

## ON UNIMODALITY FOR LINEAR EXTENSIONS OF PARTIAL ORDERS\*

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**Abstract.** R. Rivest has recently proposed the following intriguing conjecture: Let  $x^*$  denote an arbitrary fixed element in an  $n$ -element partially ordered set  $P$ , and for each  $k$  in  $\{1, 2, \dots, n\}$  let  $N_k$  be the number of order-preserving maps from  $P$  onto  $\{1, 2, \dots, n\}$  that map  $x^*$  into  $k$ . Then the sequence  $N_1, \dots, N_n$  is unimodal. This note proves the conjecture for the special case in which  $P$  can be covered by two linear orders. It also generalizes this result for  $P$  that have disjoint components, one of which can be covered by two linear orders.

**1. Introduction.** Given a finite partially ordered set  $(P, <)$ , where  $<$  is asymmetric, we say that an injection  $\lambda$  from  $P$  into the set  $Z$  of integers is a *linear extension* of  $P$  if, for all  $x, y \in P$ ,

$$x < y \Rightarrow \lambda(x) < \lambda(y).$$

We shall presume that  $P$  has  $n$  elements and, in the main part of the paper, restrict ourselves to bijections  $\lambda : P \rightarrow [n] \equiv \{1, 2, \dots, n\}$ . Generalizations are discussed later.

Let  $x^*$  be an arbitrary fixed element in  $P$ . For each  $k \in [n]$ , define  $N_k$  to be the number of linear extensions  $\lambda : P \rightarrow [n]$  for which  $\lambda(x^*) = k$ . Rivest [2] has proposed the following tantalizing conjecture.

**CONJECTURE.** *The sequence  $N_k, k \in [n]$ , is unimodal.*

By unimodal we mean that, for all  $1 \leq i < j < k \leq n$ ,

$$N_j \geq \min \{N_i, N_k\}.$$

In this note we shall prove that the conjecture is valid for the important class of partially ordered sets that can be partitioned into two linearly ordered subsets, i.e., *chains*, with  $<$ -pairs allowed between the chains. In fact, we show that the  $N_k$ 's in this case satisfy the stronger property of logarithmic concavity, i.e.,

$$N_k^2 \geq N_{k-1}N_{k+1} \quad \text{for } 1 < k < n.$$

A similar proof provides an interesting result involving the unimodality of certain sequences of integers.

**2. Lattice paths in  $Z^2$ .** We shall say that the partially ordered set  $(P, <)$  can be covered by two chains if there is a partition  $\{A, B\}$  of  $P$  such that the restriction of  $<$  on each of  $A$  and  $B$  is a linear order. To avoid the trivial case, we shall suppose that  $<$  on  $P$  is not linear, and that  $(P, <)$  can be covered by two chains, denoted as  $A = \{a_1 < \dots < a_r\}$  and  $B = \{b_1 < \dots < b_s\}$ , with  $r \geq 1, s \geq 1$  and  $r + s = n$ . There can be "cross-relations" like  $a_i < b_j$  or  $b_j < a_i$  from  $(P, <)$ , but in any event  $<$  must be asymmetric ( $x < y \Rightarrow \text{not } (y < x)$ ) and transitive.

Let  $L$  denote the set of all ordered pairs of nonnegative integers. Each linear extension  $\lambda : P \rightarrow [n]$  induces maps of  $A$  and  $B$  into  $[n]$ , with  $\lambda(a_1) < \dots < \lambda(a_r)$  and  $\lambda(b_1) < \dots < \lambda(b_s)$ . To each such  $\lambda$  we will associate a lattice path  $\pi(\lambda)$  in  $L$  as follows. The first point on  $\pi(\lambda)$  is  $(0, 0)$ . If the  $k$ th point on  $\pi(\lambda)$  is  $(x_k, y_k)$  and if  $\lambda(p) = k + 1$ , then the  $(k + 1)$ st point on  $\pi(\lambda)$  is  $(x_k + 1, y_k)$  if  $p \in A$ , and  $(x_k, y_k + 1)$  if  $p \in B$ . The terminal point on  $\pi(\lambda)$  is  $(r, s)$ . An example appears in Fig. 1.

