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On Uniqueness of KMS States of One-dimensional Quantum Lattice Systems

Huzihiro Araki*

Institut für Theoretische Physik der Universität, Göttingen, Federal Republic of Germany

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Abstract. Uniqueness of KMS states is proved for one-dimensional quantum lattice system. Sakai's theorem on uniqueness of KMS states is generalized to cases of non-commutative generators.

§ 1. Introduction

Uniqueness of equilibrium states for one-dimensional lattice system has been proved by Ruelle [7] for classical interactions and by Araki [1] for quantum interactions with a finite-range interaction. Simpler proofs have since been given for these cases (for example, see [8]. Also see Theorem 2 in [5]). It amounts to showing that any two states φ_1 and φ_2 satisfying the KMS condition are majorized by each other: $\varphi_1 \leqq \lambda\varphi_2 \leqq \lambda^2\varphi_1$ for some $\lambda > 0$.

We present here a proof of the uniqueness for one-dimensional quantum lattice system with an interaction Φ, which satisfies the same type of condition as known classical cases, namely surface energy has a bound independent of the volume. The key argument in the proof is Lemma 2 which states roughly that if the relative entropy of a state φ_1 with respect to a state φ_2 is finite, then the associated representation π_1 quasi-contains π_2.

To state the result more precisely, we use the following notation: The C^*-algebra $\mathfrak{A}$ under investigation will have the following structure as usual: For each integer v, $\mathfrak{A}$ has a subalgebra $\mathfrak{A}_v$ mutually commuting for different v. For any subset I of the set Z of all integers, $\mathfrak{A}(I)$ denotes the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_v$, $v \in I$. We assume that each $\mathfrak{A}_v$ is a type I finite factor and $\mathfrak{A}(Z) = \mathfrak{A}$. For each finite subset Λ of Z, an interaction potential $\Phi(\Lambda) \in \mathfrak{A}(\Lambda)$ is given such that

(0) $\Phi(\emptyset) = 0$,

(1) $\|\Phi\|_\alpha \equiv \sup_v \sum_\Lambda \{e^{\alpha N(\Lambda)} \|\Phi(\Lambda)\|\,;\, v \in \Lambda\} < \infty$,

where $N(\Lambda)$ denotes the number of points in Λ and $\alpha > 0$,

(2) the following element $W(\Lambda_n)$ of $\mathfrak{A}$ for an increasing sequence of finite subsets Λ_n of Z is bounded in norm uniformly in n:

$$W(\Lambda) \equiv \sum_J \{\Phi(J);\, J \subset\subset Z,\, J \cap \Lambda \neq \emptyset,\, J \cap \Lambda^c \neq \emptyset\}\,. \tag{1.1}$$

Here Λ^c denotes the complement of Λ in Z and $\subset\subset$ denotes a finite subset.

* On leave from Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

The assumption (0) and (1) are sufficient condition for the existence of the limit

$$\alpha_t(Q) \equiv \lim_{\Lambda} e^{itU(\Lambda)} Q e^{-itU(\Lambda)}, \qquad Q \in \mathfrak{A}, \tag{1.2}$$

$$U(\Lambda) \equiv \sum_J \{\Phi(J); J \subset \Lambda\}, \tag{1.3}$$

which defines a one-parameter group α_t of automorphisms of $\mathfrak{A}$.

The assumption (2) is the key condition for the uniqueness of equilibrium states and is essentially the same as the classical cases [7].

Our main result:

Theorem 1. *For any β real, $\mathfrak{A}$ has one and only one α_t-KMS state at the inverse temperature β.*

The proof will be given under more abstract setting, which leads to a generalization of Sakai's result [8]: Let $\mathfrak{A}$ be a C^*-algebra generated by an increasing sequence of C^*-subalgebras $\mathfrak{A}_n$ of $\mathfrak{A}$, which are full matrix algebras. Let α_t be a one-parameter group of automorphisms of $\mathfrak{A}$ such that $\alpha_t(Q)$ is continuous in t for each $Q \in \mathfrak{A}$. Assume that there exists $h_n = h_n^* \in \mathfrak{A}$ for each n satisfying

$$(d/dt)\alpha_t(Q)|_{t=0} = i[h_n, Q] \tag{1.4}$$

for all $Q \in \mathfrak{A}_n$. Let τ be the unique tracial state on $\mathfrak{A}$ and $\bar{h}_n \in \mathfrak{A}_n$ be the conditional expectation of h_n: $\tau(h_n Q) = \tau(\bar{h}_n Q)$, $Q \in \mathfrak{A}_n$.

