natural way. Let $\gcd(k, l)$ denote the greatest common divisor of the integers k and l, $\operatorname{lcm}(k, l)$ the least common multiple of k and l, and for $r \in \mathbb{N}$ let $f^{[r]}(x) = f(f^{[r-1]}(x))$, where $f^{[1]}(x) = f(x)$.

Theorem 2.1. *Assume that f is a real function such that $f^{[r]}(x) \equiv x$ on its domain of definition, for some $r \in \mathbb{N}$, then all well-defined solutions of the system of difference equations*

$$x_n^{(1)} = f\left(x_{n-1}^{(2)}\right), x_n^{(2)} = f\left(x_{n-1}^{(3)}\right), \ldots, x_n^{(k)} = f\left(x_{n-1}^{(1)}\right), \quad n \in \mathbb{N}_0, \tag{2.1}$$

are periodic with period $T = \operatorname{lcm}(k, r)$.

Proof. We use our method of "prolongation" described in [1]. Note that for each $s \in \mathbb{N}$, system (2.1) is equivalent to a system of ks difference equations of the same form, where

$$x_n^{(i)} = x_n^{(jk+i)}, \tag{2.2}$$

for every $n \in \mathbb{N}_0$, $i \in \{1, \dots, k\}$ and $j = 1, \dots, s$.

From (2.1) and since $f^{[r]}(x) \equiv x$, for $n \geq r-1$ we have

$$x_n^{(i+1)} = f\left(x_{n-1}^{(i+2)}\right) = f^{[2]}\left(x_{n-2}^{(i+3)}\right) = \cdots = f^{[r]}\left(x_{n-r}^{(i+r+1)}\right) = x_{n-r}^{(i+r+1)}. \tag{2.3}$$

for each $i \in \{1, 2, \dots, k\}$, and every $n \geq r-1$.

It is clear that

$$T = k \cdot r_1 = k_1 \cdot r, \tag{2.4}$$

where $r_1, k_1 \in \mathbb{N}$ are such that $\gcd(k, r_1) = 1$ and $\gcd(k_1, r) = 1$.

From (2.3) we have

$$x_n^{(i+1)} = x_{n-r}^{(i+r+1)} = \cdots = x_{n-k_1 r}^{(i+k_1 r+1)} = x_{n-kr_1}^{(i+kr_1+1)} = x_{n-T}^{(i+1)}, \tag{2.5}$$

for each $i = 0, 1, \dots, k-1$, and $n \geq T-1$, from which the result follows. □

The following result is proved similarly. Hence we omit its proof.

Theorem 2.2. *Assume that f is a real function such that $f^{[r]}(x) \equiv x$ on its domain of definition, for some $r \in \mathbb{N}$, then all well-defined solutions of the system of difference equations*

$$x_n^{(2)} = f\left(x_{n-1}^{(1)}\right), \dots, x_n^{(k)} = f\left(x_{n-1}^{(k-1)}\right), \qquad x_n^{(1)} = f\left(x_{n-1}^{(k)}\right), \quad n \in \mathbb{N}_0, \tag{2.6}$$

are periodic with period $T = \operatorname{lcm}(k, r)$.

Remark 2.3. The proof of Theorem A follows from Theorems 2.1 and 2.2. Indeed, note that the function $f(x) = x/(x-1)$ satisfies the condition $f^{[2]}(x) \equiv x$ on its domain of definition. By Theorems 2.1 and 2.2 we know that all well-defined solutions of systems (1.1) and (1.2) are periodic with period $T = \operatorname{lcm}(k, 2)$, from which the result follows.

Remark 2.4. We also have to say that the main result in [3] is a trivial consequence of a result in [1] (see Remark 5 therein). Just note that the simple change of variables $x_n^{(i)} = a y_n^{(i)}$, $i \in \{1, \dots, k\}$, transforms their system (1.3) satisfying conditions $a_1 = a_2 = \cdots = a_k = a$ and $b_1 = b_2 = \cdots = b_k = b = a^2$, into system (4) in [1].

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