ON UNIFORM BOUNDEDNESS OF SOLUTIONS TO DISCRETE VELOCITY MODELS IN SEVERAL DIMENSIONS

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1. Introduction

The global existence question for discrete velocity models in more than one space dimension remains unsolved except for initial values which are small in some sense ([2], [6], [7]). The crucial difficulty is that we do not seem to have the tools to obtain uniform L^∞–bounds in time on the local solution in terms of the initial values (this, of course, would entail global existence of a mild solution). The purpose of this article is to compare the situation with the better understood one–dimensional case, spell out some crucial differences and point out a possible way to progress.

2. Growth results for the 4-velocity Broadwell model

The initial value problem for discrete velocity models in one space dimension,

$$\frac{\partial}{\partial t} f_i + \xi_i \frac{\partial}{\partial x} f_i = Q_i(f,f)$$

$$f_i(0,x) = f_{i,0}(x), \qquad i = 1,\cdots,n \tag{1}$$

is well understood if $f_{i,0} \in L^1_+ \cap L^\infty(\mathbb{R})$. In fact, Beale [1] and later, with a more elegant method, Bony [3] proved that (1) admits a global, uniformly bounded solution, and were also able to show that the asymptotic behaviour of this solution is given by free streaming. Cabannes and Kawashima [4] obtained the global existence via the older methods pioneered by Nishida and Mimura [10] and Crandall and Tartar [12]. However, this method only proves global existence, not uniform boundedness.

If $f_{i,0} \in L^\infty_+(\mathbb{R})$ (but $\notin L^1_+(\mathbb{R})$), we can use the strict hyperbolicity of equations (1) to conclude that the solution to the Cauchy problem will still exist globally. What the methods from [1,3,4] do not show is uniform boundedness of this solution, though it is hard to imagine how the solution could grow indefinitely — we expect uniform boundedness! How difficult this problem is can be seen from the many, as yet unsuccessful attempts to solve the one-dimensional Broadwell model

$$\partial_t v + \partial_x v = z^2 - vw$$

$$\partial_t w - \partial_x w = z^2 - vw \tag{2}$$

$$\partial_t z = \tfrac{1}{2}(vw - z^2)$$

with periodic boundary conditions $v(t,0) = w(t,0)$, $v(t,1) = w(t,1)$ and smooth data on $[0,1]$ such that $v_0(0) = w_0(0)$, $v_0(1) = w_0(1)$.

This problem, as is well known, can be recast as a pure Cauchy problem for periodic initial values. The big prize is to show that as $t \longrightarrow \infty$, there is a constant $a > 0$ such that

$$\lim_{t \to \infty} (v(t,x),\ w(t,x),\ z(t,x)) = (a,a,a) \tag{3}$$

uniformly in x, and the big hurdle to this end is the lack of global L^∞-bounds for the solution. Recently, M. Slemrod [11] has proved a result on the asymptotic behaviour (more precisely, the orbital stability) of solutions to this initial boundary value problem, but in spite of skillful use of modern methods (like compensated compactness), he could not prove uniform boundedness. As a consequence, he could also not establish that the asymptotic state is a constant vector; the method only shows that v, w and z approach, in the weak topology, solutions to the collision-free system, *i.e.* waves traveling without interaction.

In more than one dimension, we know that $f_{i,0} \in L^1_+ \cap L^\infty$ is not sufficient for uniform boundedness, because these are counterexamples (see [8]). The initial values from [8] which lead to unlimited growth of the

L^∞–norm of the solution are characteristic functions and therefore discontinuous, but the geometric idea behind these examples can be applied to prove the following type of growth result for smooth data.