An abstract version of Theorem 1 is as follows:

Theorem 2. *Assume that*

$$\sup_n \|h_n - \bar{h}_n\| \equiv \lambda < \infty . \tag{1.5}$$

Then $\mathfrak{A}$ has at most one α_t-KMS state for each inverse temperature β.

Remark 1. If there exists $\hat{h}_n \in \mathfrak{A}_n$ satisfying

$$\sup_n \|h_n - \hat{h}_n\| < \infty , \tag{1.6}$$

then the condition (1.5) is satisfied: $\bar{h}_n - \hat{h}_n$ is the conditional expectation of $h_n - \hat{h}_n$, which implies

$$\|\bar{h}_n - \hat{h}_n\| \leqq \|h_n - \hat{h}_n\| .$$

Hence

$$\|h_n - \bar{h}_n\| \leqq 2\|h_n - \hat{h}_n\| .$$

Remark. 2. In the concrete case of Theorem 1, we may set $\mathfrak{A}_n = \mathfrak{A}(\Lambda_n)$, $h_n = U(\Lambda_n) + W(\Lambda_n)$, $\hat{h}_n = U(\Lambda_n)$. Then Theorem 2 and Remark 1 implies the uniqueness part of Theorem 1. The existence is well-known. Thus it is sufficient to prove Theorem 2.

§ 2. Quasi Containment

Two representations π_1 and π_2 of a C^*-algebra $\mathfrak{A}$ is said to be quasiequivalent if kernels of π_1 and π_2 coincide and the mapping $\pi_1(Q) \rightarrow \pi_2(Q)$, $Q \in \mathfrak{A}$, extends to a $*$-isomorphism of weak closures. In the present case, $\mathfrak{A}$ is simple and ker $\pi_1 =$

ker $\pi_2=0$. If a subrepresentation of π_1 is quasi-equivalent to π_2, then π_1 is said to quasi-contain π_2.

Let φ_1 and φ_2 be states of $\mathfrak{A}$. Let $\mathfrak{H}_j$, π_j and Ω_j be the space, representation and cyclic vector associated with φ_j, $j=1,2$.

Lemma 1. *If π_1 does not quasi-contain π_2, there exists a sequence of projections $e_m \in (\bigcup_n \mathfrak{A}_n)$ such that*

$$\lim_m \varphi_1(e_m)=0\,, \tag{2.1}$$

$$\lim_m \varphi_2(e_m)=a>0\,. \tag{2.2}$$

Proof. Consider the representation $\pi=\pi_1\oplus\pi_2$ on $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ with vectors $\Phi_1=\Omega_1\oplus 0$ and $\Phi_2=0\oplus\Omega_2$. Let $\mathfrak{M}=\pi(\mathfrak{A})''$, $\mathfrak{Z}$ be the center of $\mathfrak{M}$ and E_j be the $\mathfrak{Z}$-support of Φ_j, $j=1,2$. A condition that π_1 quasi-contains π_2 is equivalent to $E_1\geqq E_2$. Since this condition is not satisfied, there exists a non-zero central projection E such that $EE_1=0$ and $E\leqq E_2$ Since $\pi(\bigcup_n\mathfrak{A}_n)$ is dense in $\mathfrak{M}$, there exists a sequence $a_m\in\mathfrak{A}_{n(m)}$ (for some $n(m)$) satisfying

$$\lim_m \pi(a_m)=E\,.$$

Let e_m be the spectral projection of a_m for an interval $[1-\delta, 1+\delta]$ where $\delta\in(0,1)$ is fixed. Then $e_m\in\mathfrak{A}_{n(m)}$ and

$$\lim_m \pi(e_m)=E$$

by a theorem of Kaplansky [6]. Since $EE_1=0$, $E\Phi_1=0$. Since $E\leqq E_2$ and $E\neq 0$, $E\Phi_2\neq 0$. Hence (2.1) and (2.2) are satisfied with $a=\|E\Phi_2\|^2>0$.

§ 3. Relative Entropy

For two states φ_1 and φ_2 of a matrix algebra, the relative entropy is defined by

$$S(\varphi_1/\varphi_2)=\varphi_2(\log\varrho_2)-\varphi_2(\log\varrho_1) \tag{3.1}$$

where ϱ_j is the density matrix for φ_j.

