TSUKUBA J. MATH.
Vol. 15 No. 1 (1991), 99–104

ON TYPE NUMBER OF REAL HYPERSURFACES IN $P_n(C)$

By

Young Jin SUH*

Introduction.

Let $P_n(\boldsymbol{C})$ denote an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$. Real hypersurfaces in $P_n(\boldsymbol{C})$ have been studied by many differential geometers (See [2], [3], [4], [5] and [6]).

In particular, as for a problem with respect to the typec number t, i.e., the rank of the second fundamental form of real hypersurfaces M in $P_n(\boldsymbol{C})$, R. Takagi showed in [6] that there is a point p on M such that $t(p)\geqq 2$, and M. Kimura and S. Maeda [4] gave an example of real hypersurfaces in $P_n(\boldsymbol{C})$ satisfying $t=2$, which is non-complete. In this paper we shall prove

THEOREM 1. *Let M be a complete real hypersurface in $P_n(\boldsymbol{C})$ $(n\geqq 3)$. Then there exists a point p on M such that $t(p)\geqq 3$.*

REMAPK. It is known that a certain geodesic hypersphere in $P_2(\boldsymbol{C})$ has a property $t=2$ (cf. [2], [7]). Thus the assumption $n\geqq 3$ in Theorem 1 can not be removed.

The author would like to express his sincere gratitude to Prof. R. Takagi for his valuable suggestions and encouragement during the preparation of this paper.

1. Preliminaries.

Let M be a real hypersurface in $P_n(\boldsymbol{C})$ $(n\geqq 2)$. Let $\{e_1, \cdots, e_{2n}\}$ be a local field of orthonormal frame in $P_n(\boldsymbol{C})$ such that, restricted to M, $e_1, \cdots, e_{2n-1}$ are tangent to M. Denote its dual frame field by $\theta_1, \cdots, \theta_{2n}$. We use the following convention on the range of indices unless otherwise stated; $A, B, \cdots, =1, \cdots, 2n$ and $i, j, \cdots, =1, \cdots, 2n-1$.

The connection forms θ_{AB} are defined as the 1-forms satisfying

*) Partially supported by TGRC-KOSEF.
Received May 23, 1990.

(1.1) $$d\theta_A=-\sum\theta_{AB}\wedge\theta_B, \qquad \theta_{AB}+\theta_{BA}=0.$$

Restrict the forms under consideration to M. Then, we set $\theta_{2n}=0$ and the forms $\theta_{2n,i}$ can be written as

(1.2) $$\phi_i\equiv\theta_{2n,i}=\sum h_{ij}\theta_j, \qquad h_{ij}=h_{ji}.$$

The quadratic form $\sum h_{ij}\theta_i\otimes\theta_j$ is called the *second fundamental form* of M with direction of e_{2n}. The curvature forms Θ_{ij} of M are defined by

(1.3) $$\Theta_{ij}=d\theta_{ij}+\sum\theta_{ik}\wedge\theta_{kj}.$$

We denote by J the complex structure of $P_n(\boldsymbol{C})$, and put

$$Je_i=\sum J_{ji}e_j+f_ie_{2n}.$$

Then the almost contact structure (J_{ij}, f_k) satisfies

(1.4) $$\begin{cases}\sum J_{ik}J_{kj}=f_if_j-\delta_{ij}, & \sum f_jJ_{ji}=0\\ \sum f_i^2=1, & J_{ij}+J_{ji}=0.\end{cases}$$

(1.5) $$\begin{cases}dJ_{ij}=\sum(J_{ik}\theta_{kj}-J_{jk}\theta_{ki})-f_i\phi_j+f_j\phi_i,\\ df_i=\sum(f_j\theta_{ji}-J_{ji}\phi_j).\end{cases}$$

The equations of Gauss and Codazzi are given by

(1.6) $$\Theta_{ij}=\phi_i\wedge\phi_j+c\theta_i\wedge\theta_j+c\sum(J_{ik}J_{jl}+J_{ij}J_{kl})\theta_k\wedge\theta_l,$$

(1.7) $$d\phi_i=-\sum\phi_j\wedge\theta_{ji}+c\sum(f_iJ_{jk}+f_jJ_{ik})\theta_j\wedge\theta_k,$$

respectively.

