ON UNIQUENESS FOR TIME HARMONIC ANISOTROPIC MAXWELL'S EQUATIONS WITH PIECEWISE REGULAR COEFFICIENTS

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ABSTRACT. We are interested in the uniqueness of solutions to Maxwell's equations when the magnetic permeability μ and the permittivity ε are symmetric positive definite matrix-valued functions in $\mathbb{R}^3$. We show that a unique continuation result for globally $W^{1,\infty}$ coefficients in a smooth, bounded domain, allows one to prove that the solution is unique in the case of coefficients which are piecewise $W^{1,\infty}$ with respect to a suitable countable collection of sub-domains with C^0 boundaries. Such suitable collections include any bounded finite collection. The proof relies on a general argument, not specific to Maxwell's equations. This result is then extended to the case when within these sub-domains the permeability and permittivity are only L^∞ in sets of small measure.

1. INTRODUCTION

Suppose we are given a time-harmonic incident electric field $\mathscr{E}^i$ and magnetic field $\mathscr{H}^i$, special solutions of the time-harmonic homogeneous linear Maxwell equations of the form $\mathscr{E}^i = \Re\left(\mathbf{E}^i e^{-i\omega t}\right)$ and magnetic field $\mathscr{H}^i = \Re\left(\mathbf{H}^i e^{-i\omega t}\right)$, where $\mathbf{E}^i \in H^1_{\rm loc}\left(\mathbb{R}^3\right)^3$ and $\mathbf{H}^i \in H^1_{\rm loc}\left(\mathbb{R}^3\right)^3$ are complex-valued solutions of the homogeneous time-harmonic Maxwell equations

$$\nabla \wedge \mathbf{E}^i - i\,\omega\mu_0\mathbf{H}^i = 0 \text{ in } \mathbb{R}^3,$$
$$\nabla \wedge \mathbf{H}^i + i\,\omega\varepsilon_0\mathbf{E}^i = 0 \text{ in } \mathbb{R}^3,$$

where μ_0 and ε_0 are positive constants, representing respectively the magnetic permeability and the electric permittivity of vacuum, and $\omega \in \mathbb{R}\setminus\{0\}$. The full time-harmonic electromagnetic field $(\mathbf{E},\mathbf{H}) \in H_{\rm loc}\left(\mathrm{curl};\mathbb{R}^3\right)$, where for any domain W we define

$$H_{\rm loc}\left(\mathrm{curl};W\right) := \left\{\mathbf{u} \in L^2_{\rm loc}\left(W\right)^3 \text{ such that } \nabla\wedge\mathbf{u} \in L^2_{\rm loc}\left(W\right)^3\right\},$$

satisfies Maxwell's equations

$$\begin{aligned} \nabla \wedge \mathbf{E} - i\,\omega\mu_0\mu(x)\,\mathbf{H} &= 0 \text{ in } \mathbb{R}^3,\\ \nabla \wedge \mathbf{H} + i\,\omega\varepsilon_0\varepsilon(x)\,\mathbf{E} &= 0 \text{ in } \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where ε and μ are real matrix-valued functions in $L^\infty\left(\mathbb{R}^3\right)^{3\times3}$. Decomposing the full electromagnetic field into its incident part and its scattered part,

$$\mathbf{E}^s := \mathbf{E} - \mathbf{E}^i, \text{ and } \mathbf{H}^s := \mathbf{H} - \mathbf{H}^i, \tag{1.2}$$

we assume that the scattered field satisfies the Silver-Müller radiation condition, uniformly in all directions, that is, if $x := r\theta$ then

$$\lim_{r\to\infty} \sup_{\theta\in S^2} \left|\mathbf{H}^s(r\theta)\wedge r\theta - r\mathbf{E}^s(r\theta)\right| = 0, \tag{1.3}$$

where $S^2 := \{x \in \mathbb{R}^3 \text{ such that } |x| = 1\}$ denotes the unit sphere.