We are concerned with the solutions of the standard 4–velocity model in two space dimensions

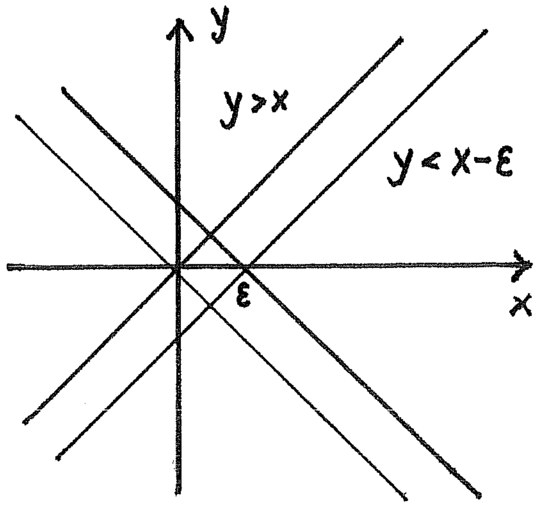
$$\begin{aligned} (\partial_t+\partial_x)f_1 &= Q(f,f) \\ (\partial_t-\partial_x)f_2 &= Q(f,f) \\ (\partial_t+\partial_y)f_3 &= -Q(f,f) \\ (\partial_t-\partial_y)f_4 &= -Q(f,f) \end{aligned} \tag{3}$$

with $Q(f,f) = f_3f_4 - f_1f_2$.

Theorem. *For any two constants* $\delta > 0$ *and* $C > 0$, *there are continuous initial values* $f_{i,0}$ *for* (3), $i = 1,\cdots,4$, *such that* $0 \le f_{i,0}(x,y) \le \delta$ *for all* x, y *and* i, *but* $\sup_{t,(x,y),i} f_i(t,x,y) \ge C$. *In addition, the* L^1*–norm of the initial values can be chosen arbitrarily small.*

Proof. Consider the initial values $f_{1,0} = f_{2,0} = 0$, and construct $f_{3,0}$ and $f_{4,0}$ as indicated in Figure 1:

Figure 1.



$$f_{4,0}(x,y) = \begin{cases} \delta \text{ for } & y > x \\ 0 \text{ for } & y < x-\epsilon \\ \text{linear for } & x-\epsilon \le y \le x \end{cases}$$

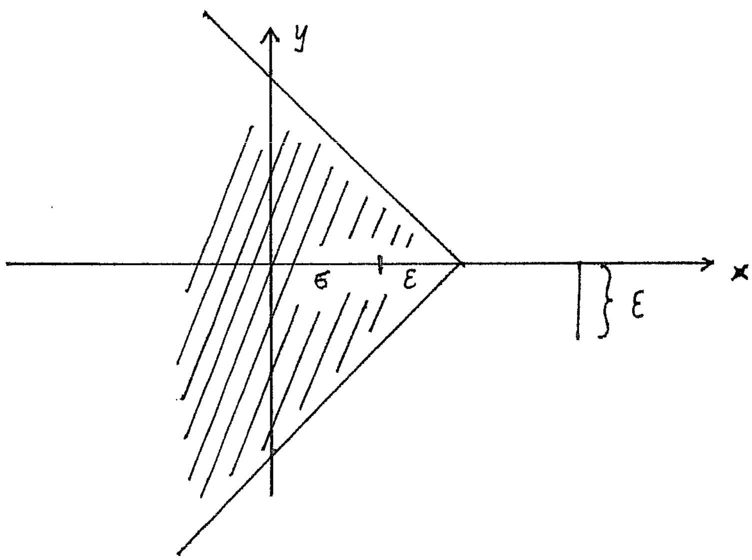
$$f_{3,0}(x,y) = \begin{cases} \delta \text{ for } y < -x \\ 0 \text{ for } y > -x+\epsilon \\ \text{linear for } -x \le y \le -x+\epsilon \end{cases}$$

ϵ is a nonnegative parameter to be chosen later. Fix $\delta > 0$ and suppose that there is a constant $C > 0$ such that for any $\epsilon > 0$, the solution to (3) with these initial values will remain bounded by C. Then consider the solution at time t at the point $x = t$, $y = 0$: We have

$$f_3(t;t,0) = f_3(t-\epsilon;t,-\epsilon) + \int_0^\epsilon (f_1f_2-f_3f_4)(t-\epsilon+\tau;t,-\epsilon+\tau)d\tau. \tag{4}$$

For $\sigma \in [0,t-\epsilon]$, f_3 is constant along the (backward) characteristic connecting $(t-\epsilon;t,-\epsilon)$ and $(0;t,-t)$, because f_1, f_2 and f_4 are all identically zero on that characteristic. This follows from the geometric setup: f_1 and f_2, being zero initially, can only become nonzero by interaction between f_3 and f_4. Such interaction has until time σ only occurred in the shaded region in Figure 2. Clearly, f_3 and f_4 do not