2. Lemmas.

Let M be a real hypersurface in $P_n(\boldsymbol{C})$. We choose an arbitary point p in M, and use the following convention on the range of indices; $a, b, \cdots, =1, \cdots, t(p)$ and $r, s, \cdots, =t(p)+1, \cdots, 2n-1$. Then we can take a field $\{e_1, \cdots, e_{2n}\}$ of orthonormal frame on a neighborhood of p in such a way that the 1-forms ϕ_i can be written as

(2.1) $$\begin{cases}\phi_a=\sum h_{ba}\theta_b, & h_{ab}=h_{ba},\\ \phi_r=0,\end{cases}$$

at p. We call such a field $\{e_1, \cdots, e_{2n}\}$ to be associated with a point p.

Under this notation we have

LEMMA 2.1. *Assume that $J_{rs}(p)=0$ at a point p on M. Then $t(p)\geqq n-1$. Furthermore, the equality holds if and only if $f_a=0$ and $J_{ab}=0$ at p.*

PROOF. By (1.4) we have

$$\sum_b J_{ab}^2+\sum_r J_{ar}^2+f_a^2=1, \tag{2.2}$$

$$\sum_a J_{ra}^2+f_r^2=1. \tag{2.3}$$

Summing up (2.2) on a, and (2.3) on r, we have

$$\sum_{a,b} J_{ab}^2+\sum_{a,r} J_{ar}^2+\sum_a f_a^2=t(p), \tag{2.4}$$

$$\sum_{a,r} J_{ar}^2+\sum_r f_r^2=2n-1-t(p). \tag{2.5}$$

Substituting (2.5) into (2.4) and making use of $\sum_a f_a^2+\sum_r f_r^2=1$, we have

$$\sum_{a,b} J_{ab}^2+2\sum_a f_a^2=2(t(p)-(n-1))\geqq 0,$$

and so our assertion follows.

This concludes the proof.

Now we consider a point p where the type number t attains the maximal value, say T. Then there is a neighborhood U of p, on which the function t is constant and the equation (2.1) holds.

Put $\theta_{ar}=\sum A_{arb}\theta_b+\sum B_{ars}\theta_s$. Then, taking the exterior derivative of $\phi_r=0$ and using (1.7), we have

$$\sum h_{ab}\theta_b\wedge\theta_{ar}-c\sum(f_r J_{ij}+f_i J_{rj})\theta_i\wedge\theta_j=0,$$

from which we have

$$\sum(h_{ac}A_{crb}-h_{bc}A_{cra})-cf_a J_{rb}+cf_b J_{ra}-2cf_r J_{ab}=0, \tag{2.6}$$

$$\sum h_{ab}B_{brs}-cf_a J_{rs}+cf_s J_{ra}-2cf_r J_{as}=0, \tag{2.7}$$

$$f_s J_{rt}-f_t J_{rs}+2f_r J_{st}=0. \tag{2.8}$$

It is easy to see that (2.8) is reduced to

$$f_r J_{st}=0. \tag{2.9}$$

Similarly, taking the exterior derivative of $\phi_a=\sum h_{ba}\theta_b$ and using the equation (1.7) of Codazzi, we have

$$\begin{aligned}\sum_b\{dh_{ab}-\sum_c(h_{ac}\theta_{cb}+h_{cb}\theta_{ca}-\sum_r h_{ac}A_{crb}\theta_r-cf_b J_{ac}\theta_c\\ +cf_c J_{ab}\theta_c-2cf_a J_{bc}\theta_c)+c\sum_r(f_b J_{ar}\theta_r-f_r J_{ab}\theta_r+2f_a J_{br}\theta_r)\}\wedge\theta_b=0.\end{aligned}$$