This paper is about the existence of a unique solution to (1.1) satisfying (1.2) and (1.3), under the following additional hypotheses on ε and μ. We assume that both permittivity and permeability are real symmetric, uniformly positive definite and bounded, that is, there exist $0 < \alpha \le \beta < \infty$ such that for all $\xi \in \mathbb{R}^3$ and almost every $x \in \mathbb{R}^3$,

$$\alpha |\xi|^2 \le \varepsilon(x)\xi\cdot\xi \le \beta|\xi|^2, \tag{1.4}$$

$$\alpha |\xi|^2 \le \mu(x)\xi\cdot\xi \le \beta|\xi|^2. \tag{1.5}$$

We suppose that ε and μ vary only in an open bounded domain Ω, so that

$$\varepsilon = \mu = \mathbf{I}_3 \text{ in } \Omega^c = \mathbb{R}^3 \setminus \Omega, \tag{1.6}$$

where $\mathbf{I}_3$ is the identity matrix in $\mathbb{R}^{3\times3}$. We assume that Ω is of the form

$$\Omega = \operatorname{int}\left(\underset{i\in I}{\cup}\bar{\Omega}_i\right), \tag{1.7}$$

where the sub-domains Ω_i, $i \in I \subset \mathbb{N}$ are disjoint and of class C^0, and int denotes the interior. The permittivity ε and the permeability μ are assumed to be piecewise $W^{1,\infty}$ with respect to the sub-domains Ω_i, so that for each $i \in I$, there exist $\varepsilon_i, \mu_i \in W^{1,\infty}\left(\mathbb{R}^3\right)^{3\times3}$ satisfying (1.4)-(1.5) and

$$\|\varepsilon_i\|_{W^{1,\infty}(\mathbb{R}^3)^{3\times3}} + \|\mu_i\|_{W^{1,\infty}(\mathbb{R}^3)^{3\times3}} \le M_i, \tag{1.8}$$

where $M_i > 0$ is a positive constant, such that

$$\varepsilon(x) = \varepsilon_i(x) \text{ and } \mu(x) = \mu_i(x), \quad \text{a.e. } x \in \Omega_i. \tag{1.9}$$

Given a bounded set $A \subset \mathbb{R}^3$, we write $\mathcal{U}(A)$ as the (unique) unbounded component of $\bar{A}^c$.

Assumption 1. *For any $J \subset I$, and $\Omega_J = \operatorname{int}\left(\underset{j\in J}{\cup}\bar{\Omega}_j\right)$, there exists $j_0 \in J$ such that $\partial\mathcal{U}(\Omega_J) \cap \partial\Omega_{j_0}$ admits an interior point relative to $\partial\mathcal{U}(\Omega_J)$. In other words, there exist $j_0 \in J$ and $x_0 \in \partial\mathcal{U}(\Omega_J) \cap \partial\Omega_{j_0}$ such that $B(x_0,\delta) \cap \partial\mathcal{U}(\Omega_J) \subset \partial\Omega_{j_0}$ for some $\delta > 0$.*

Proposition 1. *Assumption 1 holds if for all $J \subset I$, there exist $x_J \in \partial\mathcal{U}(\Omega_J)$ and $\delta_J > 0$ such that $B(x_J,\delta_J) \cap \Omega_j \ne \emptyset$ for only finitely many $j \in J$. In particular, Assumption 1 holds when I is finite.*

Proof. Given J, $x_J \in \partial\mathcal{U}(\Omega_J)$ and δ_J as in the statement of the proposition let $B_J = B(x_J,\delta_J)$ and let J' be the finite subset of J such that $B_J \cap \underset{j\in J}{\cup}\Omega_j = B_J \cap \underset{j\in J'}{\cup}\Omega_j$. We first show that

$$\partial\mathcal{U}(\Omega_J) \cap B_J = \underset{j\in J}{\cup}\left(\partial\mathcal{U}(\Omega_J) \cap B_J\right) \cap \partial\Omega_j. \tag{1.10}$$

Indeed, let $x \in \partial\mathcal{U}(\Omega_J) \cap B_J$. Then $x \notin \underset{j\in J}{\cup}\Omega_j$. We claim that there exists a sequence $x_k \in \underset{j\in J}{\cup}\Omega_j$ such that x_k tends to x. If not, for some $\eta > 0$ sufficiently

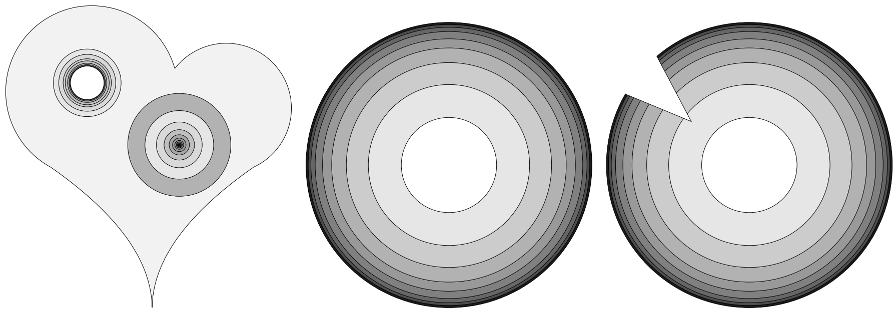

FIGURE 1. Left: an infinite collection of sub-domains satisfying Assumption 1. Centre: an infinite collection of sub-domains excluded by Assumption 1. Right: this collection satisfies Assumption 1.

