On Uncertain Lifetimes

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This paper contrasts consumer choice under uncertain lifetimes with the behavior that would arise if each individual's lifetime were announced at birth. In a model that includes life insurance and excludes investments in human capital, the expected utility under uncertain lifetimes exceeds that under known lifetimes when the latter expectation is based on preannouncement survival probabilities. This conclusion emerges, first, because the model without human capital contains no planning benefits from knowledge of the horizon and, second, because the prior announcement of lifetimes forces risk-averse consumers to undertake an extra gamble that they could otherwise avoid by using life insurance.

This paper explores the role of uncertain lifetimes in an intertemporal model of consumer choice. The analysis compares the behavior of consumption and utility under a stochastic horizon with that which would arise under certainty where each individual's length of life is announced at birth. In a model that includes life insurance and excludes investments in human capital, the expected utility under uncertain lifetimes exceeds the expected utility under known lifetimes when the latter expectation is based on preannouncement (ex ante) survival probabilities. This conclusion emerges, first, because the model without human capital accumulation contains no planning benefits from knowledge of the horizon; and, second, because the prior announcement of the time of death forces risk-averse consumers to undertake an extra gamble that could have been avoided by the use of life insurance in the uncertainty case.

The possibility of accumulating human capital implies planning benefits from knowledge of the time of death. The ability to match investments with the horizon implies that expected lifetime wealth is lower in the case of uncertain lifetime than in the case where lifetimes are

announced at birth. The effects on expected lifetime utility are ambiguous because the planning benefit from knowledge of the horizon offsets the cost associated with risk aversion.

One other result is that, contrary to some earlier suggestions, an uncertain lifetime is not a rationale for time preference when "perfect" life insurance is admitted. Although survival probabilities below unity imply a discount on future utility, they imply a corresponding discount in the budget constraint on future income and expenditure. On net, there is no effect of uncertain lifetime on the preference for current consumption relative to future consumption.[1]

In Section I a model is examined in which the individual's lifetime is uncertain and there is no human capital accumulation. Section II deals with a model in which the lifetime is known, though randomly determined at the start. In Section III the models of the earlier sections are modified to allow for the accumulation of human capital.

I. Uncertain Lifetime with No Human Capital

Consider a consumer who lives no more than T periods, with the probability of dying at the end of the ith period being p_i. It is assumed that, at the end of the first period, a random draw occurs. With probability p_1 the consumer dies. In general, if the consumer lives during period i, a random drawing with a death probability of $p_i/\sum_{j=i}^{T} p_j$ takes place. It is assumed that $p_i \geq 0$ for $i = 1, \ldots, T - 1$, $p_T > 0$ and $\sum_{i=1}^{T} p_i = 1$. The consumer is viewed as maximizing expected lifetime utility, which is given by

$$\bar{U} = u(c_1) + (1 - p_1)u(c_2) + \cdots + (1 - p_1 - \cdots - p_{T-1})u(c_T),$$
(1)

where c_i is consumption and $u(c_i)$ is the satisfaction enjoyed during the ith period of the lifetime. Let $q_i = 1 - \sum_{j=1}^{i-1} p_j (i > 1)$ be the probability as of birth that the person survives into the ith period. Then, with $q_1 = 1$, equation (1) can be written as

$$\bar{U} = \sum_{i=1}^{T} q_i u(c_i).$$
(2)

The function, $u(c)$, is assumed to be increasing, concave, and differentiable. Thus, $u' > 0$ and $u'' < 0$. Note that $u'' < 0$ is often interpreted as "risk aversion," and also that the lifetime utility function contains no discounting of future enjoyments beyond that which is included in the survival probabilities.[2]

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[1] This result can also be seen in case C of Yaari (1965), which is a continuous time version of the discrete time model in Section I.

[2] Some comments are made on the effects of discounting in n. 5 of Section II.

Assume that the market interest rate is zero (see n. 5 below) and that the individual can purchase annuities and life insurance at actuarially fair prices, as well as being able to borrow and lend. These market opportunities make possible such diverse contracts as "negative annuities," an example of which is a contract where the consumer receives \$1.00 at the start of period 1 and repays $1/(1 - p_1)$ contingent on survival at the start of period 2. To duplicate this contract with a loan secured by life insurance, the consumer borrows $1/(1 - p_1)$ dollars at the beginning of period 1 and pays a premium of $p_1/(1 - p_1)$ dollars for a 1-period life insurance contract that pays off $1/(1 - p_1)$ dollars at the beginning of period 2 should he die by that time. For an additional discussion of contracts of this sort, see Yaari (1965). The present framework is a discrete version of his (continuous time) case C.

