On uniformly homeomorphic normed spaces

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As an approach to the problem of characterising and classifying Banach spaces in terms of their geometric structure, consideration has been given to the following problem: Must two given Banach spaces always be (linearly-topologically) isomorphic if it is supposed that they are uniformly homeomorphic (i.e., that there is a non-linear bijection f between them such that f and f^{-1} are uniformly continuous)?

In the present paper it is proved that if two normed spaces are uniformly homeomorphic, then the finite-dimensional subspaces in any of them are imbeddable into the other by means of linear imbeddings T such that the numbers $\|T\|\,\|T^{-1}\|$ have a common upper bound (Section 3). Further, for the case where the spaces are separable Banach spaces and one of them is a dual space, it is proved: If the uniform homeomorphism is "well-behaved on finite-dimensional subspaces for large distances", then the two spaces are isomorphic (Section 4).

The question of isomorphy for uniformly homeomorphic spaces has been raised by Bessaga [1] and Lindenstrauss [5], [6]. Enflo [4] has given an affirmative answer in the case where one of the spaces is a Hilbert space. If a space $L^p(\mu)$ is uniformly homeomorphic to some space $L^q(\nu)$ $(1 \leqq p \leqq q < \infty)$, then $p = q$, as was proved partially by Lindenstrauss [5], partially by Enflo [3]. Several related results have been given by Mankiewicz [7]—[9].

The methods of proof employed in [4] and [7]—[9] make use of strong derivatives of Lipschitz mappings in order to produce the desired linear mapping. In this paper we take a different approach, using averages of function-values on finite point-meshes.

All spaces will be supposed to have the real number field as scalar field.

2. A combinatorial lemma

Let d be a fixed positive integer. We denote by $G_+(m)$ that subset of Z^d which consists of all d-tuples of integers $x=(\xi_1, \ldots, \xi_d)$ with $0 \leqq \xi_i < m$ $(1 \leqq i \leqq d)$.

Lemma 1. *Let m be a given positive integer, and let q be a given number such that $0<q<1$. Then there is a positive integer j_0 such that the following statement holds:*

(S) *Let j be any integer $\geqq j_0$, and let S be any subset of $G_+(m^j)$ whose cardinality is at least qm^{jd}. Then there is a subset of the form $y+m^{j'-1}G_+(m)$ (with $2 \leqq j' \leqq j-1$ and with y in $m^{j'}G_+(m^{j-j'})$) of $G_+(m^j)$ such that for every element x in that subset,*

$$S \cap (x+G_+(m^{j'-1})) \neq \emptyset.$$

Proof. To begin with we let j be a fixed integer $\geqq 4$, and i an integer variable ranging from 2 to $j-1$. We must show that if j is large (S) holds for some $i=j'$.

Let S be a given set as in (S). For each i in the mentioned range there is a unique disjoint partition of $G_+(m^j)$ into sets of the form $x+G_+(m^{i-1})$; denote by $\mathscr{C}_i$ the collection of those disjoint sets, and by $\bar{\mathscr{D}}_i$ the subcollection of those sets in $\mathscr{C}_i$ which do not meet S. Then for $i \leqq j-2$ let $\mathscr{D}_i$ be the collection of those sets in $\bar{\mathscr{D}}_i$ which are not contained in any set of $\bar{\mathscr{D}}_{i+1}$. Since the cardinality of $G_+(m^j)$ is m^{jd}, there must be a $\mathscr{D}_{j'}$ such that the union of the sets in that collection $\mathscr{D}_{j'}$ has cardinality at most $m^{jd}/(j-3)$. Thus the number of sets in $\mathscr{D}_{j'}$ is at most

$$(*) \qquad m^{jd-j'd+d}/(j-3).$$

By the assumption about the cardinality of S, the union of all sets in $\mathscr{C}_{j'+1} \setminus \bar{\mathscr{D}}_{j'+1}$ has cardinality at least qm^{jd}; so the collection $\mathscr{C}_{j'+1} \setminus \bar{\mathscr{D}}_{j'+1}$ consists of at least $qm^{jd-j'd}$ sets. Now suppose that j was initially taken larger than $2m^d/q+3$. Then the last-mentioned number of sets is strictly larger than $(*)$, and hence there must be a set $y+G_+(m^{j'})$ in $\mathscr{C}_{j'+1} \setminus \bar{\mathscr{D}}_{j'+1}$ containing no set of $\mathscr{D}_{j'}$. If we now form the set $y+m^{j'-1}G_+(m)$ we easily find that this set has the properties claimed in statement (S).