small, we would have $B(x,\eta) \subset B_J$, and $B(x,\eta) \cap \bigcup_{j\in J} \Omega_j = B(x,\eta) \cap \bigcup_{j\in J'} \Omega_j = \emptyset$. On the other hand, there exists a sequence $y_k \in \mathcal{U}(\Omega_J)$ such that y_k tends to x. But $B(x,\eta)$ is connected and contained in $\bar{\Omega}_J^c$, thus $B(x,\eta) \subset \mathcal{U}(\Omega_J)$. This contradiction proves the claim.

Next, we note that $\partial\mathcal{U}(\Omega_J)$ is closed, thus complete in the subspace topology induced by $\mathbb{R}^3$. Its intersection with the open ball B_J is an open subspace of $\partial\mathcal{U}(\Omega_J)$ by definition of the subspace topology. It is therefore a Baire space (see e.g. [11]). If a Baire Space is a countable union of closed sets, then one of the sets has an interior point. Using the identity (1.10), we obtain that there exists j_0 such that $\partial\mathcal{U}(\Omega_J) \cap B_J \cap \partial\Omega_{j_0}$ admits an interior point relative to $\partial\mathcal{U}(\Omega_J) \cap B_J$, that is, there exist $j_0 \in J$, $x_0 \in \partial\mathcal{U}(\Omega_J) \cap B_J$ and $\delta > 0$ such that $B(x_0,\delta) \cap \partial\mathcal{U}(\Omega_J) \cap B_J \subset \partial\Omega_{j_0}$.

Since B_J is open, $B(x_0,\delta) \cap B_J = B(x_0,\delta)$ when δ is sufficiently small, and we have established that Assumption 1 holds. □

An example of a collection of sub-domains excluded by Assumption 1 is a collection of concentric shells concentrating on an exterior boundary, such as

$$\Omega_i = B\left(\mathbf{0}, \frac{i}{i+1}\right) \setminus \bar{B}\left(\mathbf{0}, \frac{i-1}{i}\right), \quad i = 1, 2, 3, \ldots \tag{1.11}$$

In such a case, $\partial\mathcal{U}(\Omega)$ is the unit sphere, which is not the boundary of any of the subsets. On the other hand, Assumption 1 allows the sub-domains Ω_i to concentrate at a point or near an interior boundary. In Figure 1, we represent on the left a non-Lipschitz non-simply connected domain Ω which satisfies Assumption 1. In the centre, the domain given by (1.11) excluded by Assumption 1 is shown. On the right, we sketch a domain inspired by the one described by (1.11) which satisfies Assumption 1: near the accumulating boundary, interior points can be found on the wedge-shaped slit in the domain.

2. Main result

Our main result is the following theorem.

Theorem 2. *Assume that* (1.4)–(1.9) *and Assumption 1 hold. If for a given* $\omega \neq 0$, $\mathbf{E} \in H_{\text{loc}}\left(\text{curl}; \mathbb{R}^3\right)$ *and* $\mathbf{H} \in H_{\text{loc}}\left(\text{curl}; \mathbb{R}^3\right)$ *are solutions of* (1.1)–(1.3) *corresponding to* $\mathbf{E}^i = 0$ *and* $\mathbf{H}^i = 0$, *then* $\mathbf{E} = \mathbf{H} = 0$.

There is a very long history concerning this problem, under various assumptions on the coefficients, see e.g. [1, 2, 3, 6, 7, 10, 13, 12, 14, 16] and the references therein. The improvement provided by the result in this work is that we assume that ε, μ are matrix-valued functions and that the sub-domains Ω_i are only of class C^0. We do not assume that the sub-domains are Lipschitz as assumed for example in [6] for the isotropic (scalar) case. The authors are not aware of the existence of a general uniqueness result for the above problem when the coefficients are just $C^{0,\alpha}$ Hölder continuous, with $\alpha < 1$. For general elliptic equations, counter-examples to unique continuation, the main technique for proving uniqueness, are known in that case, see [8]. We remind the reader of the definition of a domain of class C^0.