Taking the purchase of life insurance into account, the budget constraint is a requirement of equality between expected lifetime income and expected consumption expenditures. Let y be the income received during each period of life, A the initial assets of the consumer (which may be negative), and $\bar{W}$ the expected value of lifetime wealth (which must be positive). Then the budget constraint is

$$\sum_{i=1}^{T} q_i c_i = y \sum_{i=1}^{T} q_i + A = \bar{W}. \tag{3}$$

The term $\sum_{i=1}^{T} q_i = \bar{t}$ is the expected length of life; hence, expected lifetime wealth and expected length of life are related by

$$\bar{W} = y\bar{t} + A. \tag{4}$$

It is immediate that the maximization of $\bar{U}(c_1, \ldots, c_T) = \sum_{i=1}^{T} q_i u(c_i)$, subject to the budget constraint (eq. [3]), implies the first-order conditions,

$$u'(c_1) = u'(c_2) = \cdots = u'(c_T). \tag{5}$$

That these first-order conditions correspond to a (constrained) utility maximum follows from the concavity of $\bar{U}$ and the linearity of the budget constraint.

Thus, an optimal plan requires a constant level of consumption over time, $\tilde{c} = \bar{W}/\bar{t} = y + A/\bar{t}$, and the expected value of attained lifetime utility is

$$\bar{U} = \bar{t}u(\tilde{c}) = \bar{t}u(y + A/\bar{t}). \tag{6}$$

It has been argued that the uncertainty of survival is a rationale for the inclusion of a discount factor on future utility.[3] This observation suggests, in turn, that a person living in a world such as that modeled here would exhibit a positive rate of time preference—that is, with a zero market

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[3] See Böhm-Bawerk [1891] 1971, p. 255; Pigou 1962, p. 26; and Friedman 1969, p. 22.

rate of interest, the optimal consumption plan would call for a rate of consumption that declines over time. This argument turns out to be invalid because the same "probability discount factors" that appear in the objective function, equation (2), appear also in the budget constraint, equation (3).

II. Known Lifetime with No Human Capital

Consider modifying the model of Section I to allow for a length of life that is randomly selected at the start and announced at that time. As before, the probability of dying at the end of the ith period is p_i. However, at the moment of birth a random draw is assumed to occur, and the consumer's length of life, t, is determined and announced to him. It is also announced to any economic agent who might make an insurance contract with him. After the announcement of t, the consumer's objective function is

$$U = \sum_{i=1}^{t} u(c_i), \tag{7}$$

and his budget constraint is

$$\sum_{i=1}^{t} c_i = yt + A = W(t). \tag{8}$$

Note that the ex ante survival probabilities are absent from both the objective function and the budget constraint. Of course, the ex post survival probabilities are present. They are 1 for each of the first t periods and 0 thereafter.

Again, the optimal consumption plan calls for a constant level of consumption over time. For an optimum,

$$u'(c_1) = u'(c_2) = \cdots = u'(c_t) = u'(c^*), \tag{9}$$

where

$$c^* = y + A/t. \tag{10}$$

The attained level of lifetime utility is

$$U(t) = tu(c^*) = tu(y + A/t). \tag{11}$$

The interesting comparison to make with the model of Section I concerns a consumer who is at the beginning of his life and has not yet observed the outcome of the random draw that determines t, his length of life. His expected lifetime utility under the conditions of the present section is

$$E[U(t)] = E[tu(y + A/t)] = \sum_{t=1}^{T} p_t tu(y + A/t). \tag{12}$$

Meanwhile, the maximum value of expected lifetime utility that is attained in the model of Section I is

$$\bar{U} = \bar{t}u(y + A/\bar{t}) = U[E(t)]. \tag{13}$$

Accordingly, the question of whether the consumer is better off under the risk of Section I or that of the present section comes down to a comparison between $E[U(t)]$ and $U[E(t)]$. If $U(t)$ is concave, then Jensen's inequality states that $E[U(t)] \leq U[E(t)]$.[4] It is easily verified that the concavity of $u(c)$ implies the concavity of $U(t)$; hence Jensen's inequality does hold. Given the conditions that $A \neq 0$, that u is strictly concave, and that at least two of the p_t are greater than zero, Jensen's inequality holds strictly. Under these circumstances, the regime of Section I is preferable in the sense that it yields the consumer a higher value of expected lifetime utility, as evaluated at the beginning of life. The key element that produces this result is that the consumer's ability to buy insurance gives him the same expected lifetime utility as he would have if his lifetime were known with certainty to be of length $\bar{t}$, the mean length of life.[5]

In the model of the present section, the consumer behaves optimally, making his decision after his length of life is determined. He does not escape having to take part in a gamble—namely, the gamble associated with the determination of t.[6] Under the other model, his ability to insure allows him to consume $y + A/\bar{t}$ in each period he lives, which has the effect of removing him from part of the gamble into which nature placed him. He does not escape the presence of uncertainty of length of life, but he does escape the utility cost associated with having to bear the consequent risk of an uncertain value of W/t—the per period amount of wealth.[7]

The two regimes yield the same utility if $A = 0$; however, the expected utility derived from the regime of Section I would again be higher if income, y, were not constant. The simplest case of nonconstant income occurs when the consumer is assumed to "work" for a number of years, having the same income per year, then to "retire" at some given age and have no employment income thereafter. Such assumptions, without con-

[4] On Jensen's inequality, see Mood, Graybill, and Boes (1974), p. 72.