Figure 2.



interact along the mentioned characteristic until $\sigma = t - \epsilon$. In particular, $f_3(t-\epsilon;t,-\epsilon) = 1$, and from (4) and the assumed bounds we estimate

$$f_3(t;t,0) \geq 1 - \epsilon C^2,$$

and similarly

$$f_4(t;t,0) \geq 1 - \epsilon C^2.$$

Choose ϵ such that $1 - \epsilon C^2 \geq \frac{1}{2}$, then $f_3(t;t,0) \cdot f_4(t;t,0) \geq \frac{1}{4}$. For $f_2(t;t,0)$, we have the simple estimate

$$f_2(t;t,0) \leq f_2(t-\epsilon;t+\epsilon,0) + \int_0^{\xi} f_3 f_4(t-\epsilon+\tau,t+\epsilon-\tau,0)\,d\tau$$

$$\leq \epsilon \cdot C^2.$$

Now consider the equation for f_1 in mild form,

$$f_1(t;t,0) = f_{1,0}(0,0) + \int_0^t [f_3 f_4 - f_1 f_2](\tau;\tau,0)\,d\tau.$$

By the estimate from below of $f_3 \cdot f_4$ and the (assumed) estimate from above on $f_1 \cdot f_2$ by $\epsilon \cdot C^3$ we get

$$f_1(t;t,0) \geq 0 + \frac{t}{4} - t \cdot \epsilon C^3.$$

By choosing ϵ so small that $\epsilon \cdot C^3 < \frac{1}{4}$, we could conclude that f_1 would grow indefinitely along the characteristic $(t;t,0)$. This contradicts our assumption on boundedness. The proof is complete. ■

Remark. The principle of this proof can certainly be applied to many other discrete velocity models. For discontinuous data, the underlying geometric idea can also be used to show that the semi-discrete velocity models suggested by Cabannes [5] admit solutions which grow without bounds. The data constructed above have infinite L^1-norm, but the proof is easily modified to show that they can actually be chosen to have arbitrarily small L^1-norm.

3. Tentative steps towards growth control

The data constructed in Section 2 are, of course, very pathological. The unlimited growth is predetermined by the fact that the integrals $\int_{L_1} f_{4,0}$ and $\int_{L_2} f_{3,0}$, where L_1 and L_2 are the lines given by $y = x$ and $y = -x$ respectively, are divergent. In [7], it was shown that if $\sup_{L,i} \int_L f_{i,0}$ is small enough, where $i = 1,\cdots,4$ and L is any line parallel to L_1 or L_2, then there is indeed a uniformly bounded global solution.

Remark. This result is one example of a general global existence result proved in [7]. While all the examples from [7] remain of interest, Bony [2] has recently pointed out that the conditions imposed on the initial values are only true for the zero function (unless the admitted velocities satisfy restrictive conditions). This flaw was removed by Bony in [2]; his method does not use the conservation laws intrinsic to the model, but rather the convolution structure forced into the collision terms by the flow terms.

We return to the 4-velocity model. It is unknown whether the solution to the Cauchy problem even exists globally if $\sup_{L,i} \int_L f_{i,0}$ is finite (but large). Our growth result shows, however, that control of mixed norms of the above type is essential to obtain L^∞-control of the solution.

This is an important observation with respect to a possible generalization of the "potential for interaction" which Bony introduced in [5] for the one-dimensional case. For any one-dimensional discrete velocity model with mass and momentum conservation and for which $\xi_i \neq \xi_j$ if $i \neq j$, let

$$B[f] := \sum_{i,j} \iint_{y<x} (\xi_i-\xi_j)\ f_i(t,x)\ f_j(t,y)\ dx\ dy.$$

$B[f]$ is clearly bounded in terms of the largest x-component of the velocities and $\max_i \int f_{i,0}(x)dx$, and

$$\frac{d}{dt} B[f] = - \sum_{i,j} (\xi_i-\xi_j)^2 \int f_i f_j(t,x)\ dx.$$

This last identity is the key to very efficient control of the L^∞-norm of the solution in terms of the initial "mass" $\sum_i \int f_{i,0}(x)dx$. $B[f]$ has become known as "potential for interaction".

