# ON UNIQUENESS OF MULTI-BUBBLE BLOW-UP SOLUTIONS AND MULTI-SOLITONS TO $L^{2}$-CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS 

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#### Abstract

We are concerned with the focusing $L^{2}$-critical nonlinear Schrödinger equations in $\mathbb{R}^{d}$ for $d=1,2$. The uniqueness is proved for a large energy class of multi-bubble blow-up solutions, which converge to a sum of $K$ pseudo-conformal blow-up solutions particularly with low rate $(T-t)^{0+}$, as $t \rightarrow T, 1 \leq K<\infty$. Moreover, we also prove the uniqueness in the energy class of multisolitons which converge to a sum of $K$ solitary waves with convergence rate $(1 / t)^{2+}$, as $t \rightarrow \infty$. The uniqueness class is further enlarged to contain the multi-solitons with even lower convergence rate $(1 / t)^{\frac{1}{2}+}$ in the pseudo-conformal space. The proof is mainly based on the pseudo-conformal invariance and the monotonicity properties of several functionals adapted to the multi-bubble case, the latter is crucial towards the upgradation of the convergence to the fast exponential decay rate.


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## 1. Introduction and formulation of main results

1.1. Introduction. We are concerned with the focusing $L^{2}$-critical nonlinear Schrödinger equations in $\mathbb{R}^{d}$ for $d=1,2$,

$$
\begin{equation*}
i \partial_{t} u+\Delta u+|u|^{\frac{4}{d}} u=0, \tag{NLS}
\end{equation*}
$$

where $u: I \times \mathbb{R}^{d} \rightarrow \mathbb{C}, I \subseteq \mathbb{R}$ is a time interval.
Equation (NLS) has a variety of applications in nonlinear optics, Bose-Einstein condensation and plasma physics. It can be a model for the propagation of intense laser beams in bulk media with Kerr nonlinearity, and has also relationship with the weak turbulence theory. See, e.g., [17, 21, 54].

Mathematically, it is well-known (see, e.g., [8, 55]) that equation (NLS) is locally well-posed in the space $H^{1}$. An important role is played by the ground state $Q$, which is the unique positive radial solution to the elliptic equation

$$
\begin{equation*}
\Delta Q-Q+Q^{1+\frac{4}{d}}=0 \tag{1.1}
\end{equation*}
$$

The mass of ground state is exactly the threshold of global well-posedness and blow-up of solutions to (NLS): the solutions exist globally in the subcritical mass case where $\|u\|_{L^{2}}<\|Q\|_{L^{2}}$, while singularities can be formed in the (super-)critical mass case where $\|u\|_{L^{2}} \geq\|Q\|_{L^{2}}$.

Moreover, the solutions to (NLS) satisfy the following three conservation laws:

- Mass conservation:

$$
\begin{equation*}
M(u)(t):=\int_{\mathbb{R}^{d}}|u(t)|^{2} d x=M\left(u_{0}\right) . \tag{1.2}
\end{equation*}
$$

- Energy conservation:

$$
\begin{equation*}
E(u)(t):=\frac{1}{2} \int_{\mathbb{R}^{d}}|\nabla u(t)|^{2} d x-\frac{d}{2 d+4} \int_{\mathbb{R}^{d}}|u(t)|^{2+\frac{4}{d}} d x=E\left(u_{0}\right) . \tag{1.3}
\end{equation*}
$$

- Momentum conservation:

$$
\begin{equation*}
\operatorname{Mom}(u)(t):=\operatorname{Im} \int_{\mathbb{R}^{d}} \nabla u(t) \bar{u}(t) d x=\operatorname{Mom}\left(u_{0}\right) \tag{1.4}
\end{equation*}
$$

Equation (NLS) also admits the invariance under the translation, scaling, phase rotation and Galilean transform, i.e., if $u$ solves (NLS), then so does

$$
\begin{equation*}
\mathscr{T} u(t, x)=\lambda_{0}^{-\frac{d}{2}} u\left(\frac{t-t_{0}}{\lambda_{0}^{2}}, \frac{x-x_{0}}{\lambda_{0}}-\frac{\beta_{0}\left(t-t_{0}\right)}{\lambda_{0}}\right) e^{i \frac{\beta_{0}}{2} \cdot\left(x-x_{0}\right)-i \frac{\left.\beta_{0}\right|^{2}}{4}\left(t-t_{0}\right)+i \theta_{0}}, \tag{1.5}
\end{equation*}
$$

where $\left(\lambda_{0}, \beta_{0}, \theta_{0}\right) \in \mathbb{R}^{+} \times \mathbb{R}^{d} \times \mathbb{R}, x_{0} \in \mathbb{R}^{d}, t_{0} \in \mathbb{R}$. In particular, the $L^{2}$-norm of solutions is preserved under the scaling, and thus (NLS) is called the $L^{2}$-critical equation.

Another invariance, particularly important in the $L^{2}$-critical case, is the pseudo-conformal invariance related to the pseudo-conformal transformation, defined by

$$
\begin{equation*}
P_{T}(u)(t, x):=\frac{1}{(T-t)^{\frac{d}{2}}} u\left(\frac{1}{T-t}, \frac{x}{T-t}\right) e^{-i \frac{\left.k\right|^{2}}{4(T-t)}}, t \neq T, u \in \Sigma, \tag{1.6}
\end{equation*}
$$

where $\Sigma$ denotes the pseudo-conformal space $\Sigma:=\left\{u \in H^{1}:\|u\|_{\Sigma}:=\|u\|_{H^{1}}+\|x u\|_{L^{2}}<\infty\right\}$.
Furthermore, the pseudo-conformal invariance relates both multi-solitons and multi-bubble blowup solutions with pseudo-conformal blow-up rate. These two special families of solutions are indeed of significant importance to describe the dynamics of solutions to (NLS): one in the large time behavior regime, and the other in the singularity regime. Hence, the pseudo-conformal invariance provides an alternative way to study the uniqueness of multi-solitons from that of multi-bubble
blow-up solutions, to which more sharper singularity analysis can be performed. This is actually the main motivation of present work.