Definition 3. *A bounded domain Ω of $\mathbb{R}^3$ is of class C^0 if for any point x_0 on the boundary $\partial\Omega$, there exists a ball $B(x_0,\delta)$ and an orthogonal coordinate system (x_1,x_2,x_3) with origin at x_0 such that there exists a continuous function $f : C^0\left(\mathbb{R}^2;\mathbb{R}\right)$ that satisfies*

$$\Omega \cap B(x_0,\delta) = \{x \in B(x_0,\delta) : x_3 > f(x_1,x_2)\}.$$

We define B_0 as the smallest open ball containing Ω. Note that the uniqueness of the solution outside B_0 is well known, due to the so-called Rellich's Lemma, see e.g. [2].

Lemma 4 (Rellich's Lemma)**.** *If for a fixed ω, $\mathbf{E} \in H_{\text{loc}}\left(\text{curl};\mathbb{R}^3\right)$ and $\mathbf{H} \in H_{\text{loc}}\left(\text{curl};\mathbb{R}^3\right)$ are solutions of* (1.1)-(1.3) *corresponding to $\mathbf{E}^i = 0$ and $\mathbf{H}^i = 0$, then $\mathbf{E} = \mathbf{H} = 0$ in $\bar{B}_0^c$.*

Our proof relies on a recent unique continuation result [13] proved for globally $W^{1,\infty}$ regular coefficients.

Theorem 5 ([13])**.** *Let V be a connected open set in $\mathbb{R}^3$. Assume that ε and μ are two real symmetric matrix valued functions in V satisfying* (1.4)–(1.5)*, and*

$$\|\varepsilon\|_{W^{1,\infty}(V)^{3\times 3}} + \|\mu\|_{W^{1,\infty}(V)^{3\times 3}} \leq M,$$

for some constant $M > 0$. Suppose $(\mathbf{E},\mathbf{H}) \in \left(L^2_{\text{loc}}(V)\right)^2$ satisfy

$$\nabla \wedge \mathbf{E} - i\,\omega\mu_0\mu(x)\,\mathbf{H} = 0 \text{ in } V,$$
$$\nabla \wedge \mathbf{H} + i\,\omega\varepsilon_0\varepsilon(x)\,\mathbf{E} = 0 \text{ in } V.$$

Then, there exist $s > 0$ independent of V, $\mathbf{E}$ and $\mathbf{H}$, such that if for some $x_0 \in V$, and for all $N \in \mathbb{N}$ and all $\delta > 0$ sufficiently small,

$$\int_{B(x_0,\delta)} \left(|\mathbf{E}|^2 + |\mathbf{H}|^2\right) dx \leq C_N \exp\left(-N\delta^{-s}\right)$$

for some constant $C_N > 0$, then $\mathbf{E} = \mathbf{H} = 0$ in V.

The proof of Theorem 2 consists of three steps. The first two steps are given by the two propositions below.

Proposition 6. *Under the hypothesis of Theorem 2, suppose that $A \subset \mathbb{R}^3$ is a bounded open set and that for almost every $x \in \mathcal{U}(A)$ either $\varepsilon(x) = \mu(x) = \mathbf{I}_3$ or $\mathbf{E}(x) = \mathbf{H}(x) = 0$. Then $\mathbf{E} = \mathbf{H} = 0$ in $\mathcal{U}(A)$.*

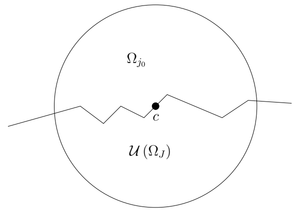

FIGURE 2. A ball centred at an interior point on the boundary of $\mathcal{U}(\Omega_J)$.

Proof. For any $\mathbf{v}, \mathbf{w} \in L^2(\mathcal{U}(A))^2$, we have

$$\int_{\mathcal{U}(A)} \nabla \wedge \mathbf{E} \cdot \mathbf{v}\, dx - i\omega\mu_0 \int_{\mathcal{U}(A)} \mu(x)\mathbf{H} \cdot \mathbf{v}\, dx = 0, \tag{2.1}$$

$$\int_{\mathcal{U}(A)} \nabla \wedge \mathbf{H} \cdot \mathbf{w}\, dx + i\omega\varepsilon_0 \int_{\mathcal{U}(A)} \epsilon(x)\mathbf{E} \cdot \mathbf{w}\, dx = 0, \tag{2.2}$$

where the integrals (2.1) and (2.2) are well defined by Rellich's Lemma 4. Since for almost every x in $\mathcal{U}(A)$, either $\varepsilon(x) = \mu(x) = \mathbf{I}_3$ or $\mathbf{E} = \mathbf{H} = 0$, the solutions of the system (2.1)-(2.2) can be written also in the form