Suppose we call a potential for interaction in higher dimensions any functional of the system state which will yield L^∞-control of the solution. The above growth results show that such a functional could not be bounded in terms of the total mass. It can, at least for the 4-velocity model, at best be bounded in terms of integrals $\int_L f_{i,0}$.

Finally, we point out that the integrals $\int_L f_{i,0}$ arise, for the 4-velocity model, quite naturally from simple cancellations of the collision terms: It is easily checked that if $f_1,\cdots,f_4$ is a solution to (3) and if $L_{1,+}(t)$ is any line parallel to $y = x$ and moving with velocity $(1,0)$ (or, equivalently, with $(0,1)$), then

$$\frac{d}{dt} \int_{L_{1,+}(t)} (f_1+f_3)(t,\cdot) = 0. \tag{5}$$

Similarly, if $L_{2,-}(t)$ is a line parallel to L_2 and moving with velocity $(-1,0)$ or $(0,1)$, then

$$\frac{d}{dt}\int_{L_{2,-}(t)}(f_2+f_3)(t,\cdot) = 0.$$

$f_1 + f_4$ and $f_2 + f_4$ also satisfy such conservation laws. These conservations are quite clear from a mechanical point of view, given the underlying particle model. Equivalent conservation laws exist for the corresponding lattice gas model; as was pointed out to me at the Symposium, this fact is known to at least some of the experts in cellular automata theory [9].

The conservation law (5) is an example for a general principle, which we now explain. Let $u_1,\cdots,u_n \in \mathbb{R}^3$ be the admissible velocities for a certain discrete velocity model

$$\frac{\partial f_i}{\partial t} + u_i \cdot \nabla_x f_i = Q_i(f,f),$$

and suppose that $M \subset \{1,\cdots,n\}$ is an index set such that there are real numbers α_i, $i \in M$ with

$$\sum_{i \in M} \alpha_i\, Q_i(f,f) = 0$$

(mass, momentum and energy conservation are special cases with $M = \{1,\cdots,n\}$).

Now suppose that L is a linear submanifold of $\mathbb{R}^3$ such that the sets

$$L_i(t) = \{x \in \mathbb{R}^3;\ x \in L + tu_i\}$$

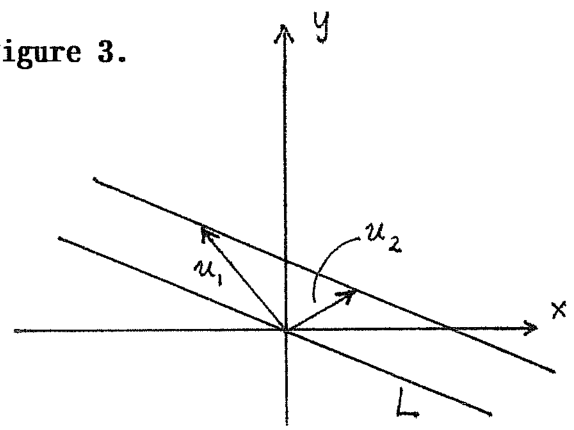
are all identical (say $L_i(t) = L(t)$). Then clearly

$$\frac{d}{dt} \sum_{i \in M} \alpha_i \int_{L(t)} f_i(t,x)\ dx = 0. \tag{6}$$

It is easily checked that (5) is just a special case of (6). We suggest that a potential of interaction, if it exists in 2 or 3 dimensions, would have to involve integrals like the ones in (6) in some way, with submanifolds of dimensions 1, 2 and 3.

If $u_1, u_2 \in \mathbb{R}^2$ are linearly independent, then there is exactly one line L through the origin such that $L + tu_1$ and $L + tu_2$ are identical for all t:

Figure 3.



Similarly, if u_1, u_2 and $u_3 \in \mathbb{R}^3$ are linearly independent, then there is exactly one plane P through the origin such that $p + tu_i$ is the same set for $i = 1, 2$ and 3.

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