3. Uniform representability

Theorem 1. *For any two normed spaces which are uniformly homeomorphic, there is a number $C>0$ with the property that every finite-dimensional subspace of one of the given spaces is imbeddable into the other by means of a linear mapping T such that $\|T\|\,\|T^{-1}\| \leqq C$.*

In view of the triangle inequality we easily obtain Theorem 1 from the following:

Theorem 1A. *For two normed spaces E and F, let there be given a (non-linear) mapping $f\colon E\to F$ which for some number $b>0$ fulfils the inequality*

$$b^{-1}\|x-y\| \leqq \|f(x)-f(y)\| \leqq b\|x-y\| \quad \text{whenever} \quad \|x-y\| \geqq 1.$$

Then there is a number $C>0$ such that every finite-dimensional subspace of E is imbeddable into F by means of a linear mapping T such that $\|T\|\,\|T^{-1}\|\leqq C$.

Notation. For the proof of Theorem 1A we need some definitions. Given some points $x_1, \dots, x_d$ $(d\geqq 1)$ in a linear space and an integer $m\geqq 1$, we denote by $G(x_1, \dots, x_d|m)$ [resp. $G_+(x_1, \dots, x_d|m)$] the set of all linear combinations $\xi_1x_1+\dots+\xi_dx_d$ with ξ_i integers, $|\xi_i|\leqq m$ [resp. $0\leqq\xi_i<m$].

For a normed space E we let $\mathsf{S}(E)$ be the set of all d-tuples $(x_1, \dots, x_d)\subset E$ such that $\|x_i\|=1$ and $\operatorname{dist}(x_i, \operatorname{lin}(x_1, \dots, x_{i-1}))=1$.

Assumptions. To begin with, we consider a given (non-linear) mapping $f\colon E\to F$, where E and F are normed linear spaces, such that for some number $b>0$ we have

$$\|f(x)-f(y)\| \leqq b\|x-y\| \quad \text{for} \quad x, y \quad \text{in} \quad E, \quad \|x-y\| \geqq 1.$$

Further, let $c>0$ be another fixed number.

Notation. With these assumptions, let x in E and u in F' be given points. (F' is the dual, or conjugate space, of F.) We denote by $\mathscr{A}(x, u)$ the class of all sets S in E such that whenever y is a point in S and k is any positive integer such that $y+kx$ is also in S, we have

$$u\big(f(y+kx)-f(y)\big) \geqq c\|u\|\,\|x\|k.$$

Lemma 2. *With these assumptions, let $d\geqq 1$ be a given integer. Then there is an integer $m_0(d, b/c)=m_0\geqq 3$ such that for $m\geqq m_0$ there is an integer $j_0(d, m, b/c)=j_0\geqq 1$ with this property: Let $(x_1, \dots, x_d)$ be a d-tuple of $\mathsf{S}(E)$ and let $j\geqq j_0$; suppose that y^0 in $G(x_1, \dots, x_d|[m^{3j}/3])$, z in $G(x_1, \dots, x_d|m)$, and u in F' are elements for which*

$$u\big(f(y^0+[m^{3j-1}/3]z)-f(y^0)\big) \geqq 5c(m^{3j-1}/3)\|u\|\,\|z\|.$$

Then the set $G(x_1, \dots, x_d|m^{3j})$ contains a subset which is of the form

$$y^- + m^{j^- -1}G(x_1, \dots, x_d|m)$$

(where $1\leqq j^-\leqq 3j-1$), and which belongs to the class $\mathscr{A}(m^{j^- -1}z, u)$.

