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On uniform convergence of Hermite series

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Abstract. It is proved in this note that almost uniform convergence of the series $\sum_{n=1}^{\infty} |a_n h_n(x)|$, where h_n are Hermite functions, is equivalent to uniform convergence of the series $\sum_{n=1}^{\infty} (1+|x|^{5/2})^{-1} |a_n h_n(x)|$ and to convergence of series $\sum_{n=1}^{\infty} n^{-1/4}|a_n|$.

1. According to [1], by *Hermite functions* we mean the functions

$$h_n(x) = (-1)^n \left(\sqrt{2\pi}\; n^{\boldsymbol{R}}\right)^{-1/2} \exp\left(\frac{x^2}{4}\right) \exp\left(-\frac{x^2}{2}\right)^{(n)}$$

for $x \in \boldsymbol{R}$ and $n = 0, 1, 2, \ldots$ The functions h_n form an orthonormal system in the space $L^2(\boldsymbol{R})$. Thus, by the Riesz–Fischer theorem, the series $\sum_{n=0}^{\infty} a_n h_n(x)$ is convergent in $L^2(\boldsymbol{R})$ iff the series $\sum_{n=0}^{\infty} a_n^2$ is convergent (see [1]).

We are going to prove that the series $\sum_{n=1}^{\infty} |a_n h_n(x)|$ is convergent almost uniformly on $\boldsymbol{R}$ iff the series $\sum_{n=1}^{\infty} n^{-1/4}|a_n|$ is convergent. More precisely, we have

THEOREM. *The following conditions are equivalent:*

(i) *the series* $\sum_{n=1}^{\infty} n^{-1/4}|a_n|$ *is convergent,*

(ii) *the series* $\sum_{n=1}^{\infty} (1+|x|^{5/2})^{-1} |a_n h_n(x)|$ *is uniformly convergent on* $\boldsymbol{R}$,

(iii) *the series* $\sum_{n=1}^{\infty} |a_n|\, |h_n(x)|$ *is almost uniformly convergent on* $\boldsymbol{R}$,

(iv) *the series* $\sum_{n=1}^{\infty} n^{-1/4}|a_n|\, |\cos(\beta_n x - n\pi/2)|$ *with* $\beta_n = \sqrt{n+1/2}$ *is almost uniformly convergent on* $\boldsymbol{R}$.

2. Before proving the theorem we give some relations needed in the sequel.

We define a sequence of numbers V_n ($n = 1, 2, \ldots$):

$$V_1 = 1, \quad V_n = \sqrt{\frac{1\cdot 3\cdot \ldots \cdot (n-1)}{2\cdot 4\cdot \ldots \cdot n}} \quad \text{for every } n,$$

$$V_n = \beta_n^{-1} \sqrt{\frac{3\cdot 5\cdot \ldots \cdot n}{2\cdot 4\cdot \ldots \cdot (n-1)}} \quad \text{with} \quad \beta_n = \sqrt{n+1/2} \quad \text{for odd } n > 1.$$

By the Wallis formula, we have

$$\lim_{n\to\infty} n^{1/4} V_n = 2^{1/4}\pi^{-1/4}. \tag{1}$$

Using similar arguments and calculations as in [2], p. 89, we obtain

$$h_n(x) = (2\pi)^{-1/4} V_n \cos(\beta_n x - n\pi/2) + R_n(x), \tag{2}$$

where

$$R_n(x) = (4\beta_n)^{-1} \int_0^x y^2 h_n(y) \sin(\beta_n(x-y))\, dy.$$

By the Schwarz inequality we have

$$|R_n(x)| \leqslant (4\beta_n)^{-1} \Big(\int_0^x y^4\, dy\Big)^{1/2} \Big(\int_{-\infty}^{\infty} h_n^2(y)\, dy\Big)^{1/2} = \big(4\sqrt{5}\,\beta_n\big)^{-1} |x|^{5/2} \leqslant n^{-1/4}|x|^{5/2}$$

for all $x \in \boldsymbol{R}$ and almost all $n \in N$.