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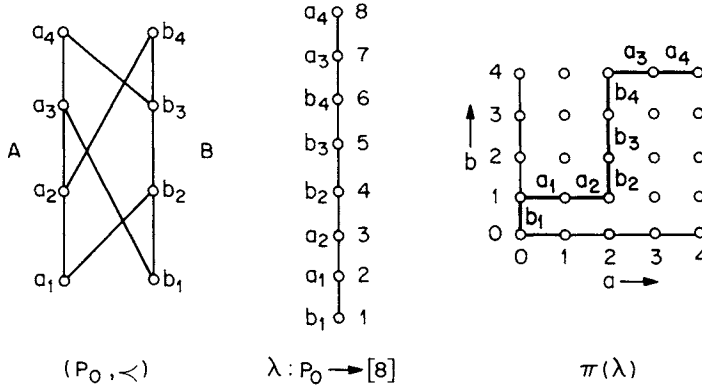


FIG. 1. The correspondence between  $\lambda$  and  $\pi(\lambda)$ .

The fact that  $\lambda$  preserves the linear orders on  $A$  and  $B$  is reflected in the fact that the indices of the  $a_i$  and  $b_j$  increase as we move along  $\pi(\lambda)$  from  $(0, 0)$  to  $(r, s)$ . But how do the *other*  $<$ -pairs show up in  $\pi(\lambda)$ ? For Fig. 1, what constraint does  $a_1 < b_2$  (which forces  $\lambda(a_1) < \lambda(b_2)$ ) place on  $\pi(\lambda)$ ? The answer is very simple. Each  $a_i < b_j$  corresponds to a rectangular “barrier” which the path  $\pi(\lambda)$  is not allowed to penetrate. This barrier is defined to be all lattice points  $(x, y)$  in  $L$  for which  $x \leq i$  and  $y \geq j - 1$ , as illustrated in Fig. 2.

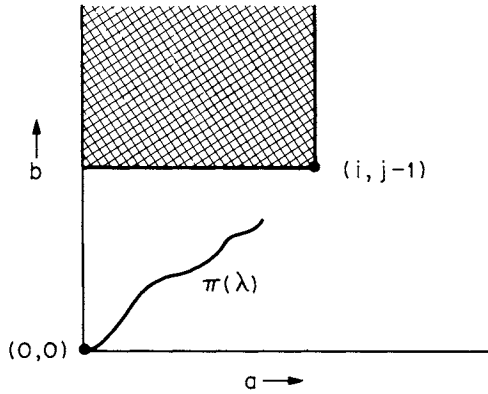


FIG. 2. The barrier for  $a_i < b_j$ .

The barrier for  $a_i < b_j$  forces  $\pi(\lambda)$  to reach a lattice point with  $x$ -coordinate  $i$  before it reaches one with  $y$ -coordinate  $j$ , i.e.,  $a_i$  occurs before  $b_j$  on  $\pi(\lambda)$ . This is precisely what is needed for  $\lambda(a_i) < \lambda(b_j)$ .

In a similar manner,  $b_j < a_i$  corresponds to a rectangular barrier consisting of all  $(x, y)$  in  $L$  for which  $x \geq i - 1$  and  $y \leq j$ . For  $\lambda$  to be a linear extension of  $P$ ,  $\pi(\lambda)$  must not penetrate *any* of the barriers formed from the cross-relations in  $(P, <)$ . Fig. 3 shows the union of the barriers for  $(P_0, <)$  from Fig. 1.

The next point we consider is how  $\lambda(x^*) = k$  is reflected in  $\pi(\lambda)$ . Without loss of generality, we assume that  $x^* = a_i$ , so that  $x^* \in A$ . Then it is easy to see that  $\lambda(a_i) = k$  iff  $\pi(\lambda)$  contains the two points  $(i - 1, k - i)$  and  $(i, k - i)$ . (Similarly,  $\lambda(b_j) = k$  iff  $\pi(\lambda)$  contains  $(k - j, j - 1)$  and  $(k - j, j)$ .)