$$\int_{\mathcal{U}(A)} \nabla \wedge \mathbf{E} \cdot \mathbf{v}\, dx - i\omega\mu_0 \int_{\mathcal{U}(A)} \mathbf{H} \cdot \mathbf{v}\, dx = 0,$$
$$\int_{\mathcal{U}(A)} \nabla \wedge \mathbf{H} \cdot \mathbf{w}\, dx + i\omega\varepsilon_0 \int_{\mathcal{U}(A)} \mathbf{E} \cdot \mathbf{w}\, dx = 0,$$

which is the weak formulation of

$$\nabla \wedge \mathbf{E} - i\,\omega\mu_0\,\mathbf{H} = 0 \text{ in } \mathcal{U}(A),$$
$$\nabla \wedge \mathbf{H} + i\,\omega\varepsilon_0\,\mathbf{E} = 0 \text{ in } \mathcal{U}(A).$$

Next, since A is bounded, thanks to Rellich's Lemma 4, $\mathbf{E} = \mathbf{H} = 0$ in $\mathcal{U}(A) \cap \left(\mathbb{R}^3 \setminus \bar{B}(R)\right)$, for R large enough. In particular, $\mathbf{E}$ and $\mathbf{H}$ vanish in a ball contained in $\mathcal{U}(A)$, which is open and connected, and the conclusion follows from Theorem 5, applied with $\varepsilon(x) = \mu(x) = \mathbf{I}_3$, which in this case reduces to a well known result concerning the Helmholtz equation. □

Proposition 7. *Let*

$$J := \{i \in I : |\mathbf{E}(x)|^2 + |\mathbf{H}(x)|^2 > 0 \textit{ on a set of positive measure in } \Omega_i\}.$$

Then $J = \emptyset$.

Proof. Suppose for contradiction that J is nonempty. Then, by Assumption 1 there exists $x_0 \in \partial\mathcal{U}(\Omega_J) \cap \partial\Omega_{j_0}$ such that $B(x_0, \delta) \cap \partial\mathcal{U}(\Omega_J) \subset \partial\Omega_{j_0}$ for some $j_0 \in J$ and $\delta > 0$. To simplify notation, set $j_0 = 1$.

Let us show that there exist a point c on $\partial\mathcal{U}(\Omega_J) \cap \partial\Omega_1$ and a radius $\tilde{\delta} > 0$ such that

$$\Omega_J \cap B(c, \tilde{\delta}) = \Omega_1 \cap B(c, \tilde{\delta}). \tag{2.3}$$

Figure 2 sketches the configuration we have at hand around c.

Since Ω_1 has a C^0 boundary, for some (smaller) $\delta > 0$ there exists a continuous map f and a suitable orientation of axes such that $B(x_0, \delta) \cap \partial\Omega_J \subset \partial\Omega_1$ and

$$\Omega_1 \cap B(x_0, \delta) = \{x \in B(x_0, \delta) : x_3 > f(x_1, x_2)\}.$$

This alone does not prove our claim, since $B(x_0,\delta)$ could still intersect Ω_J when $x_3 \leq f(x_1,x_2)$. Since $x_0 \in \partial\mathcal{U}(\Omega_J)$, there exists a sequence $\{y_j\} \subset \mathcal{U}(\Omega_J) \cap B(x_0,\delta)$ such that y_j tends to x_0. Consider for a fixed and sufficiently large j the line segment $\{y_j + te_3, t \geq 0\}$, and let $\tau > 0$ be the least value of t such that $y_j + te_3 \in \partial\Omega_J$. Then, $y_j + te_3 \notin \bar{\Omega}_J$, for $t < \tau$, and $y_j + \tau e_3 \in \partial\Omega_1$. Hence $y_j + \tau e_3 \notin \underset{k\in J, k>1}{\cup} \Omega_k$. Since the sets Ω_k are disjoint, the line segment does not intersect $\underset{k\in J, k>1}{\cup} \Omega_k$ in $B(x_0,\delta)$. The same argument applies to any line segment $\{z + te_3, t \geq 0\}$ for z sufficiently close to y_j. Introducing $c = y_j + \tau e_3$ we have established that there exists a ball $B\left(c,\tilde{\delta}\right)$ such that $\Omega_1 \cap B\left(c,\tilde{\delta}\right) = \Omega_J \cap B\left(c,\tilde{\delta}\right)$, which is (2.3).