Hence

$$|a_n h_n(x)| \leqslant n^{-1/4}|a_n|\,(1+|x|^{5/2}) \quad (x \in \boldsymbol{R}), \tag{3}$$

in view of (2) and (1).

Moreover,

$$|a_n R_n(x)| \leqslant \int_0^{|x|} y^2 |a_n h_n(y)|\, dy \leqslant \alpha^2 \int_0^{\alpha} |a_n h_n(y)|\, dy \tag{4}$$

for all x such that $|x| < \alpha$. Finally, identity (2) leads to the inequality

$$|(2\pi)^{-1/4} V_n \cos(\beta_n x - n\pi/2)| \leqslant |h_n(x)| + |R_n(x)| \quad (x \in \boldsymbol{R}),$$

which together with (1) yields

$$n^{-1/4}|a_n|\, |\cos(\beta_n x - n\pi/2)| \leqslant C\big(|a_n h_n(x)| + |a_n R_n(x)|\big), \tag{5}$$

where C is an arbitrary constant greater than $\sqrt{2}$.

3. **Proof of Theorem.** By (3), condition (i) implies condition (ii). Implication (ii) $\Rightarrow$ (iii) is obvious.

To prove implication (iii) $\Rightarrow$ (iv), suppose that the series $\sum_{n=1}^{\infty} |a_n h_n(x)|$ is uniformly convergent on every interval I. Hence the series

$$\sum_{n=1}^{\infty} \int_I |a_n h_n(y)|\, dy$$

is convergent. Hence, by (4) and (5), the series

$$\sum_{n=1}^{\infty} n^{-1/4} |a_n|\, |\cos(\beta_n x - n\pi/2)|$$

is uniformly convergent on every interval I, i.e., condition (iv) is fulfilled.

Finally, let us assume that (iv) holds. Then the series

$$\sum_{n=1}^{\infty} (2n)^{-1/4} |a_{2n}|\, |\cos(\beta_{2n} x)|, \qquad \sum_{n=1}^{\infty} (2n-1)^{-1/4} |a_{2n-1}|\, |\sin(\beta_{2n-1} x)|$$

are uniformly convergent on every interval, in particular on intervals of the form $[0, \alpha]$, $\alpha > \pi$. Letting $x = 0$ in the first of the above series, we get

$$\sum_{n=1}^{\infty} (2n)^{-1/4} |a_{2n}| < \infty. \tag{6}$$

On the other hand, we have

$$\sum_{n=1}^{\infty} (2n-1)^{-1/4} |a_{2n-1}| \int_0^{\alpha} |\sin(\beta_{2n-1} x)|\, dx < \infty. \tag{7}$$

Note that

$$\int_0^{\alpha} |\sin(\beta_k x)|\, dx = \beta_k^{-1} \int_0^{\alpha\beta_k} |\sin y|\, dy \geqslant 2\beta_k^{-1} \left[\frac{\alpha\beta_k}{\pi}\right] \geqslant \frac{2\alpha}{\pi} - 2\beta_k^{-1},$$

which results in

$$\int_0^{\alpha} |\sin(\beta_{2n-1} x)|\, dx \geqslant \frac{2\alpha}{\pi} - 2 > 0 \quad (n = 1, 2, \ldots).$$

Hence

$$\sum_{n=1}^{\infty} (2n-1)^{-1/4} |a_{2n-1}| < \infty, \tag{8}$$

in virtue of (7).

Relations (6) and (8) mean that condition (i) is satisfied. This ends the proof of implication (iv) $\Rightarrow$ (i) and of the whole theorem.

References

[1] P. Antosik, J. Mikusiński, R. Sikorski, *Theory of distributions. The sequential approach*, Elsevier–PWN, Amsterdam–Warszawa 1973.
[2] N. N. Lebedev, *Special functions and their applications*, in Russian, Moskva.

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