To be precise, on one hand, given $K \in \mathbb{N} \backslash\{0\}$, the solitary waves $W_{k}, 1 \leq k \leq K$, are defined by

$$
\begin{equation*}
W_{k}(t, x):=\omega_{k}^{-\frac{d}{2}} Q\left(\frac{x-v_{k} t}{\omega_{k}}\right) e^{i\left(\frac{1}{2} v_{k} \cdot x-\frac{1}{4}\left|v_{k}\right|^{2} t+\omega_{k}^{-2} t+\vartheta_{k}\right)} \tag{1.7}
\end{equation*}
$$

where the parameters $\omega_{k} \in \mathbb{R}^{+}, v_{k} \in \mathbb{R}^{d}$ and $\vartheta_{k} \in \mathbb{R}$, corresponding to the frequency, propagation speed and phase, respectively, $1 \leq k \leq K$. A multi-soliton (or, multi-solitary wave solution) is a solution to (NLS) defined on $\left[T_{0}, \infty\right.$ ) for some $T_{0} \in \mathbb{R}$ and such that

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{H^{1}}=o(1), \text { as } t \rightarrow \infty \tag{1.8}
\end{equation*}
$$

where and hereafter $o(1)$ means small quantities that converge to zero. This means that the multisolitons behave exactly as a sum of solitary waves without loss of mass by dispersion. We would like to mention that, multi-solitions with space translations are also studied in literature, see, e.g., $[11,12,13,39,43]$. For the simplicity of exposition, we focus on the multi-solitons of form (1.7).

Multi-solitons have been studied extensively in the integrable case, see, e.g., [49, 57]. For the nonintegrable equations, the construction of multi-solitons to (NLS) was initiated by Merle [43]. The proof in [43] is based on the pseudo-conformal invariance and the construction of multi-bubble blow-up solutions. Later, multi-solitons are constructed in the subcritical case by Martel and Merle [39], and in the supercritical case by Côte, Martel and Merle [13]. Moreover, Martel, Merle and Tsai [41] proved that multi-solitons are stable by the perturbation of initial data in the energy space. We also refer to $[31,32]$ for the construction of the infinite trains of solitons.

Multi-solitons have been also constructed in various other settings. For the generalized Kortewegde Vries (gKdV) equations, we refer to [35] for the subcritical and critical cases, and [9] for the super-critical case. See also [14] for the Klein-Gordon equation, [28] for the Hartree equation and [48] for the water-waves system.

Despite the extensive study of constructions, it remains open for the uniqueness or classification of multi-solitons to nonlinear Schrödinger equations. This was first pointed out by Martel and Merle [39] in the $L^{2}$-subcritical case, and later was raised as an open problem by Martel [36] in both the $L^{2}$-subcritical and critical cases. It is also expected that no uniqueness holds in the $L^{2}$-supercritical case, see Côte and Le Coz [12].

To our knowledge, the only complete study of the uniqueness problem of multi-solitons was done for the gKdV equations in the pioneering work of Martel [35] for the $L^{2}$-subcritical and critical cases. Later, multi-solitons were classified by Cômbet [9] in the $L^{2}$-supercritical case. An important ingredient in the proof of [35,9] is the almost monotonicity of local mass and energy, which allows to gain the fast exponential decay rate.

However, this property fails for NLS, which makes it quite difficult to study the uniqueness of multi-solitons to NLS. The recent progress has been made by Côte and Friederich [11] in the $L^{2}$ subcritical and critical cases. In [11], the uniqueness class is achieved for the multi-solitons $u$ to NLS that converge to the sum of solitary waves with a high power of $1 / t$, i.e.,

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{H^{1}}=O\left(\frac{1}{t^{N}}\right), \text { as } t \rightarrow \infty \tag{1.9}
\end{equation*}
$$

for some $N$ large enough. Note that, this result allows to break the class of exponential convergence. Moreover, the smoothness of multi-solitons and general nonlinearities are also studied in [11]. The challenge is hence to prove the uniqueness of multi-solitons in the low convergence regime.

Furthermore, the above uniqueness problem in the single bubble case ( $K=1$ ) is closely related to the solitary wave conjecture, which states that non-scattering $L^{2}$ global solutions to equation (NLS) with critical mass $\|Q\|_{L^{2}}^{2}$ shall coincide with the solitary wave up to the symmetries (1.5). This conjecture is affirmative in the pseudo-conformal space by the rigidity result of Merle [44] and the pseudo-conformal invariance. In the Sobolev space, when $d=2,3$ it is proved in [34] for $H^{1}$ radial solutions, and when $d \geq 4$ it is proved in [26] and [33], respectively, for $H^{1}$ and $H^{s}(s>0)$ radial solutions.