Therefore, we can put

$$\begin{aligned}dh_{ab}-\sum_c(h_{ac}\theta_{cb}+h_{cb}\theta_{ca}-\sum_r h_{ac}A_{crb}\theta_r-cf_b J_{ac}\theta_c+cf_c J_{ab}\theta_c\\ -2cf_a J_{bc}\theta_c)+c\sum_r(f_b J_{ar}\theta_r-f_r J_{ab}\theta_r+2f_a J_{br}\theta_r)=\sum C_{abc}\theta_c,\end{aligned} \tag{2.10}$$

where $C_{abc}=C_{acb}=C_{bac}$.

Under such a situation we have

LEMMA 2.2. *If $J_{rs}=0$ on U, then $T\geqq n$ on U.*

PROOF. If $T<n$, then by Lemma 2.1 we have $T=n-1$, and $f_a=0$ on U. For a suitable choice of a field $\{e_r\}$ of orthonormal frames, if necessary, we may set $f_{2n-1}=1$ and $f_r=0$ for $r=n, \cdots, 2n-2$. Then from (1.5) we have

$$0=df_r=-\sum J_{ar}\phi_a.$$

But, since rank $J=2n-2$, we have $\det(J_{ar})\neq 0\,(a=1, \cdots, n-1, r=n, \cdots, 2n-2)$. Thus the above equation implies $\phi_a=0$, which contradicts the fact that $\det(h_{ab})\neq 0$.

This concludes the proof.

In the remainder of this section we restrict the forms under consideration to the following open set V_T defined by

$$V_T=\{p\in M \mid J_{rs}(p)\neq 0,\ t(p)=T\},$$

where $J_{rs}(p)\neq 0$ means "$J_{rs}(p)\neq 0$ for some $r, s=T+1, \cdots, 2n-1$". First from (2.9) we have $f_r=0$. Thus we may set $f_1=1$, and $f_a=0$ for $a\geqq 2$. Hence we have

$$J_{1a}=0, \qquad J_{1r}=0. \tag{2.11}$$

Furthermore, $df_a=0$ and $df_r=0$ give

$$\theta_{1a}=\sum J_{ba}\phi_b, \tag{2.12}$$

$$A_{1ra}=\sum h_{ab}J_{br}, \tag{2.13}$$

$$B_{1rs}=0. \tag{2.14}$$

The equation (2.7) amounts to

$$\sum h_{ab}B_{brs}=cf_aJ_{rs}. \tag{2.15}$$

LEMMA 2.3. $\det(h_{ab})=0\,(a, b=2, \cdots, T)$ *on* V_T.

PROOF. Here indices a, b run from 2 to T. If $\det(h_{ab})\neq 0$, then by (2.15) we have $B_{ars}=0$, which together with (2.14) gives $J_{rs}=0$. A contradiction to the fact $J_{rs}(p)\neq 0$ on V_T.

This concludes the proof.

3. Proof of Theorem 1.

We keep the notation in section 2. If $J_{rs}=0$ on a nonempty open set, then

Lemma 2.2 proves Theorem 1. Therefore, we have only to consider the case where the open set V_T defined in section 2 is not empty.

Assume $T=2$. Then we shall derive a contradiction. First by Lemma 2.3 we have $h_{22}=0$.

Now we put $F=h_{12}$. Then from (2.13) we have

$$A_{1r1}=FJ_{2r}, \qquad A_{1r2}=0. \tag{3.1}$$

Put $a=1$ and $b=2$ in (2.6) to get

$$FA_{2r2}-FA_{1r1}-cJ_{r2}=0. \tag{3.2}$$

Then (3.1) and (3.2) give

$$FA_{2r2}=(F^2-c)J_{2r}. \tag{3.3}$$

According to (2.12), we have

$$\theta_{12}=0.$$

Put $a=1$ and $b=2$ in (2.10). Then, together with (3.3) we find

$$dF+(F^2+c)\sum J_{2r}\theta_r=\sum D_{12a}\theta_a, \tag{3.4}$$

where we have put $D_{12a}=C_{12a}-2cJ_{2a}$.