[5] Since the time pattern of consumption is uniform under both known and unknown horizons, there are no "planning benefits" that derive from knowledge of the time of death. Introduction of an interest rate or of a subjective rate of discount on future utility would eliminate the constancy of consumption over time but would not eliminate the correspondence of the time pattern of consumption for the two cases. An example of planning benefits from known lifetimes that are associated with human capital accumulation is described in Section III.

[6] There is assumed to be no possibility of risk pooling before the announcement on lifetime occurs. Insurance possibilities are irrelevant after the announcement when the length of life is known.

[7] Hirshleifer (1971, p. 568) discusses some analogous situations in which information may not be socially useful. The length of life can be considered part of the individual's endowment in Hirshleifer's model.

sideration of a randomized length of life, occur in Modigliani and Brumberg (1954).

In sum, the conclusion can be expressed as follows. Consider two regimes: regime 1 in which all individuals are born with a common probability density that determines the length of life; and regime 2 in which individuals are labeled at birth with a value of t that is drawn from the same distribution. The model implies that regime 1 is preferable—in the sense that, if both regimes were technically feasible, all (risk-averse) members of society would opt for regime 1.

The analysis is also suggestive for the optimal amount of investment to undertake in activities that yield information about individual life expectancy. The type of argument may provide a rationale for maintaining some ignorance about the state of health, especially where terminal illness may be involved.

III. Human Capital

The planning benefits from known lifetimes can be illustrated by introducing a variable amount of human capital. In a simple case all of the (nondepreciable) amount of human capital, K, is acquired during the first period of life. Suppose that income in each period is still constant over time but is now a function of the amount of human capital, as expressed by the function, $f(K)$, where $f' > 0$ and $f'' < 0$.

In the case of an uncertain horizon (the model of Section I), the only change is a modification of expected lifetime wealth (eq. [4]) to

$$\bar{W} = f(K)\bar{t} + A - K. \tag{14}$$

Choosing K to maximize expected lifetime utility (when an interior maximum exists) results in the first order condition,

$$f'(K) = 1/\bar{t}. \tag{15}$$

Letting $\hat{K}$ denote the optimal quantity of human capital, the optimal rate of consumption is

$$\hat{c} = f(\hat{K}) + (A - \hat{K})/\bar{t} = \hat{W}/\bar{t}, \tag{16}$$

and the expected value of lifetime utility is

$$\hat{U} = \bar{t}u(\hat{W}/\bar{t}) = \bar{t}u[f(\hat{K}) + (A - \hat{K})/\bar{t}]. \tag{17}$$

Turning now to the model in which the lifetime is determined before any actions are taken, lifetime wealth is

$$W(t) = f(K)t + A - K, \tag{18}$$

and the optimal value of human capital, $K(t)$, is the solution to

$$f'(K) = 1/t. \tag{19}$$

The assumptions on f' assure that equation (19) has at most one solution with $K > 0$. With a knowledge of the lifetime, individuals can match K with t, rather than basing the investment decision on $\bar{t}$. Optimal consumption and the maximized value of attained lifetime utility are

$$c(t) = W(t)/t = f[K(t)] + [A - K(t)]/t, \tag{20}$$

$$U(t) = tu[W(t)/t] = tu\{f[K(t)] + [A - K(t)]/t\}. \tag{21}$$

Consider, again, an ex ante calculation of expected utility when t is distributed in accordance with $(p_1, \ldots, p_T)$. It is straightforward to show that expected lifetime wealth is higher under the case of known horizon—that is,

$$E[W(t)] > W[E(t)]. \tag{22}$$

This conclusion follows from the convexity of W in t and the application of Jensen's inequality. [8] Intuitively, it may be thought of as resulting from the ability to make the investment decision after the lifetime is known, which provides a perfect match between K and t.

It is now not clear whether, in the presence of human capital investment (or analogous considerations that produce planning benefits from knowledge of the horizon), the consumer is better off having his length of life randomly determined prior to any decisions at the start of period 1, or having a random drawing at the end of each period that he survives. In the former situation he can make a perfect match of human capital investment to his (known) lifetime, but he cannot take out insurance that guarantees him a per period amount of consumption that corresponds to living the mean length of life. Obviously, which of these two elements predominates depends on the particular utility function, investment function, and probability distribution that apply.

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[8] Convexity of $W(t)$ can be seen by differentiating eq. (18) and using eq. (19): $W''(t) = -(t^3 f'')^{-1} > 0$.