Suppose  $N_{k-1}$  and  $N_{k+1}$  are both positive, and let  $\lambda^+$  and  $\lambda^-$  be linear extensions of  $P$  such that  $\lambda^+(a_i) = k + 1$  and  $\lambda^-(a_i) = k - 1$ . Thus,  $\pi(\lambda^+)$  contains points  $(i - 1, k + 1 -$

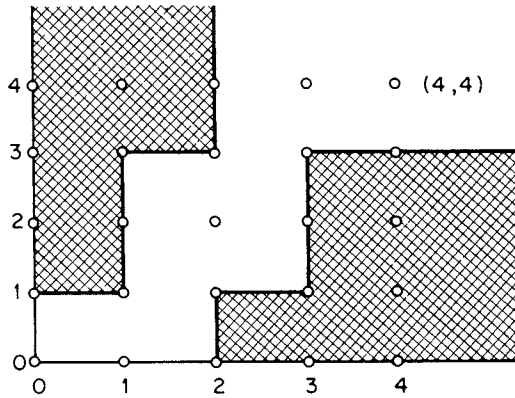


FIG. 3. The union of barriers for  $(P_0, <)$ .

$i$ ) and  $(i, k + 1 - i)$ , and  $\pi(\lambda^-)$  contains  $(i - 1, k - 1 - i)$  and  $(i, k - 1 - i)$ . Let  $x_0$  be the largest integer that is  $\leq i - 1$  such that, for some  $y$ ,  $(x_0, y + 1)$  is on  $\pi(\lambda^+)$  and  $(x_0, y)$  is on  $\pi(\lambda^-)$ , and let  $y_0$ , which cannot exceed  $k - 1 - i$ , be the largest integer such that  $(x_0, y_0 + 1)$  is on  $\pi(\lambda^+)$  and  $(x_0, y_0)$  is on  $\pi(\lambda^-)$ . Similarly, let  $x_1$  be the smallest integer  $\geq i$  such that, for some  $y$ ,  $(x_1, y + 1)$  is on  $\pi(\lambda^+)$  and  $(x_1, y)$  is on  $\pi(\lambda^-)$ , and let  $y_1$ , which cannot be less than  $k - i$ , be the smallest integer such that  $(x_1, y_1 + 1)$  is on  $\pi(\lambda^+)$  and  $(x_1, y_1)$  is on  $\pi(\lambda^-)$ .

We now form two new lattice paths  $\pi(\lambda_1)$  and  $\pi(\lambda_2)$  as follows. Let  $\pi(\lambda_1)$  consist of the points on  $\pi(\lambda^-)$  from  $(0, 0)$  to  $(x_0, y_0)$ , plus the points on  $\pi(\lambda^+)$  from  $(x_0, y_0 + 1)$  to  $(x_1, y_1 + 1)$  translated by  $-1$  in the  $y$ -direction, plus the points on  $\pi(\lambda^-)$  from  $(x_1, y_1)$  to  $(r, s)$ . Let  $\pi(\lambda_2)$  consist of the points on  $\pi(\lambda^+)$  from  $(0, 0)$  to  $(x_0, y_0 + 1)$ , plus the points on  $\pi(\lambda^-)$  from  $(x_0, y_0)$  to  $(x_1, y_1)$  translated by  $+1$  in the  $y$ -direction, plus the points on  $\pi(\lambda^+)$  from  $(x_1, y_1 + 1)$  to  $(r, s)$ . It is of course possible to have  $\pi(\lambda_1) = \pi(\lambda_2)$ , or, equivalently,  $\lambda_1 = \lambda_2$ , but this will not affect our conclusions. We observe that:

- (i)  $\pi(\lambda_1)$  and  $\pi(\lambda_2)$  are lattice paths from  $(0, 0)$  to  $(r, s)$  which contain  $(i, k - i)$  and  $(i - 1, k - i)$ , and, therefore,  $\lambda_1(a_i) = \lambda_2(a_i) = k$ ;
- (ii) since  $\pi(\lambda^+)$  lies strictly above  $\pi(\lambda^-)$  in the region where the translations occur in the construction, neither  $\pi(\lambda_1)$  nor  $\pi(\lambda_2)$  penetrates any of the barriers formed by  $(P, <)$ . It follows that  $\lambda_1$  and  $\lambda_2$  are linear extensions of  $P$ ;
- (iii) if two ordered pairs of the form  $(\lambda^+, \lambda^-)$  are distinct, then their associated  $(\lambda_1, \lambda_2)$  pairs are distinct. This follows from the construction: if two  $(\pi(\lambda^+), \pi(\lambda^-))$  differ prior to  $i$  on the abscissa, then their associated  $(\pi(\lambda_1), \pi(\lambda_2))$  will differ before  $i$ ; if two  $(\pi(\lambda^+), \pi(\lambda^-))$  differ after  $i - 1$ , then their associated  $(\pi(\lambda_1), \pi(\lambda_2))$  will differ after  $i - 1$ .