For two faithful states of a von Neumann algebra $\mathfrak{M}$ the definition has been extended with a help of relative modular operators [2], [3]. In particular, for a state φ^h obtained from a faithful state φ by a perturbation $h=h^*\in\mathfrak{M}$, we have

$$S(\varphi^h/\varphi)=-\varphi(h)\,, \tag{3.2}$$

$$S(\varphi/\varphi^h)=\varphi^h(h)\,. \tag{3.3}$$

If N is a von Neumann subalgebra of $\mathfrak{M}$ and φ_j^N denotes the restriction of φ_j to $\mathfrak{N}$, the monotonicity

$$0\leqq S(\varphi_1^N/\varphi_2^N)\leqq S(\varphi_1/\varphi_2) \tag{3.4}$$

has been proved for hyperfinite $\mathfrak{M}$ and $\mathfrak{N}$ [2]. (For finite matrices, non-faithful φ_j are allowed.)

If $e \in \mathfrak{M}$ is a projection operator, the inequality (3.4) for $\mathfrak{N}$ generated by e and $(1-e)$ yield

$$S(\varphi_1/\varphi_2) \geqq \varphi_2(e) \log \{\varphi_2(e)/\varphi_1(e)\} + \varphi_2(1-e) \log \{\varphi_2(1-e)/\varphi_1(1-e)\} . \tag{3.5}$$

Lemma 2. *Let* φ_1 *and* φ_2 *be states of* $\mathfrak{A}$ *and* φ_j^n *denote the restriction of* φ_j *to* $\mathfrak{A}_n$. *If*

$$\sup_n S(\varphi_1^n/\varphi_2^n) \equiv \lambda_1 < \infty , \tag{3.6}$$

then π_1 *quasi-contains* π_2 *where* π_j *is the cyclic representation of* $\mathfrak{A}$ *associated with* φ_j.

Proof. Assume that π_1 does not quasi-contain π_2. By Lemma 1, there exists a sequence of projections $e_m \in \mathfrak{A}_{n(m)}$ such that $\varphi_1(e_m) \to 0$ and $\varphi_2(e_m) \to a > 0$. Then

$$-\varphi_2(e_m) \log \varphi_1(e_m) \to +\infty ,$$

while

$$\varphi_2(e_m) \log \varphi_2(e_m) + \varphi_2(1-e_m) \log \varphi_2(1-e_m) \geqq -\log 2 ,$$
$$-\varphi_2(1-e_m) \log \varphi_1(1-e_m) \geqq 0 .$$

These estimates contradicts with the bound (3.6) when $\varphi_j^{n(m)}$ and e_m are substituted into φ_j and e of the inequality (3.5).

§ 4. Gibbs Condition

Let $\mathfrak{A}'_N$ denote the commutant of $\mathfrak{A}_N$ in $\mathfrak{A}$. Then $\mathfrak{A} = \mathfrak{A}_N \otimes \mathfrak{A}'_N$. Let τ_N and τ'_N denote the restriction of the tracial state τ of $\mathfrak{A}$ to $\mathfrak{A}_N$ and $\mathfrak{A}'_N$. Let

$$\varphi_N^G(Q) = \tau_N(e^{-\beta \bar{h}_N} Q)/\tau_N(e^{-\beta \bar{h}_N}) . \tag{4.1}$$

Let $W(N) \equiv h_N - \bar{h}_N$. A state φ of $\mathfrak{A}$ is said to satisfy the Gibbs condition at β if

(i) The normal extension $\hat{\varphi}$ of φ to the weak closure $\mathfrak{M} = \pi_\varphi(\mathfrak{A})''$ of the associated representation is faithful on $\mathfrak{M}$ and

(ii) for every N, $\varphi^{\beta W(N)} = \varphi_N^G \otimes \varphi'_N$ for some linear positive functional φ'_N on $\mathfrak{A}'_N$.

Theorem 3. *If* φ *satisfies the KMS condition at* β, *it satisfies the Gibbs condition at* β.

Proof. The condition (i) is known to follow from the KMS condition. Let

$$\psi = \{\varphi^{\beta W(N)}(1)\}^{-1} \varphi^{\beta W(N)} \tag{4.2}$$

be a state on $\mathfrak{A}$ obtained from φ by a perturbation $\beta W(N) - \{\log \varphi^{\beta W(N)}(1)\} 1$.