Now, thanks to Proposition 6, and noting that (by Fubini's Theorem) each $\partial\Omega_i$ is of measure zero, $\mathbf{E} = \mathbf{H} = 0$ almost everywhere in $\mathcal{U}(\Omega_J)$. Thus, for almost every $x \in B(c,\tilde{\delta})$, either $\mathbf{E} = \mathbf{H} = 0$ or $\varepsilon(x) = \varepsilon_1(x)$, and $\mu(x) = \mu_1(x)$. Considering the weak formulation of Maxwell's equations, and arguing as in the proof of Proposition 6, we note that $\mathbf{E}$ and $\mathbf{H}$ are weak solutions of

$$\nabla \wedge \mathbf{E} - i\,\omega\mu_0\mu_1(x)\,\mathbf{H} = 0 \text{ in } B(c,\tilde{\delta}),$$
$$\nabla \wedge \mathbf{H} + i\,\omega\varepsilon_0\varepsilon_1(x)\,\mathbf{E} = 0 \text{ in } B(c,\tilde{\delta}),$$

and vanish on the connected non-empty open set $B(c,\tilde{\delta}) \cap \{x_3 < f(x_1,x_2)\}$. Since ϵ_1 and μ_1 satisfy (1.8), that is,

$$\|\varepsilon_1\|_{W^{1,\infty}(\mathbb{R}^3)^{3\times 3}} + \|\mu_1\|_{W^{1,\infty}(\mathbb{R}^3)^{3\times 3}} \leq M_1,$$

Theorem 5 shows that $\mathbf{E} = \mathbf{H} = 0$ in $B(c,\tilde{\delta})$. This in turn shows that $\mathbf{E}$ and $\mathbf{H}$ vanish on a ball inside Ω_1, and applying Theorem 5 in Ω_1 we obtain $\mathbf{E} = \mathbf{H} = 0$ almost everywhere in Ω_1. This contradiction concludes the proof. $\square$

We now turn to the final step. We have obtained that $\mathbf{E} = \mathbf{H} = 0$ almost everywhere in Ω, and therefore either $\mathbf{E} = \mathbf{H} = 0$ or $\varepsilon(x) = \mu(x) = \mathbf{I}_3$ almost everywhere in $\mathbb{R}^3$. Arguing as above, we deduce that $(\mathbf{E},\mathbf{H})$ is a weak solution of (1.1) with $\varepsilon(x) = \mu(x) = \mathbf{I}_3$ everywhere and the conclusion of Theorem 2 follows from Rellich's Lemma.

3. The case of a medium with defects

We extend our result to the case when defects of small measure are allowed in the medium. One application is to liquid crystals (see [15] for more details). Namely, we assume that the permittivity and permeability are of the form

$$\begin{aligned} \varepsilon_D &= (1 - \mathbf{1}_D)\,\varepsilon + \mathbf{1}_D\tilde{\varepsilon}, \\ \mu_D &= (1 - \mathbf{1}_D)\,\mu + \mathbf{1}_D\tilde{\mu}, \end{aligned} \tag{3.1}$$

where ε and μ satisfy (1.4)-(1.9), $\mathbf{1}_D$ is the indicator function of a measurable bounded set D, such that

$$D \subset \underset{i\in I}{\cup}\Omega_i, \quad \overline{D \cap \Omega_i} \subset \Omega_i \text{ and } \Omega_i \setminus \overline{D \cap \Omega_i} \text{ is connected for each } i \in I, \tag{3.2}$$

and $\tilde{\epsilon}$ and $\tilde{\mu}$ are real symmetric positive definite matrices in $L^\infty\left(\mathbb{R}^3\right)^{3\times 3}$ satisfying (1.4)-(1.5).

Theorem 8. *Suppose that the electric and magnetic fields* $\mathbf{E}_D \in H_{\mathrm{loc}}\left(\mathrm{curl};\mathbb{R}^3\right)$ *and* $\mathbf{H}_D \in H_{\mathrm{loc}}\left(\mathrm{curl};\mathbb{R}^3\right)$ *are solutions of*

$$\begin{aligned}\nabla\wedge\mathbf{E}_D - i\,\omega\mu_0\mu_D(x)\,\mathbf{H}_D = 0 \text{ in } \mathbb{R}^3,\\ \nabla\wedge\mathbf{H}_D + i\omega\varepsilon_0\varepsilon_D(x)\,\mathbf{E}_D = 0 \text{ in } \mathbb{R}^3,\end{aligned}\tag{3.3}$$

together with the Silver-Müller radiation condition (1.3)*, and that* ε_D *and* μ_D *are given by* (3.1)*, with* D *satisfying* (3.2)*. Suppose Assumption 1 holds. Then, there exists a constant* $d_0 > 0$ *depending only on the measure* $|B_0|$ *of* B_0*,* $|\omega|$ *and the lower and upper bounds* α *and* β *given in* (1.4)*-*(1.5) *such that if the measure of* D *satisfies* $|D| < d_0$*, then* $\mathbf{E}_D = \mathbf{H}_D = 0$ *almost everywhere.*

To prove Theorem 8, we use the following variant of Theorem 2.