Thus, our construction provides an injection from the ordered pairs  $(\lambda^+, \lambda^-)$  into pairs  $(\lambda_1, \lambda_2)$ , where  $\lambda^+$  and  $\lambda^-$  are any linear extensions of  $P$  for which  $\lambda^+(a_i) = k + 1$  and  $\lambda^-(a_i) = k - 1$ , and  $\lambda_1$  and  $\lambda_2$  are linear extensions of  $P$  that satisfy  $\lambda_1(a_i) = \lambda_2(a_i) = k$ . If  $\alpha$ ,  $\beta$  and  $\gamma$  are the number of linear extensions of  $P$  for which  $\lambda(a_i) = k + 1$ ,  $\lambda(a_i) = k - 1$ , and  $\lambda(a_i) = k$ , respectively, then such an injection requires  $\gamma^2 \geq \alpha\beta$ , for otherwise two  $(\lambda_1, \lambda_2)$  pairs associated with distinct  $(\lambda^+, \lambda^-)$  pairs would have to be identical.

The preceding argument applies analogously when  $x^* = b_j$ . Thus, we have proved the following result.

**THEOREM 1.** *Let  $x^*$  be a fixed element in a partially ordered set  $(P, <)$  on  $n$  elements, and suppose  $(P, <)$  can be covered by two chains. For  $k \in \{1, 2, \dots, n\}$ , let  $N_k$  be the*

number of linear extensions  $\lambda : P \rightarrow \{1, 2, \dots, n\}$  for which  $\lambda(x^*) = k$ . Then

$$N_k^2 \geq N_{k-1}N_{k+1} \quad \text{for } k = 2, \dots, n-1.$$

**COROLLARY.** *Given the hypotheses of Theorem 1, the sequence  $N_1, N_2, \dots, N_n$  is unimodal.*

The same basic argument for Theorem 1 can be used to prove the following result for sequences of integers. Let  $A = (a_1 \geq a_2 \geq \dots)$  be a nonincreasing sequence of nonnegative integers. Given  $A$ , let  $S_n$  be the number of nonincreasing sequences  $x = (x_1 \geq x_2 \geq \dots \geq x_n)$  of integers for which  $0 \leq x_k \leq a_k$ , for  $k = 1, \dots, n$ .

**THEOREM 2.** *The sequence  $S_1, S_2, \dots$  is logarithmically concave, i.e.,*

$$S_n^2 \geq S_{n-1}S_{n+1} \quad \text{for all } n \geq 2.$$

When  $A$  is constant, say  $A = (t, t, t, \dots)$ , Theorem 2 shows the (easily proved) logarithmic concavity of the binomial coefficients  $\binom{t+k}{k}$  for  $k = 1, 2, \dots$ .

**3. A generalization.** We now generalize our analysis of logarithmic concavity by considering disjoint partial orders along with linear extensions that map  $P$  into  $[m] \equiv \{1, \dots, m\}$  when  $m$  exceeds the cardinality of  $P$ . The following lemma provides a basis for the generalization.

**LEMMA.** *Let  $(P, <)$  and  $(P \cup C, <)$  be partially ordered sets on  $n$  and  $n + \alpha$  elements, respectively, that have the same ordered pairs in their partial orders with  $C \cap P = \emptyset$ . Let  $x^* \in P$  be fixed, and let  $N_k$  and  $N'_k$ , respectively, be the number of linear extensions  $\lambda : P \rightarrow [n]$  and  $\lambda' : P \cup C \rightarrow [n + \alpha]$  that have  $\lambda(x^*) = k$  and  $\lambda'(x^*) = k$ . If  $N_1, \dots, N_n$  is logarithmically concave, then so is  $N'_1, \dots, N'_{n+\alpha}$ .*