Let p be any point of V_2 and let $\alpha: I\to V_2$ be the maximal integral curve of the unit dual vector field $\sum J_{2r}\theta_r$ on V_2 such that $\alpha(0)=p$, $\sum_r J_{2r}\theta_r(\alpha'(t))=1$ and $D_{12a}\theta_a(\alpha'(t))=0$, where $(0\in)I$ denotes an open interval of R.

Assume that there exists $\sup I$, say t_0. Since M is complete, we have a point $p_0=\lim_{t\to t_0}\alpha(t)$ on M. We assert $F(p_0)=\lim_{t\to t_0}F(\alpha(t))=0$. In order to prove this assertion, it suffices to show that $t(p_0)\leqq 1$, because $\det(h_{ij})=-F^2$ by $h_{22}=0$ at p_0. For this we assume $t(p_0)=2$. Thus we can consider that our frame field $\{e_i\}$ is defined also on the neighborhood of p_0. Since $p_0\in\overline{V}_2$, $J_{rs}(p_0)=0$ for any $r, s\geqq 3$. Then by Lemma 2.1 we have $t(p_0)\geqq n-1\geqq 3$ for $n\geqq 4$, which is a contradiction. For a case where $n=3$ also by using Lemma 2.1 we get $J_{ab}=0$ and $f_a=0$ at p_0 for all $a, b=1, 2$. This also contradicts to the fact that $f_1=1$ at p_0, which proves our assertion.

Now we shall show that $\inf I=-\infty$. Indeed, if there exists $t_1=\inf I$, then we find $\lim_{t\to t_1}F(\alpha(t))=0$ by an argument similar to the above. Thus there is a real number t' such that $t_1<t'<t_0$, $dF=0$ at $\alpha(t')$. Then (3.4) gives $J_{2r}=0$. From this together with $J_{1r}=0$ in (2.11) it follows that $\operatorname{rank} J\leqq 2n-3$, which makes a contradiction to the fact that $\operatorname{rank} J=2n-2$.

Now the function $|F|$ defined on the interval $(-\infty, t_0)$ satisfies by (3.4)

$$\frac{d|F|}{dt}=F^2+c \quad \text{or} \quad \frac{1}{F^2+c}\frac{d|F|}{dt}=1.$$

Solving the above differential equation, we have

$$F(\alpha(t))=\sqrt{c}\tan\sqrt{c}(t-t_0),$$

which is a contradiction, because $F(\alpha(t))$ is defined on $(-\infty, t_0)$ but the right hand side can not be defined at the points such that $\sqrt{c}(t-t_0)=(2k+1)\pi/2$, where k is an integer.

For a case where $\sup I=\infty$, we can take a point $\alpha(t_0)\in V_2$ such that $F(\alpha(t_0))=F_0\neq 0$ for $t_0<\infty$. This case also contains the situation such that $F(\alpha(t))$ is defined on $(-\infty, \infty)$ and $\alpha(t)$ is contained in V_2. Using the similar method to the above, we also get

$$F(\alpha(t))=\sqrt{c}\tan\sqrt{c}(t-t_0+s_0) \quad \text{on} \quad (-\infty, t_0]$$

where we have put $\sqrt{c}\,s_0=\tan^{-1}(F_0/\sqrt{c})$. This also makes a contradiction as in the above case.

It completes the proof of Theorem 1.

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University of Tsukuba
Institute of Mathematics
Ibaraki, Tsukuba-shi, 305
Japan

Andong University
Department of Mathematics
Andong, 760-749, Korea