Proposition 9. *Under the same assumptions as Theorem 2, and assuming that* (3.2) *holds,*

$$\operatorname{supp}\mathbf{H}_D \cup \operatorname{supp}\mathbf{E}_D \subset \bar{D}.$$

Proof. The proof follows from that of Theorem 2, since by assumption for each $i \in I$, $\overline{D\cap\Omega_i} \subset \Omega_i$, and the boundary of $\Omega\setminus\Omega_i$ is unaltered by the defects. □

Proof of Theorem 8. Since (3.3) admits a weak formulation, arguing as before we see using Proposition 9 that $\mathbf{E}_D \in H\left(\mathrm{curl};B_0\right)$ and $\mathbf{H}_D \in H\left(\mathrm{curl};B_0\right)$ have compact support in B_0 and are also solutions of

$$\begin{aligned}\nabla\wedge\mathbf{E}_D - i\omega\mu_0\hat{\mu}\,\mathbf{H}_D = 0 \text{ in } B_0,\\ \nabla\wedge\mathbf{H}_D + i\omega\varepsilon_0\hat{\epsilon}\,\mathbf{E}_D = 0 \text{ in } B_0,\end{aligned}$$

where $\hat{\varepsilon} = \mathbf{I}_3 + \mathbf{1}_D\left(\tilde{\varepsilon} - \mathbf{I}_3\right)$, and $\hat{\mu} = \mathbf{I}_3 + \mathbf{1}_D\left(\tilde{\mu} - \mathbf{I}_3\right)$. Note that $i\omega\mu_0\hat{\mu}\,\mathbf{H}_D$ has compact support and is divergence free. Thus the Helmholtz decomposition (see e.g. [4, 5, 9]) of $i\omega\mu_0\hat{\mu}\,\mathbf{H}_D$ shows there exists a unique $\mathbf{A}_H \in H^1\left(B_0\right)^3$ such that $\mathbf{A}_H\cdot\nu = 0$, on ∂B_0, $\mathrm{div}\left(\mathbf{A}_H\right) = 0$ and such that $i\omega\mu_0\hat{\mu}\,\mathbf{H}_D = \nabla\wedge\mathbf{A}_H$. Furthermore, $\mathbf{A}_H$ satisfies

$$\begin{aligned}\left\|\nabla\mathbf{A}_H\right\|_{L^2(B_0)^{3\times 3}} &\leq C\left(\left\|\nabla\wedge\mathbf{A}_H\right\|_{L^2(B_0)^3} + \left\|\mathbf{A}_H\right\|_{L^2(B_0)^3}\right)\\ \text{and } \left\|\mathbf{A}_H\right\|_{L^2(B_0)^3} &\leq C|B_0|^{1/3}\left\|\nabla\wedge\mathbf{A}_H\right\|_{L^2(B_0)^3},\end{aligned}$$

where C is a universal constant. Altogether this yields

$$\left\|\nabla\mathbf{A}_H\right\|_{L^2(B_0)^{3\times 3}} \leq C\beta\mu_0|\omega|\left(|B_0|+1\right)^{1/3}\left\|\mathbf{H}_D\right\|_{L^2(B_0)^3}.\tag{3.4}$$

Since $\mathbf{E}_D - \mathbf{A}_H$ is curl free, we deduce that there exists $p \in H^1\left(B_0\right)$ such that $\mathbf{E}_D = \mathbf{A}_H + \nabla p$, and p is uniquely defined by setting $\int_{B_0} p\,dx = 0$. Noticing that $\hat{\varepsilon}\mathbf{E}_D$ is divergence free, and $\hat{\varepsilon} - \mathbf{I}_3$ is compactly supported in B_0 we have that p is the solution of

$$\begin{aligned}&\mathrm{div}\left(\hat{\varepsilon}\nabla p\right) = -\mathrm{div}\left(\hat{\varepsilon}\mathbf{A}_H\right) \text{ in } B_0,\\ &\nabla p\cdot n = 0 \text{ on } \partial B_0,\\ &\int_{B_0} p\,dx = 0.\end{aligned}$$