Let σ_t^φ and σ_t^ψ be modular automorphisms of $\mathfrak{M}$ for states $\hat{\varphi}$ and $\hat{\psi}$ (the normal extensions of φ and ψ to $\mathfrak{M}$). Then

$$(d/dt)\{\sigma_t^\psi(x) - \sigma_t^\varphi(x)\}_{t=0} = i\beta[\pi_\varphi(W(N)), x] \tag{4.3}$$

for $x \in \mathfrak{M}$. The KMS condition implies

$$\sigma_t^\varphi(\pi_\varphi(Q)) = \pi_\varphi(\alpha_{-\beta t}(Q)), \qquad Q \in \mathfrak{A} . \tag{4.4}$$

By (1.4), (4.4) and (4.3), we obtain

$$(d/dt)\sigma_t^{\psi}(\pi_{\varphi}(Q))|_{t=0} = -i\beta\pi_{\varphi}([\bar{h}_N, Q]) \tag{4.5}$$

for $Q \in \mathfrak{A}_N$. By the group property,

$$(d/dt)\sigma_t^{\psi}(x) = \sigma_t^{\psi}\{(d/ds)\sigma_s^{\psi}(x)|_{s=0}\}\,. \tag{4.6}$$

Let

$$\alpha_t^N(Q) = e^{it\bar{h}_N} Q e^{-it\bar{h}_N}\,. \tag{4.7}$$

Then

$$(d/dt)\alpha_t^N(Q) = i[\bar{h}_N, \alpha_t^N(Q)]\,. \tag{4.8}$$

From (4.5), (4.6) and (4.8), we obtain

$$(d/dt)\sigma_t^{\psi}(\pi_{\varphi}\{\alpha_{\beta t}^N(Q)\}) = 0 \tag{4.9}$$

for $Q \in \mathfrak{A}_N$. This implies

$$\sigma_t^{\psi}\{\pi_{\varphi}(Q)\} = \pi_{\varphi}\{\alpha_{-\beta t}^N(Q)\}, \qquad Q \in \mathfrak{A}_N\,. \tag{4.10}$$

In particular

$$\pi_{\varphi}(\bar{h}_N) \in \mathfrak{M}^{\psi}\,. \tag{4.11}$$

where the centralizer $\mathfrak{M}^{\psi}$ is the set of $x \in \mathfrak{M}$ invariant under σ_t^{ψ}. If we set $\psi_1 = \psi^{\beta\bar{h}_N}$, then (4.11) implies

$$\psi_1(Q) = \psi(e^{\beta\bar{h}_N}Q), \qquad Q \in \mathfrak{A}\,, \tag{4.12}$$

and

$$\sigma_t^{\psi_1}(Q) = e^{i\beta\pi_{\varphi}(\bar{h}_N)}\sigma_t^{\psi}(x)e^{-i\beta\pi_{\varphi}(\bar{h}_N)} \tag{4.13}$$

for $x \in \mathfrak{M}$. The last equation together with (4.10) imply

$$\pi_{\varphi}(\mathfrak{A}_N) \in \mathfrak{M}^{\psi_1}\,. \tag{4.14}$$

If $Q_1, Q_2 \in \mathfrak{A}_N$ and $Q' \in \mathfrak{A}_N'$, then

$$\begin{aligned}\psi_1(Q_1(Q_2Q')) &= \psi_1((Q_2Q')Q_1) \quad \text{(by (4.14))}\\ &= \psi_1(Q_2Q_1Q')\end{aligned}$$

which implies

$$\psi_1([Q_1, Q_2]Q') = 0\,. \tag{4.15}$$

Since $\mathfrak{A}_N$ is a full matrix algebra, any element $Q \in \mathfrak{A}_N$ can be written as

$$Q = \tau_N(Q)\mathbf{1} + \textstyle\sum_j [Q_{j1}, Q_{j2}] \tag{4.16}$$

for some $Q_{j1}, Q_{j2} \in \mathfrak{A}_N$. Hence (4.15) implies

$$\psi_1(QQ') = \tau_N(Q)\psi_1(Q') \tag{4.17}$$

for $Q \in \mathfrak{A}_N$, $Q' \in \mathfrak{A}_N'$. Namely $\psi_1 = \tau_N \otimes \psi_1'$ where ψ_1' is the restriction of ψ_1 to $\mathfrak{A}_N'$. Because of (4.12), we obtain (ii) of the Gibbs condition.

Remark. What we need in the subsequent application is a part of the Gibbs condition, which says that the restriction of $\varphi^{\beta W(N)}$ to $\mathfrak{A}_N$ is the Gibbs state φ_N^G up to a normalization constant $\varphi^{\beta W(N)}(1)$. This much is deduced immediately from (4.10) by the uniqueness of KMS states for full matrix algebra.

§ 5. Proof of Theorem 2

Let φ_∞ be any one of the accumulation points of the sequence of states $\varphi_n^G \otimes \tau'_n$ at $n=\infty$. Let φ be an arbitrary extremal α_t-KMS state at β. By a known general result, φ is primary.