On the other hand, given $T \in \mathbb{R}$, the pseudo-conformal blow-up solutions $S_{k}, 1 \leq k \leq K$, are defined by

$$
\begin{equation*}
S_{k}(t, x):=\left(\omega_{k}(T-t)\right)^{-\frac{d}{2}} Q\left(\frac{x-x_{k}}{\omega_{k}(T-t)}\right) e^{-\frac{i}{4} \frac{x-\left.x_{k}\right|^{2}}{T-t}+\frac{i}{\omega_{k}^{(T-t)}}+i \vartheta_{k}} . \tag{1.10}
\end{equation*}
$$

Note that, $S_{k}$ has the critical mass $\|Q\|_{L^{2}}^{2}$ and blows up at time $T$ with the blow-up rate $\left\|\nabla S_{k}(t)\right\|_{L^{2}} \sim$ $(T-t)^{-1}$. More importantly, by the seminal work of Merle [44], the pseudo-conformal blow-up solutions are exactly the unique minimal mass blow-up solutions to (NLS), up to the symmetries (1.5). Furthermore, via the the pseudo-conformal transformation, the pseudo-conformal blow-up solutions are closely related to the solitary waves

$$
\begin{equation*}
S_{k}=P_{T}\left(W_{k}\right), \text { with } x_{k}=v_{k}, 1 \leq k \leq K . \tag{1.11}
\end{equation*}
$$

Thus, a natural question, as in the case of multi-solitons with asymptotic behavior (1.8), is whether there exists a unique multi-bubble blow-up solution $v$ to (NLS) such that

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}}=o(1), \text { as } t \rightarrow T . \tag{1.12}
\end{equation*}
$$

It should be mentioned that, the multi-bubble blow-up solutions to (NLS) with pseudo-conformal blow-up rate were first constructed by Merle [43]. Bubbling phenomena have been also exhibited in various other settings. We would like to refer to [18] for the multi-bubble solutions with log-log blow-up rate, [23] for the energy-critical NLS, and [42] for the blow-up solutions with multiple bubbles concentrating at the same point. We also refer to [10, 30] for the gKdV equations, [24, 29] for the wave maps, and [51] for the nonlinear Schrödinger system.

The understanding of above uniqueness problem enables to enlarge the uniqueness class of multisolitons to (NLS) particularly in the low convergence regime.

Furthermore, this uniqueness problem provides an illustration of the rigidity of equation (NLS) around the pseudo-conformal blow-up solutions. It seems also helpful to understand the mass quantization conjecture in [46]. It is conjectured by Merle and Raphaël [46] that, the blow-up solutions to (NLS) shall concentrate the mass $m_{k}\left(\geq\|Q\|_{L^{2}}^{2}\right)$ at the singularities and converge strongly in $L^{2}$ to a residue $u^{*}$ away from the singularities. Then, intuitively, the blow-up solutions with mass $K\|Q\|_{L^{2}}^{2}$ and $K$ distinct singularities shall distribute the mass $\|Q\|_{L^{2}}^{2}$ to each blow-up bubble, and thus the residue vanishes (i.e., $u^{*}=0$ ). Inspired by the rigidity result in [44], each blow-up bubble with critical mass $\|Q\|_{L^{2}}^{2}$ is expected to behave as a pseudo-conformal blow-up solution near the corresponding singularity. Thus, the above intuition leads naturally to the multi-bubble blow-up solutions with asymptotic behavior (1.12) (and an extra energy information), and the uniqueness problem states that this class of blow-up solutions shall be unique.

Let us also mention that, this kind of uniqueness problem shares interesting similarities with the local uniqueness problem of peak or bubbling solutions to nonlinear elliptic equations, which has attracted a lot of attentions during the last decades. For instance, Cao and Heinz [5] studied the local uniqueness of multi-lump solutions $u_{\varepsilon}$ to stationary nonlinear Schrödinger equations such that

$$
\left\|u_{\varepsilon}-\sum_{k=1}^{K} Q_{k}\left(\frac{x-x_{k, \varepsilon}}{\varepsilon}\right)\right\|_{H^{1}}=O\left(\varepsilon^{\frac{d}{2}+2}\right), \text { and } x_{k, \varepsilon} \rightarrow x_{k}, \text { as } \varepsilon \rightarrow 0
$$

where $\left\{x_{1}, \ldots, x_{K}\right\}$ are the distinct nondegenerate critical points of potential $V, Q_{k}(x)=Q\left(\sqrt{V\left(x_{k}\right)} x\right)$. See also [6] for the local uniqueness of multi-peak solutions $u_{\varepsilon}$ to Brezis-Nirenberg problem concentrating at different points $\left\{x_{1}, \ldots, x_{K}\right\}$ satisfying

$$
\left\|u_{\varepsilon}-\sum_{k=1}^{K} P U_{x_{k, \varepsilon}, \lambda_{k, k}}\right\|_{H^{1}}=o(1), \quad x_{k, \varepsilon} \rightarrow x_{k}, \text { and } \lambda_{k, \varepsilon} \rightarrow+\infty, \text { as } \varepsilon \rightarrow 0
$$

where $P$ is the projection from $H^{1}(\Omega)$ onto $H_{0}^{1}(\Omega), \Omega$ is a smooth and bounded domain in $\mathbb{R}^{d}$, and $U_{x, \lambda}$ solves the elliptic equation $\Delta u+u^{\frac{d+2}{d-2}}=0$ in $\mathbb{R}^{d}, d \geq 5$. We refer to $[4,7]$ and the references therein for more recent progresses on local uniqueness problem for other elliptic equations.

As mentioned above, the unconditional uniqueness of critical mass blow-up solutions to (NLS) has been completely understood by the seminal work of Merle [44]. Such strong rigidity result has been also proved for the minimal mass blow-up solutions to the inhomogeneous NLS by Raphaël and Szeftel [50], under the sharp non-degenerate condition of nonlinearity. In particular, a robust modulation method and upgradation procedures to upgrade the estimates of remainder have been developed in [50], particularly in the absence of the pseudo-conformal invariance. The uniqueness of minimal mass blow-up solutions has been also completely studied for the $L^{2}$-critical gKdV equation by Martel, Merle and Raphaël [40]. For the conditional uniqueness results in the single bubble case, we would like to refer to [47] for the Bourgain-Wang blow-up solutions to NLS, and [27] for the Chern-Simons-Schrödinger equations.