In the proof of this we shall use an elementary fact:

Sublemma. *Let $a_0, \dots, a_K$ be a finite real number sequence such that $a_K-a_0\geqq 2cK$ and $a_{k+1}-a_k\leqq b$ $(0\leqq k\leqq K-1)$ for some given b, $c>0$. Put*

$$Q = \big\{k\,|\,a_i-a_k \geqq c(i-k) \quad \text{for} \quad k \leqq i \leqq K\big\}.$$

Then the cardinality of Q is at least $\big(c/(b-c)\big)K$.

Proof of Sublemma. Form the sequence $m_k = \min_{k \leqq i \leqq K} (a_i - ci)$. Then $m_K - m_0 \geqq \geqq cK$ and $m_{k+1} - m_k \leqq b - c$. Since $m_{k+1} > m_k$ only when k in Q, we are done.

Proof of Lemma 2. Let m and j be fixed integers large enough to meet the requirements specified later; and let y^0, z, and u be given as in the statement of the lemma. Denote by S the set of those points x in $G(x_1, \dots, x_d | [2m^{3j}/3])$ for which

$$(\dagger) \qquad u\big(f(x+iz) - f(x)\big) \geqq 2c\|u\|\,\|z\|\,i$$

when $$0 \leqq i \leqq [m^{3j-2}/3].$$

If B denotes the closed unit ball in E, consider

$$V = G(x_1, \dots, x_d | [2m^{3j}/3]) \cap \big(y^0 + (cm^{3j-1}/6b)B\big).$$

Then take a set $Y \subset V$ so that for every line parallel to z and having non-empty intersection with V, the set Y has precisely one point in that intersection. The definition of $\mathrm{S}(E)$ implies that $\|z\| \geqq 1$, so by the definition of V we must have

$$u\big(f(y + (m^{3j-1}/3)z) - f(y)\big) \geqq 4c(m^{3j-1}/3)\|u\|\,\|z\|,$$

for all y in Y.

Making use of the latter estimate, for each y in Y we now apply the preceding Sublemma to the sequence $i \to u\big(f(y+iz)\big)$. If $l(y)$ is the set of points $y+iz$ with $0 \leqq i \leqq [m^{3j-1}/3]$, we then find that $l(y) \cap S$ contains more than $(2c/b)m^{3j-1}/6$ points. But the definitions of V and $\mathrm{S}(E)$ imply that there is also a number q, $0<q<1$, which depends only on the numbers d, m, b/c, but not on j, and which is such that the union of all the sets $l(y)$, with y running through Y, has at least qm^{3j} points. Summing up we find that there is a number q', $0<q'<1$, not depending on j, such that S has at least $q'm^{3j}$ points.

In view of this conclusion we can apply Lemma 1 of Section 2. Assuming that j was taken large enough, we thus find that $G(x_1, \dots, x_d | m^{3j})$ has a subset which is of the form

$$y + m^{3j'-3} G_+(x_1, \dots, x_d | m^3),$$

where $2 \leqq j' \leqq j-1$, and in which every point x is such that

$$S \cap \big(x + G_+(x_1, \dots, x_d | m^{3j'-3})\big) \neq \emptyset.$$

Assume that we have taken $m \geqq 2bd/c$. Then the definition of $\mathrm{S}(E)$ and the assumption about f imply that for every point x in the set

$$y + m^{3j'-2} G_+(x_1, \dots, x_d | m^2),$$

the inequality ($\dagger$), without factor 2, must hold whenever $m^{3j'-2} \leqq i \leqq [m^{3j'}/3]$.

This means that the mentioned set is of class $\mathscr{A}(m^{3j'-2}z, u)$. Then it must clearly contain a subset of the desired kind, with $j^- = 3j'-1$.

Proof of Theorem 1A. Now let $f\colon E\to F$ be as in the statement of the theorem. Let classes $\mathscr{A}^*(x, u)$ of subsets in E be defined as the $\mathscr{A}(x, u)$ just before Lemma 2, but with the given coefficient c replaced by $b/5$.