Let p be a fixed positive integer. Since $\mathfrak{A}_p$ is of finite dimension and φ_∞ is a weak accumulation point of $\varphi_n^G \otimes \tau'_n$, there exists an integer $N(\varepsilon)$ for any given $\varepsilon > 0$ such that $N(\varepsilon) \geqq p$ and

$$\|(\varphi_{N(\varepsilon)}^G)_p - (\varphi_\infty)_p\| < \varepsilon \tag{5.1}$$

where $(\varphi)_p$ denotes the restriction of φ to $\mathfrak{A}_p$. Note that $(\varphi_N^G)_p = (\varphi_N^G \otimes \tau'_N)_p$ for $N \geqq p$.

By (3.4), we have

$$0 \leqq S((\varphi)_p/(\psi_N)_p) \leqq S(\varphi/\psi_N) \tag{5.2}$$

where ψ_N denotes the state ψ given by (4.2). By (3.3), we have the following estimate:

$$\begin{aligned} S(\varphi/\psi_N) &= \psi_N(\beta W(N)) - \log \varphi^{\beta W(N)}(1) \\ &\leqq \psi_N(\beta W(N)) - \varphi(\beta W(N)) \\ &\leqq 2|\beta|\lambda \end{aligned} \tag{5.3}$$

where we have used (1.5) and the following Peierls-Bogolubov inequality [4]

$$\log \varphi^{\beta W(N)}(1) \geqq \varphi(\beta W(N))$$

which follows from $S(\psi_N/\varphi) \geqq 0$ for example.

By the Gibbs condition, $(\psi_N)_p = (\varphi_N^G)_p$ for $N \geqq p$. Hence (5.2) and (5.3) imply

$$0 \leqq S((\varphi)_p/(\varphi_N^G)_p) \leqq 2|\beta|\lambda\,. \tag{5.4}$$

The function $\mathrm{tr}(\varrho \log \varrho)$ of the density matrices ϱ for a finite dimensional case is bounded and continuous. If σ is strictly positive, $\mathrm{tr}(\varrho \log \sigma)$ is also bounded and continuous as a function of ϱ. Hence

$$S((\varphi)_p/(\varphi_\infty)_p) = \lim_{\varepsilon \to 0} S((\varphi)_p/(\varphi_{N(\varepsilon)}^G)_p)$$

due to (5.1). By (5.4), we obtain

$$0 \leqq S((\varphi)_p/(\varphi_\infty)_p) \leqq 2|\beta|\lambda\,. \tag{5.5}$$

Since p is any positive integer, Lemma 2 implies that the cyclic representation π associated with φ quasi-contains the cyclic representation π_∞ associated with φ_∞. Since π is primary, this implies that π and π_∞ are quasiequivalent. Since φ_∞ is fixed, any primary KMS states are mutually quasiequivalent. The proof of Theorem 2 is then completed by the following Lemma.

Lemma 3. *If two extremal KMS-states φ and φ' of a C^*-algebra $\mathfrak{A}$ at the same β have quasi-equivalent associated cyclic representations, then $\varphi=\varphi'$.*

Proof. Let $\mathfrak{H}$, π and Ω be canonically associated with φ and $\mathfrak{M}=\pi(\mathfrak{A})''$. Since φ is a KMS-state, Ω is separating (and cyclic by definition). By quasi-equivalence, there exists $\Omega'\in V_\Omega^{1/4}$ such that the associated vector states is φ', where $V_\Omega^{1/4}$ denotes the natural positive cone (see [3], for example). Since φ' is a KMS-state, Ω' is separating for $\mathfrak{M}$ and hence is also cyclic (see [3], for example). Let the unitary cocycle (the intertwining operator for modular automorphisms) be denoted by

$$u_t^{\varphi\varphi'}=\Delta_{\Omega',\Omega}^{it}\Delta_\Omega^{-it}.$$

Since the KMS condition characterizes the modular automorphisms, we have $\sigma_t^{\varphi'}=\sigma_t^{\varphi}(=\pi_\varphi\alpha_{-\beta t}\pi_\varphi^{-1}$ on $\pi_\varphi(\mathfrak{A}))$ and hence

$$u_t^{\varphi\varphi'}\in\mathfrak{M}\cap\mathfrak{M}'.$$

Since φ is an extremal KMS state, the center $\mathfrak{M}\cap\mathfrak{M}'$ is trivial and hence $u_t^{\varphi\varphi'}=e^{ict}$ for some real c. By analytic continuation, we have

$$\Omega'=u_{-i/2}^{\varphi'\varphi}\Omega=e^{c/2}\Omega.$$

Hence $\varphi=\varphi'$.

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H. Araki
Research Institute
for Mathematical Sciences
Kyoto University
Kyoto 606, Japan