In the recent work [53], we study the multi-bubble blow-up solutions to stochastic nonlinear Schrödinger equations driven by Wiener processes, in the absence of the pseudo-conformal invariance and the conservation law of energy due to the presence of noise. The multi-bubble blow-up solutions are constructed to converge exponentially fast in $\Sigma$ to a sum of $K$ pseudo-conformal blowup solutions, $1 \leq K<\infty$. Moreover, the conditional uniqueness is proved in [53] for the class of multi-bubble blow-up solutions $v$ such that

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}}=O\left((T-t)^{3+}\right), \text { as } t \rightarrow T \tag{1.13}
\end{equation*}
$$

provided that the frequencies $\left\{\omega_{k}\right\}$ are close or the relative distances $\left|x_{j}-x_{k}\right|$ are large, $j \neq k$ (see Case (I) and Case (II) below). Here $(T-t)^{3+}$ means that $(T-t)^{3+\zeta}$ for any given $\zeta>0$. In particular, these results are applicable to equation (NLS), without the driven noise. By virtue of the pseudo-conformal invariance, the above conditional uniqueness result also gives the uniqueness of multi-solitons to (NLS) either with convergence rate $(1 / t)^{5+}$ in the energy space or with rate $(1 / t)^{4+}$ in the pseudo-conformal space. See Remark 1.2 (iii) below.

In this paper, we obtain the uniqueness in a large energy class of multi-bubble blow-up solutions $v$ to (NLS) particularly in the low asymptotic regime where

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}}=O\left((T-t)^{0+}\right), \text { as } t \rightarrow T \tag{1.14}
\end{equation*}
$$

in both Case (I) and Case (II). Note that, the convergence rate in (1.14) almost reaches the one $o(1)$ in (1.12). The condition (1.14) can be even weakened by the asymptotic behavior

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{L^{2}}+(T-t)\left\|\nabla v(t)-\sum_{k=1}^{K} \nabla S_{k}(t)\right\|_{L^{2}}=o(1), \text { as } t \rightarrow T \tag{1.15}
\end{equation*}
$$

plus additionally a double average condition (see (1.23) below). In both cases, we show that the convergence rate indeed can be upgraded to the much faster exponential decay rate $e^{-\frac{\delta}{T-t}}, \delta>0$, in the more regular pseudo-conformal space $\Sigma$. In particular, these results apply to the case where the blow-up points $\left\{x_{k}\right\}$ can be arbitrarily distinct when the frequencies are the same.

Furthermore, the above results also allow to enlarge the uniqueness class of multi-solitons to (NLS). By virtue of the pseudo-conformal invariance, we are able to return back to multi-solitons and prove that the uniqueness holds in the energy class of multi-solitons $u$ to (NLS) such that

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{H^{1}}=O\left(\frac{1}{t^{2+}}\right), \text { for tlarge enough }, \tag{1.16}
\end{equation*}
$$

in both Case (I) and Case (II), where the blow-up points $\left\{x_{k}\right\}$ are replaced by the speeds $\left\{v_{k}\right\}$. Again, the convergence rate in (1.16) can be upgraded to the exponential decay rate in $\Sigma$. One interesting application is that the speeds $\left\{v_{k}\right\}$ can be arbitrarily distinct when the frequencies $\left\{\omega_{k}\right\}$ are the same.

In the single soliton case (i.e., $K=1$ ), this result in particular shows that the solitary wave conjecture is affirmative for general $H^{1}$ solutions with asymptotic behavior (1.16). Additionally, if the propagation speed is zero, the uniqueness class can be further enlarged to contain the solutions with lower convergence rate $(1 / t)^{1+}$.

Moreover, we also prove the uniqueness of multi-solitons $u$ to (NLS) with even lower convergence rate $(1 / t)^{\frac{1}{2}+}$ in the pseudo-conformal space, i.e.,

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{\Sigma}=O\left(\frac{1}{t^{\frac{1}{2}+}}\right), \text { for t large enough }, \tag{1.17}
\end{equation*}
$$

provided additionally that the speeds $\left\{v_{k}\right\}$ are non-zero.
The main idea of proof is to reduce, via the pseudo-conformal invariance, the proof of the uniqueness of multi-solitons to that of multi-bubble blow-up solutions. The main effort is hence dedicated to the latter issue, to which, inspired by [50], the sharp singularity analysis is applied. We show that the convergence indeed can be upgraded to the much faster exponential decay rate, which in particular is beyond the third order $(T-t)^{3+}$ in (1.13) and thus yields the desirable uniqueness of multi-bubble blow-up solutions.

The crucial ingredients in the upgradation procedure are the monotonicity properties of different functionals adapted to the multi-bubble case. More delicately, in each upgradation step the monotonicity of functionals relies on suitable estimates of the remainder and geometrical parameters in the previous step. We show that the initial low convergent rate $(T-t)^{0+}$ in (1.14) is effective to run the whole upgradation procedure.

One major difficulty, particularly in the multi-bubble case, is that in all the controls of functionals arises the localized mass which may restrict the upgradation strength. Such a problem is absence in the single bubble case, because the localized mass vanishes due to the conservation law of mass. One keypoint is that two more orders (i.e., $(T-t)^{2+}$ ) can be explored for the localized mass by the local virial identities in [44]. This convergence rate is effective to serve as the basis of estimates of localized mass. More importantly, we relate the localized mass and the remainder together
and upgrade their estimates simultaneously by iteration arguments through several Gronwall type inequalities.

Let us also mention that, another challenge in the low convergence regime is to identify the exact value of the energy of solutions. The key fact here is that the remainder exhibits dispersion in the energy space along a sequence, which enables us to obtain the energy quantization, that is, the energy of multi-bubble blow-up solutions admits the quantization into the sum of the energies of pseudo-conformal blow-up solutions. This is the key towards the derivation of refined energy estimate in the upgradation procedure.