To begin with, let $(x_1, \dots, x_d)$ be a given element in $S(E)$ and $m\geqq 1$ a given integer. Let $N\geqq 1$ be an integer which is fixed but chosen large enough to meet the requirements specified later; consider the set

$$G = G(x_1, \dots, x_d | m^{3N}).$$

Let $z_1, \dots, z_n$ (where $n=(2m+1)^d-1$) be an enumeration of the non-zero points in $G(x_1, \dots, x_d|m)$. In view of the assumption for f a recursive application of Lemma 2 gives a sequence of sets $G\supset G_1\supset \dots \supset G_n$, which are of the form

$$G_k = y_k + m^{3N(k)(j(k)-1)} G(x_1, \dots, x_d | m^{3N(k)}),$$

with integers $N\geqq N(1)\geqq \dots \geqq N(n)\geqq 1$ and $j(k)\geqq 1$, and which belong to the classes

$$\bigcap\nolimits_{i\leqq k} \mathscr{A}^*(m^{3N(k)(j(k)-1)} z_i, u_i),$$

resp., for some suitable $u_i\neq 0$ in F'. This is certainly possible if only N was taken large enough, and we may also assume that the number $m^{3N(n)}=M$, say, is suitably large for our later purposes. (Of course, the $N(k)$ have to be determined in the order $N(n-1), N(n-2), \dots, N(1), N$; but this is clearly permissible. Also notice that the choice of the point y^0 mentioned in Lemma 2 is actually without importance here.)

With the aid of the set G_n thus found, we can quickly prove: Given an $\varepsilon>0$ (to be specified shortly), there is a mapping $h\colon G(x_1, \dots, x_d|m)\to F$ fulfilling the conditions

(i) $\|h(x)+h(y)-h(x+y)\| \leqq \varepsilon$

(ii) $(10b)^{-1}\|x\| \leqq \|h(x)\| \leqq b\|x\|$

for all x and y. Namely, we define

$$h(x) = (2M+1)^{-d} M^{-j(n)+1} \sum\nolimits_{x'} \big(f(x'+M^{j(n)-1}x)-f(x')\big),$$

where the summation index x' runs through the set G_n. The right-hand inequality of (ii) is immediate. To establish the left-hand inequality of (ii), first notice that for any $0<t<1$, by assuming M/m to be large enough we can achieve that for a proportion of at least t of the number of all points x' in G_n, also the point $x'+M^{j(n)-1}x$ is in G_n (for all fixed x). In view of this observation, the mentioned inequality follows from the fact proved above that G_n is of class

$$\bigcap\nolimits_{i\leqq n} \mathscr{A}^*(M^{j(n)-1}z_i, u_i),$$

for some $u_i\neq 0$ in F'.

To verify (i), we similarly observe that by assuming M/m to be large enough, we achieve this: If we write out the defining sums of $h(x)$, $h(y)$, and $h(x+y)$, and then form the difference $h(x)+h(y)-h(x+y)$, then the number of terms which do not cancel out must become suitably small compared to the denominator $(2M+1)^d$. This gives the desired inequality (in view of the right-hand inequality in the hypothesis of the theorem).

By a modification of h we can obtain a mapping $h^-: G(x_1, \ldots, x_d|m) \to F$ fulfilling the conditions

(i)$^-$ $h^-(x+y) = h^-(x)+h^-(y)$

(ii)$^-$ $(20b)^{-1}\|x\| \leqq \|h^-(x)\| \leqq 2b\|x\|$

for all x and y. For if ε was taken small enough, it will do with the definition

$$h^-(\xi_1 x_1 + \ldots + \xi_d x_d) = \xi_1 h(x_1) + \ldots + \xi_d h(x_d).$$

We can now complete the proof. Let K be a given finite-dimensional subspace of E. Suppose that $(x_1, \ldots, x_d)$ is a sequence of $S(E)$ which spans K. There must be an integer $m \geqq 1$ such that if $h^-: G(x_1, \ldots, x_d|m) \to F$ is *any* given mapping which fulfils the conditions (i)$^-$ and (ii)$^-$ just stated, then its unique linear extension $T: K \to F$ must satisfy the inequalities

$$(30b)^{-1}\|x\| \leqq \|T(x)\| \leqq 3b\|x\|$$

for all x. Since the existence of such an h^- has just been proved, the assertion follows (with $C=90b^2$; but cf. Section 5).