Since $\mathbf{A}_H$ is divergence free, the right-hand side becomes

$$-\mathrm{div}\left(\hat{\varepsilon}\mathbf{A}_H\right) = -\mathrm{div}\left(\mathbf{1}_D\left(\tilde{\varepsilon} - \mathbf{I}_3\right)\mathbf{A}_H\right).$$

To proceed, we compute using the Cauchy-Schwarz inequality the following bound

$$\alpha \|\nabla p\|^2_{L^2(B_0)^3} \leq \int_{B_0} \hat{\epsilon}\nabla p \cdot \nabla p \, dx = -\int_{B_0} \mathbf{1}_D \left(\tilde{\varepsilon} - \mathbf{I}_3\right) \mathbf{A}_H \cdot \nabla p \, dx$$
$$\leq (\beta+1) \|\mathbf{A}_H\|_{L^2(D)^3} \|\nabla p\|_{L^2(B_0)^3},$$

and we have obtained that

$$\|\nabla p\|_{L^2(B_0)^3} \leq \frac{\beta+1}{\alpha} \|\mathbf{A}_H\|_{L^2(D)^3}.$$

Next note using Proposition 9 that

$$\|\mathbf{E}_D\|_{L^2(B_0)^3} = \|\mathbf{E}_D\|_{L^2(D)^3} \leq \|\nabla p\|_{L^2(B_0)^3} + \|\mathbf{A}_H\|_{L^2(D)^3} \leq \frac{2\beta+1}{\alpha} \|\mathbf{A}_H\|_{L^2(D)^3}.$$

The Sobolev-Gagliardo-Nirenberg inequality in B_0 shows that

$$\|\mathbf{A}_H\|_{L^6(B_0)^3} \leq C \left(|B_0|+1\right)^{1/3} \|\mathbf{A}_H\|_{H^1(B_0)^3},$$

where C is a universal constant. Therefore, using Hölder's inequality, together with the Poincaré-Friedrichs estimate (3.4), we have

$$\|\mathbf{A}_H\|_{L^2(D)^3} \leq |D|^{\frac{1}{3}} \|\mathbf{A}_H\|_{L^6(B_0)^3} \leq C\beta \left(|B_0|+1\right)^{2/3} \mu_0|\omega| \, |D|^{\frac{1}{3}} \|\mathbf{H}_D\|_{L^2(B_0)^3}.$$

Altogether we have obtained

$$\|\mathbf{E}_D\|_{L^2(B_0)^3} \leq C\frac{\beta(\beta+1)}{\alpha} \left(|B_0|+1\right)^{2/3} \mu_0|\omega| \, |D|^{\frac{1}{3}} \|\mathbf{H}_D\|_{L^2(B_0)^3}. \tag{3.5}$$

Repeating the same argument, but starting with $\mathbf{H}_D$, we obtain also

$$\|\mathbf{H}_D\|_{L^2(B_0)^3} \leq C\frac{\beta(\beta+1)}{\alpha} \left(|B_0|+1\right)^{2/3} \varepsilon_0|\omega| \, |D|^{\frac{1}{3}} \|\mathbf{E}_D\|_{L^2(B_0)^3}. \tag{3.6}$$

The inequalities (3.5) and (3.6) imply that $\mathbf{H}_D = \mathbf{E}_D = 0$ when

$$|D| < d_0 := C\frac{\alpha^3}{\beta^3(\beta+1)^3(|B_0|+1)^2\left(\sqrt{\varepsilon_0\mu_0}|\omega|\right)^3}, \tag{3.7}$$

where C is a universal constant. □

Remark 10. The dependence of the threshold constant d_0 given by (3.7) on $|\omega|$ and $|B_0|$ shows that for a permeability μ and a permittivity ε satisfying (1.4), (1.5) and (1.6) only, uniqueness for Maxwell's equations holds provided, if ω is fixed, the domain Ω is of small measure and bounded diameter, or, for a given Ω, when the absolute value of the frequency $|\omega|$ is sufficiently small. In such cases, the whole domain Ω can be taken as a defect D (and a fictitious ball containing D plays the role of Ω). We do not claim that the dependence of d_0 in terms of $|\omega|$ or $|B_0|$ in (3.7) is optimal. In contrast, Theorem 2 requires additional regularity assumptions on μ and ε, but does not depend on the frequency or the size of the domain.

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