Notations. We use the standard Sobolev spaces $H^{s, p}\left(\mathbb{R}^{d}\right), s \in \mathbb{R}, 1 \leq p \leq \infty$. In particular, $L^{p}:=H^{0, p}\left(\mathbb{R}^{d}\right)$ is the space of $p$-integrable (complex-valued) functions, $L^{2}$ denotes the Hilbert space endowed with the scalar product $\langle v, w\rangle=\int_{\mathbb{R}^{d}} v(x) \bar{w}(x) d x$, and $H^{s}:=H^{s, 2}$. As usual, if $B$ is a Banach space, $L^{q}(0, T ; B)$ means the space of all integrable $B$-valued functions $f:(0, T) \rightarrow B$ with the norm $\|\cdot\|_{L^{q}(0, T ; B)}$, and $C([0, T] ; B)$ denotes the space of all $B$-valued continuous functions on $[0, T]$ with the sup norm over $t$.

We also use the notation $\dot{g}=\frac{d}{d t} g$ for any $C^{1}$ function $g$ defined on an open time interval. As $t \rightarrow T$ or $t \rightarrow \infty, f(t)=O(g(t))$ means that $|f(t) / g(t)|$ stays bounded, and $f(t)=o(g(t))$ means that $|f(t) / g(t)|$ converges to zero.

Throughout this paper, the positive constants $C$ and $\delta$ may change from line to line.
1.2. Formulation of main results. Throughout this paper we mainly consider two cases below:

Case (I). $\left\{x_{k}\right\}_{k=1}^{K}$ are arbitrarily distinct points in $\mathbb{R}^{d}$, and $\left\{\omega_{k}\right\}_{k=1}^{K}\left(\subseteq \mathbb{R}^{+}\right)$satisfy that for some $\omega>0,\left|\omega_{k}-\omega\right| \leq \varepsilon$ for every $1 \leq k \leq K$, where $\varepsilon>0$;

Case (II). $\left\{\omega_{k}\right\}_{k=1}^{K}$ are arbitrary points in $\mathbb{R}^{+}$, and $\left\{x_{k}\right\}_{k=1}^{K}\left(\subseteq \mathbb{R}^{d}\right)$ satisfy that $\left|x_{j}-x_{k}\right| \geq \varepsilon^{-1}$ for every $1 \leq j \neq k \leq K$, where $\varepsilon>0$.

One interesting application is the case where $\left\{x_{k}\right\}$ are arbitrarily distinct points in $\mathbb{R}^{d}$ and the frequencies $\left\{\omega_{k}\right\}$ are the same. In the single bubble case where $K=1$, the blow-up point $x$ and the frequency $\omega$ can be arbitrary points in $\mathbb{R}^{d}$ and $\mathbb{R}^{+}$, respectively.

Let us first consider the multi-bubble blow-up solutions. As mentioned above, the multi-bubble blow-up solutions with pseudo-conformal blow-up rate were first constructed in the pioneering work of Merle [43]. Theorem 1.1 below gives the uniqueness class of multi-bubble blow-up solutions with convergence rate $(T-t)^{3+}$.

Theorem 1.1. ([53, Theorems 2.7 and 2.15]) Consider equation (NLS) in $\mathbb{R}^{d}$ for $d=1,2$. Let $T \in \mathbb{R}, K \in \mathbb{N} \backslash\{0\}$. Let $\left\{\vartheta_{k}\right\}_{j=k}^{K} \subseteq \mathbb{R},\left\{x_{k}\right\}_{k=1}^{K}$ and $\left\{\omega_{k}\right\}_{k=1}^{K}$ satisfy either Case (I) or Case (II). Then, for any $\zeta \in(0,1)$, there exists $\varepsilon^{*}>0$, such that for any $0<\varepsilon<\varepsilon^{*}$, there exists a unique multi-bubble blow-up solution $v$ to (NLS) such that

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}}=O\left((T-t)^{3+\zeta}\right), \text { for } t \text { close to } T \tag{1.18}
\end{equation*}
$$

where $S_{k}, 1 \leq k \leq K$, are the pseudo-conformal blow-up solutions given by (1.10). Moreover, the unique multi-bubble blow-up solution $v$ satisfies that for some $\delta>0$.

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{\Sigma}=O\left(e^{-\frac{\delta}{T-t}}\right), \text { for t close to } T . \tag{1.19}
\end{equation*}
$$

Remark 1.2. (i) In the recent work [53], the construction and conditional uniqueness of multibubble blow-up solutions are also obtained for stochastic nonlinear Schrödinger equations (SNLS)
driven by Wiener processes, particularly in the absence of the pseudo-conformal invariance and the conservation law of energy. For the interested readers, we would like to refer to $[19,52]$ for the construction of stochastic blow-up solutions with loglog blow-up rate or with critical mass, and $[15,16]$ for the noise effect on blow-up. For other related topics of SNLS, see [1, 2, 3, 20, 22, 58] and the references therein.
(ii) For equation (NLS), the smallness of T assumed in [53] can be removed. It is assumed there simply because the Wiener processes start moving at time zero. One may replace the smallness of $T$ by taking $t$ close enough to $T$ in the proof of [53, Theorem 2.15].
(iii) By virtue of the pseudo-conformal invariance, Theorem 1.1 also yields the uniqueness of multi-solitons u to (NLS) satisfying either

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{H^{1}}=O\left(\frac{1}{t^{5+}}\right), \text { for tlarge enough }, \tag{1.20}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{\Sigma}=O\left(\frac{1}{t^{4+}}\right), \text { for targe enough. } \tag{1.21}
\end{equation*}
$$

Actually, by Lemma 5.1 and the inequalities (5.22) and (5.24) below, the condition (1.20) or (1.21) suffices to verify (1.18) for $v=P_{T}(u)$ given by (1.6). Hence, Theorem 1.1 yields the uniqueness of multi-bubble blow-up solutions, and thus of the corresponding multi-solitons.