4. An isomorphy criterion

When there is a uniform homeomorphism which is "well-behaved on finite-dimensional subspaces" we can sometimes infer that the two spaces must be isomorphic. To make the assertion precise, we introduce some notations.

Notation. For a normed space E we let Φ_E be the set of all its finite-dimensional subspaces, and Ψ_E the set of all its closed subspaces of finite codimension. If $f: E \to F$ is a mapping between two normed spaces, and if K is in Φ_E and L in Ψ_F, we denote by $f_{K,L}: K \to F/L$ the composition of f with the canonical inclusion and quotient maps: $K \to E \to F \to F/L$.

Theorem 2. *Let E and F be separable Banach spaces, and let F be the dual of some Banach space. Suppose that there is a uniformly continuous surjection $f: E \to F$, for some $c>0$ fulfilling the conditions:*

(C) *For every K_E in Φ_E there is an L in Ψ_F and a $\lambda_0>0$ such that*

$$\|f_{K,L}(x)-f_{K,L}(y)\| \geqq c\|x-y\| \quad \textit{when} \quad \|x-y\| \geqq \lambda_0.$$

(D) *Conversely, for every L in Ψ_F there is a K in Φ_E and a $\lambda_0>0$ such that for any x, y in F/L with $\|x-y\|\geqq\lambda_0$, there are always points x' in $f_{K,L}^{-1}(x)$ and y' in $f_{K,L}^{-1}(y)$ such that*

$$\|x-y\| \geqq c\|x'-y'\|.$$

Then E and F are isomorphic as Banach spaces.

Proof (somewhat sketchy). Let there be given finite-dimensional subspaces $K_1\subset K_2\subset\ldots$ in E, such that their union is dense in E. Let $u_1, u_2, \ldots$ be a sequence which is dense in the set of elements of norm one in a space to which F is dual. In our notation we regard the u_i as functionals $u_i(.)$ on F.

First, by conditions (C) and (D) it can be seen that there are sequences of integers $1\leqq r(1)\leqq r(2)\leqq\ldots$ and $1\leqq s(1)\leqq s(2)\leqq\ldots$ such that if we take $K=K_k$, then condition (C), with c replaced by $c/2$, is fulfilled with $L=\bigcap_{i\leqq r(k)} u_i^{-1}(0)$; and if we take $L=\bigcap_{i\leqq k} u_i^{-1}(0)$, then (D), with c replaced by $c/2$, is fulfilled with $K=K_{s(k)}$.

Using the same reasoning as in the proof of Theorem 1A in the preceding section, we can prove that for some $C>0$ there are linear mappings $T_k\colon K_k\to F$ $(k\geqq 1)$ such that

(i) $\|T_k\|\leqq C$.

(ii) For z in K_k and $j\geqq k$, we have $u_i(T_j(z))\geqq C^{-1}\|z\|$ for some $i\leqq r(k)$.

(iii) For each integer $k\geqq 1$, we have for each $j\geqq s(k)$ that $u_k(T_j(z))\geqq C^{-1}\|z\|$ for some $z\neq 0$ in $K_{s(k)}$.

In view of Alaoglu's theorem we can use a standard Arzelà—Ascoli argument to find a point-wise weak-star convergent subsequence of T_k. The limit mapping thus found extends by continuity to a mapping $T\colon E\to F$. The mapping T is clearly linear, and on account of statements (i)—(iii) it is quickly checked that $\|T\|\,\|T^{-1}\|\leqq C^2$, and that the domain of T^{-1} is the whole of F.

5. Sharp estimates

In the proofs of Sections 3—4 we refrained from making the best possible estimates of the norms of the linear mappings. However, by modifying the proofs in a way which is quite straightforward but which would look ugly in print, it is obtained that in Theorem 1A we can actually get $C=b^2+\varepsilon$ for any $\varepsilon>0$. In the proof of Theorem 2 we can get $\|T\|\,\|T^{-1}\|\leqq b/c+\varepsilon$ (where b is as in the Assumption before Lemma 2).

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