If  $C$  is empty, there is nothing to prove; so suppose initially that  $C = \{c\}$ , with  $\alpha = 1$ . Since neither  $c < x$  nor  $x < c$  for each  $x \in P$ , each  $\lambda$  for  $P$  generates  $n + 1$   $\lambda'$  for  $P \cup \{c\}$  according to the  $n + 1$  placements of  $c$ . With  $N_0 = N_{n+1} = 0$ ,

$$N'_k = (k - 1)N_{k-1} + (n - k + 1)N_k \quad \text{for } k = 1, \dots, n + 1.$$

Using this relationship,  $(N'_k)^2 - N'_{k-1}N'_{k+1}$ , for  $2 \leq k \leq n$ , reduces to

$$k(k - 2)[N_{k-1}^2 - N_{k-2}N_k] + (n - k)(n - k + 2)[N_k^2 - N_{k-1}N_{k+1}] \\ + (k - 2)(n - k)[N_{k-1}N_k - N_{k-2}N_{k+1}] + (N_{k-1} - N_k)^2,$$

which must be nonnegative if  $\{N_k\}$  is logarithmically concave.

This completes the proof of the lemma if  $\alpha \leq 1$ , so suppose in this paragraph that  $\alpha \geq 2$  with  $C = \{c_1, \dots, c_\alpha\}$ . The  $\lambda' : P \cup C \rightarrow [n + \alpha]$  can be generated from the  $\lambda : P \rightarrow [n]$  by adding one  $c_i$  at a time. For a given  $\lambda$ , we first add  $c_1$  to obtain  $n + 1$  linear extensions from  $P \cup \{c_1\}$  onto  $[n + 1]$ ; for each of these  $n + 1$ , we then add  $c_2$  to obtain  $n + 2$  linear extensions from  $P \cup \{c_1, c_2\}$  onto  $[n + 2]$ ; and so forth. If  $\{N_m\}$  is logarithmically concave, then successive applications of the result obtained in the preceding paragraph for each  $c_i$  addition show that  $\{N'_k\}$  must be logarithmically concave. The lemma is thus proved.

We now state our generalization, discuss its features, and then conclude this section with its proof.

**THEOREM 3.** *Suppose  $(P_1, <_1)$ ,  $(P_2, <_2)$  and  $(P, <)$  are partially ordered sets on  $n_1$ ,  $n_2$  and  $n$  elements respectively such that  $0 < n_1 \leq n$ ,  $P_1 \cup P_2 = P$ ,  $P_1 \cap P_2 = \emptyset$  and  $<_1 \cup <_2 = <$ . Let  $x^* \in P_1$  be fixed, and let  $N_k$  ( $k = 1, \dots, n_1$ ) be the number of linear extensions  $\lambda : P_1 \rightarrow [n_1]$  for which  $\lambda(x^*) = k$ . In addition, given  $m \geq n$ , let  $M_k$  ( $k =$*

$1, \dots, m$ ) be the number of linear extensions  $\lambda^* : P \rightarrow [m]$  for which  $\lambda^*(x^*) = k$ . If  $N_1, \dots, N_{n_1}$  is logarithmically concave, then so is  $M_1, \dots, M_m$ .

When  $n_2 = 0$  and  $m > n$ , this shows that logarithmic concavity for  $\lambda : P \rightarrow [n]$  carries over to  $\lambda^* : P \rightarrow [m]$ . When  $n_2 > 0$  and  $m = n$ , Theorem 3 says that logarithmic concavity for the elements within a part of  $(P, <)$ , namely  $(P_1, <_1)$ , carries over to all of  $(P, <)$  for those same elements, provided that the rest of  $(P, <)$  is not connected to the first part. The combination of these two cases provides the generalization stated in the theorem.

Theorems 1 and 3 together yield the following result.

**THEOREM 4.** *If an  $n$ -element partially ordered set  $(P, <)$  can be partitioned into partially ordered sets  $(P_1, <_1)$  and  $(P_2, <_2)$  with no  $<$ -connection between  $P_1$  and  $P_2$ , if  $(P_1, <_1)$  can be covered by two chains, and if  $x^* \in P_1$ ,  $m \geq n$ , and  $M_k$  is the number of linear extensions  $\lambda : P \rightarrow [m]$  for which  $\lambda(x^*) = k$ , then  $M_1, \dots, M_m$  is logarithmically concave and unimodal.*

We now sketch the proof of Theorem 3 using the notation in its statement. In addition, let  $T_k$  be the number of linear extensions  $\lambda_0 : P \rightarrow [n]$  for which  $\lambda_0(x^*) = k$ , and if  $n_2 > 0$ , let  $\beta$  be the number of linear extensions  $\lambda_2 : P_2 \rightarrow [n_2]$ , and let  $N'_k$  be the number of linear extensions  $\lambda' : P_1 \cup C \rightarrow [n]$  that have  $\lambda'(x^*) = k$  when  $C$  is a completely unordered  $n_2$ -element set (see the lemma) that is disjoint from  $P_1$ .