The first main result of this paper is concerned with the uniqueness class of multi-bubble blow-up solutions to (NLS). The precise statements are formulated below.

Theorem 1.3. Consider equation (NLS) in dimensions $d=1,2$. Let $T \in \mathbb{R}, K \in \mathbb{N} \backslash\{0\}$. Let $\left\{\vartheta_{k}\right\} \subseteq \mathbb{R},\left\{\omega_{k}\right\}$ and $\left\{x_{k}\right\}$ satisfy either Case (I) or Case (II). Then, for any $\zeta \in(0,1)$, there exists $\varepsilon^{*}>0$ such that the following holds. For any $0<\varepsilon<\varepsilon^{*}$, there exists a unique multi-bubble blow-up solution $v$ to (NLS) such that

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{L^{2}}+(T-t)\left\|\nabla v(t)-\sum_{k=1}^{K} \nabla S_{k}(t)\right\|_{L^{2}}=o(1), \text { as } t \text { close to } T, \tag{1.22}
\end{equation*}
$$

and additionally

$$
\begin{equation*}
\frac{1}{T-t} \int_{t}^{T} \frac{1}{T-s} \int_{s}^{T}\left\|v(r)-\sum_{k=1}^{K} S_{k}(r)\right\|_{H^{1}}^{2} d r d s=\mathcal{O}\left((T-t)^{\zeta}\right), \tag{1.23}
\end{equation*}
$$

where $S_{k}, 1 \leq k \leq K$, are the pseudo-conformal blow-up solutions given by (1.10). Moreover, the unique solution $v$ converges exponentially fast to $\sum_{k=1}^{K} S_{k}$ in the pseudo-conformal space, i.e., there exists $\delta>0$ such that

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{\Sigma}=O\left(e^{-\frac{\delta}{T-t}}\right), \text { for t close to } T \tag{1.24}
\end{equation*}
$$

In particular, the above results hold for the multi-bubble blow-up solutions v to (NLS) such that

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}}=O\left((T-t)^{\zeta}\right), \text { for } t \text { close to } T \tag{1.25}
\end{equation*}
$$

Remark 1.4. (i) The conditions (1.22) and (1.23) (or, (1.25)) are much weaker than the previous one (1.18). Actually, the proof of Theorem 2.15 in [53] is based on the control of modified generalized energy, and the convergence rate $(T-t)^{3+}$ is enough to provide a Lyapunov type rigidity. However, the proof of Theorem 1.3 is much more delicate in the low convergence regime. It requires several upgradation procedures and relies crucially on the monotonicity of several new functionals, including the (modified) localized virial functionals.
(ii) It is conjectured by Merle and Raphaël [46] that, the general blow-up solutions to (NLS) can be decomposed into multi blow-up bubbles and a regular residue. Theorem 1.3 gives one uniqueness class of such blow-up solutions in the case where the mass is $K\|Q\|_{L^{2}}^{2}$ and the asymptotic behaviors (1.22) and (1.23) (or, (1.25)) are satisfied.

Our next goal is to enlarge the uniqueness class of multi-solitons to (NLS). The precise statement for the uniqueness in energy class is formulated below.

Theorem 1.5. Consider equation (NLS) in dimensions $d=1,2$. Let $K \in \mathbb{N} \backslash\{0\}$. Let $\left\{\vartheta_{k}\right\} \subseteq \mathbb{R}$, $\left\{\omega_{k}\right\}$ and $\left\{v_{k}\right\}$ satisfy either Case (I) or Case (II) with $v_{k}$ replacing $x_{k}, 1 \leq k \leq K$. Then, for any $\zeta \in(0,1)$, there exists $\varepsilon^{*}>0$ such that the following holds. For any $0<\varepsilon<\varepsilon^{*}$, there exists a unique multi-soliton $u$ to (NLS) such that

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{H^{1}}=O\left(\frac{1}{t^{2+\zeta}}\right), \text { for tlarge enough } \tag{1.26}
\end{equation*}
$$

where $\left\{W_{k}\right\}$ are the solitons given by (1.7). Moreover, the unique multi-soliton $u$ converges exponentially fast to $\sum_{k=1}^{K} W_{k}$ in the more regular pseudo-conformal space, i.e., for some $\delta>0$,

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{\Sigma}=O\left(e^{-\delta t}\right), \text { for targe enough } \tag{1.27}
\end{equation*}
$$

Remark 1.6. (i) The uniqueness of multi-solitons to NLS was first obtained by Côte and Friederich [11] in the $L^{2}$-subcritical and critical cases, provided that the convergence rate is $(1 / t)^{N}$ for $N$ large enough. Theorem 1.5 shows that the uniqueness class of multi-solitons to $L^{2}$-critical (NLS) can be enlarged in the low convergence regime with rate $(1 / t)^{2+}$. Differently from [11], the proof of Theorem 1.5 is based on the pseudo-conformal invariance and the uniqueness of multi-bubble blow-up solutions in Theorem 1.3.
(ii) Thanks to the smoothness result in [11], the unique multi-solitons in Theorem 1.5 belong to the space $C\left([T, \infty) ; H^{\infty}\right)$ for some $T \in \mathbb{R}$ and satisfy that for all integer $s \geq 0$,

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{H^{s}} \leq C e^{-\delta t}, \text { for t large enough } \tag{1.28}
\end{equation*}
$$

where $C, \delta>0$.
In the pseudo-conformal space, the uniqueness class of multi-solitons to (NLS) can be further enlarged with even lower convergence rate $(1 / t)^{\frac{1}{2}+}$. This is the content of Theorem 1.7 below.
Theorem 1.7. Assume the conditions in Theorem 1.5 to hold. Assume additionally that $v_{k} \neq 0$, $1 \leq k \leq K$. Then, for any $\zeta \in(0,1)$, there exists $\varepsilon^{*}>0$ such that for any $0<\varepsilon<\varepsilon^{*}$, there exists a unique multi-soliton и to (NLS) such that