If  $n_2 = 0$  then  $T_k = N_k$ , so assume henceforth in this paragraph that  $n_2 > 0$ . We shall apply the lemma with  $\alpha = n_2$ . Consider a fixed  $\lambda_2 : P_2 \rightarrow [n_2]$  along with a generic  $\lambda_1 : P_1 \rightarrow [n]$ . The  $n_2$  numbers in  $[n]$  that are not in  $\lambda_1(P_1)$  can be bijectively assigned to the elements in  $P_2$  in exactly one way that preserves the  $\lambda_2$  order and yields a  $\lambda_0 : P \rightarrow [n]$ —as compared to the  $n_2!$  ways this could be done for the unordered set  $C$ . Since this is true for each such  $\lambda_1$ , it follows that the number of  $\lambda_0 : P \rightarrow [n]$  that have  $\lambda_0(x^*) = k$  and have  $P_2$  in its  $\lambda_2$  order is  $N'_k/n_2!$ . Since there are  $\beta$  such  $\lambda_2$ ,  $T_k = \beta N'_k/n_2!$ . If  $N_1, \dots, N_{n_1}$  is logarithmically concave, then the lemma says that  $T_1, \dots, T_n$  is too.

This proves Theorem 3 if  $m = n$ . If  $m > n$ , we reapply the lemma with  $\alpha = m - n$ . In this case let  $C'$  be a completely unordered  $(m - n)$ -element set disjoint from  $P$  and, with respect to  $(P \cup C', <)$ , let  $T'_k$  be the number of linear extensions  $\lambda' : P \cup C' \rightarrow [m]$  for which  $\lambda'(x^*) = k$ . By the lemma, if  $\{T_k\}$  is logarithmically concave then so is  $\{T'_k\}$ . Since the  $m - n$  numbers in  $[m]$  that aren't in a  $\lambda'(P)$  can be bijectively assigned to  $C'$  in  $(m - n)!$  ways, it follows that  $M_k$  as defined in Theorem 3 equals  $T'_k/(m - n)!$ . When this is combined with preceding conclusions, we see that if  $N_1, \dots, N_{n_1}$  is logarithmically concave, then so is  $M_1, \dots, M_m$ .

**4. Concluding remarks.** The preceding techniques can be used to prove other unimodality results for restricted lattice path problems. For example, consider lattice paths  $\pi$  that are not allowed to penetrate barriers of the type shown in Fig. 3, so that  $\pi$  is bounded between two increasing staircases. Let  $D_{n,k}$  be the number of such paths that go through point  $(k, n - k)$ . Then, for each  $n$ , the sequence  $D_{n,k}$ ,  $0 \leq k \leq n$ , is logarithmically concave and therefore unimodal. (Of course, here we are just looking at the intersections of lattice paths with the line  $x + y = n$ .) The reader is referred to the recent paper of Graham, Yao, and Yao [1] for similar applications of these ideas.

Finally, we note another open conjecture that is suggested by our analysis. Within the context used for the earlier conjecture, we propose:

**CONJECTURE\*.** *The sequence  $N_k$ ,  $k \in [n]$ , is logarithmically concave.*

Conjecture\* is stronger than Rivest's Conjecture since unimodality follows from logarithmic concavity, but not conversely. Thus, a counterexample for Conjecture\* need not disprove unimodality, while verification of Conjecture\* would establish Rivest's Conjecture.

*Note added in proof.* R. Stanley has just proved Conjecture\* using a very ingenious application of the Alexandroff-Fenchel theorem (which guarantees the logarithmic concavity of certain coefficients arising from the volume of weighted sums of  $n$ -dimensional polytopes).

## REFERENCES

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