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{\Sigma}=O\left(\frac{1}{t^{\frac{1}{2}+\zeta}}\right), \text { for targe enough } . \tag{1.29}
\end{equation*}
$$

Moreover, the unique multi-soliton $u$ converges exponentially fast to $\sum_{k=1}^{K} W_{k}$ in $\Sigma$.
Remark 1.8. By the pseudo-conformal invariance, it is easy to see from (1.25) and the inequalities (5.22)-(5.24) below that, the uniqueness holds for the multi-solitons $u$ to (NLS) such that

$$
\begin{equation*}
\left\|u(t)-\sum_{k=1}^{K} W_{k}(t)\right\|_{\Sigma}=O\left(\frac{1}{t^{1+}}\right), \text { for } t \text { large enough } \tag{1.30}
\end{equation*}
$$

However, the regime of low convergence rates from $\frac{1}{2}+$ to 1 requires more delicate proof. The key ingredients are the monotonicity of virial functional and the refined space time estimate of the gradient of multi-solitons, which allow to verify the weaker double average condition (1.23) for the corresponding multi-bubble blow-up solutions, rather than the pointwise decay condition (1.25).

We conclude this subsection with an interesting application to the single soliton case $K=1$, which is related to the solitary wave conjecture.

Corollary 1.9. Consider equation (NLS) in dimensions $d=1,2$. For any $\zeta \in(0,1)$, assume that $u$ is a solution to (NLS) satisfying

$$
\begin{equation*}
\left\|u(t)-\omega^{-\frac{d}{2}} Q\left(\frac{\cdot-v t}{\omega}\right) e^{i\left(\frac{1}{2} \cdot x \cdot x-\frac{1}{4}|v|^{2} t+\omega^{-2} t+\vartheta\right)}\right\|_{H^{1}}=O\left(\frac{1}{t^{2+\zeta}}\right), \text { for tlarge enough } \tag{1.31}
\end{equation*}
$$

where $v, \vartheta \in \mathbb{R}, \omega>0$. Then, $u$ coincides with the solitary wave solution, i.e.,

$$
\begin{equation*}
u(t, x)=\omega^{-\frac{d}{2}} Q\left(\frac{x-v t}{\omega}\right) e^{i\left(\frac{1}{2} v \cdot x-\frac{1}{4}|v|^{2} t+\omega^{-2} t+\vartheta\right)} \tag{1.32}
\end{equation*}
$$

Moreover, if $v=0$, then the same uniqueness holds for solutions with lower convergence rate

$$
\begin{equation*}
\left\|u(t)-\omega^{-\frac{d}{2}} Q\left(\frac{\dot{1}}{\omega}\right) e^{i\left(\omega^{-2} t+\vartheta\right)}\right\|_{H^{1}}=O\left(\frac{1}{t^{1+\zeta}}\right), \text { for tlarge enough. } \tag{1.33}
\end{equation*}
$$

Remark 1.10. The solitary wave conjecture is affirmative in the pseudo-conformal space by the rigidity result in [44] and the pseudo-conformal invariance. In the Sobolev space, when $d=2,3$ it is proved in [34] for $H^{1}$ radial solutions, and when $d \geq 4$ it is proved in [26] and [33], respectively, for $H^{1}$ and $H^{s}(s>0)$ radial solutions. Corollary 1.9 shows that, in dimensions $d=1,2$, this conjecture is affirmative for general $H^{1}$ solutions satisfying either the asymptotic behavior (1.31) or (1.33) when $v=0$.
1.3. Strategy of the proof. We make use of the pseudo-conformal invariance to reduce the uniqueness of multi-solitons to that of multi-bubble blow-up solutions, and then perform several upgradation procedures to upgrade the convergence to the exponential decay rate. Let us give a sketch of the proof below.

Step 1. Geometrical decomposition and preliminary controls.
We first obtain the geometrical decomposition of multi-bubble blow-up solution $v$ to (NLS), i.e.,

$$
\begin{equation*}
v(t, x)=\sum_{k=1}^{K} \lambda_{k}^{-\frac{d}{2}}(t) Q_{k}\left(t, \frac{x-\alpha_{k}(t)}{\lambda_{k}(t)}\right) e^{i \theta_{k}(t)}+R(t, x)\left(:=\sum_{k=1}^{K} U_{k}(t, x)+R(t, x)\right) \tag{1.34}
\end{equation*}
$$

where $Q_{k}(t, y)=Q(y) e^{i\left(\beta_{k}(t) \cdot y-\frac{1}{4} \gamma_{k}(t)|y|^{2}\right)}$ and the remainder $R$ satisfies the orthogonality conditions in (2.3) below. The geometrical decomposition enables us to perform the robust modulation method and to reduce the analysis of the blow-up dynamics of $v$ to that of finite dimensional geometrical parameters $\mathcal{P}=(\lambda, \alpha, \beta, \gamma, \theta)$ and the remainder $R$.

It should be mentioned that, the geometrical decomposition (1.34) should be valid for any time close to the blow-up time, which, however, may give rise to the singularity of Jacobian matrix with respect to $\mathcal{P}$. This is quite different from the construction of blow-up solutions (see, e.g., [52, 53]), where the geometrical decomposition is performed on the intervals away from the blow-up time.

To fix this problem, inspired by [45], the idea is to introduce a new family of parameters $\widetilde{\mathcal{P}}=$ $(\widetilde{\lambda}, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\theta})$ to measure the small deformations of pseudo-conformal blow-up solutions, implied by (1.22), and to obtain a universal bound of the corresponding determinant of Jacobian matrix with respect to these new parameters.

As a byproduct, the preliminary controls of remainder and geometrical parameters are obtained as well, which serve as the basic estimates at the beginning of the upgradation procedure.

Step 2. Controls of functionals adapted to the multi-bubble case.
The first upgradation step relies on the coercivity control of energy around the blow-up profile:

$$
\begin{align*}
& \sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}}\|Q\|_{2}^{2}+\sum_{k=1}^{K} \frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|y Q\|_{L^{2}}^{2}+C_{1} \frac{D^{2}}{(T-t)^{2}} \\
\leq & E(v)+\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}+\sum_{k=1}^{K} \frac{1}{2 \lambda_{k}^{2}} M_{k}+C_{2}\left(\sum_{k=1}^{K} \frac{M_{k}^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) . \tag{1.35}
\end{align*}
$$

Here $C_{1}, C_{2}, \delta>0, D(t):=\|R(t)\|_{L^{2}}+(T-t)\|\nabla R(t)\|_{L^{2}}$, and $M_{k}$ denotes the localized mass

$$
M_{k}:=2 \operatorname{Re}\left\langle R_{k}, U_{k}\right\rangle+\int|R|^{2} \Phi_{k} d x, \text { with } R_{k}=R \Phi_{k}, 1 \leq k \leq K
$$

where $\left\{\Phi_{k}\right\}$ are the localization functions (see (2.49) below).
Note that, the quantity $D$ measures the smallness of remainder and indeed plays the key role in the proof of the uniqueness of multi-bubble blow-up solutions. As a matter of fact, the upgradation procedure is mainly dedicated to upgrading the convergence rate of $D$.

In view of the conservation law of energy, estimate (1.35) suffices to upgrade the convergence rate to the first order, i.e., $D(t)=O(T-t)$.

Two major problems, however, arise in the above energy estimate (1.35). The first one is, that the exact value of energy $E(v)$ is a priori unclear due to the low convergence rate in (1.22).

In order to identify the energy of solutions, inspired by [50], we introduce the localized virial functional adapted to the multi-bubble case

$$
\begin{equation*}
\mathcal{L}:=\sum_{k=1}^{K} \frac{1}{2} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x-\frac{\gamma_{k}}{4 \lambda_{k}}\|x Q\|_{L^{2}}^{2}, \tag{1.36}
\end{equation*}
$$

where $\chi_{A}$ is a suitable localized function (see Subsection 3.2 below).
Let us mention that, localized virial functionals were first introduced by Martel and Merle [38] to study the $L^{2}$-critical gKdV equation, and later used by Raphaël and Szeftel [50] in the inhomogeneous NLS setting. The corresponding monotonicity property in particular yields a space time estimate of remainder around the singularity.

In the multi-bubble case, the localization functions $\left\{\Phi_{K}\right\}$ in (1.36) are introduced in this particular way mainly to ensure the monotonicity property (see Theorem 3.8 below). Actually, one difficulty in the multi-bubble case is to beat the interactions between different localized remainders $R_{j}$ and $R_{k}, j \neq k$, which are usually not easy to control, due to the few knowledge of remainder. The keypoint here is, that cancellations up to high orders can be gained for these coupling terms, by suitably choosing the localization functions and test functions. See, e.g., the proof of $H^{1}$ dispersion
in Lemma 3.18, see also the proof of the monotonicity of localized virial functional in Theorem 3.8 and the related arguments of [53, Lemma 5.12].

The refined space time estimate of remainder yields the precise asymptotic order of parameter $\gamma_{k}$ which, however, can not be obtained from the previous geometrical decomposition. More importantly, it reveals the key $H^{1}$ dispersion of remainder along a sequence $\left\{t_{n}\right\}$ to $T$, which enables us to obtain the crucial energy quantization in Theorem 3.15.

As a consequence, the refined energy estimate is derived: for some $C_{1}, C_{2}>0$,

$$
\begin{align*}
\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}}\|Q\|_{2}^{2}+C_{1} \frac{D^{2}}{(T-t)^{2}} \leq & \frac{\|y Q\|_{L^{2}}^{2}}{8} \sum_{k=1}^{K}\left(\omega_{k}^{2}-\frac{\gamma_{k}^{2}}{\lambda_{k}^{2}}\right)+\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}+\sum_{k=1}^{K} \frac{1}{2 \lambda_{k}^{2}} M_{k} \\
& +C_{2}\left(\sum_{k=1}^{K} \frac{M_{k}^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) . \tag{1.37}
\end{align*}
$$

The second problem of energy estimate is that, at this stage, the first term on the R.H.S. above is merely of order $O(1)$, which indeed ceases the further upgradation.

This leads us to introduce another new modified localized virial functional adapted to the multibubble case

$$
\begin{equation*}
\mathscr{L}:=\sum_{k=1}^{K} \frac{\gamma_{k}}{2 \lambda_{k}} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x-\frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|x Q\|_{2}^{2}, \tag{1.38}
\end{equation*}
$$

and to derive the corresponding monotonicity property, which in particular yields the coercivity type control

$$
\begin{equation*}
\tilde{c} \sum_{k=1}^{K} \int_{t}^{T} \frac{\left\|R_{k}\right\|_{L^{2}}^{2}}{\lambda_{k}^{3}} d s \leq \frac{\|y Q\|_{L^{2}}^{2}}{8} \sum_{k=1}^{K}\left(\frac{\gamma_{k}^{2}}{\lambda_{k}^{2}}-\omega_{k}^{2}\right)+C\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}+\left|\int_{t}^{T} \sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k} d s\right|+C \int_{t}^{T} E r d s, \tag{1.39}
\end{equation*}
$$

where $\tilde{c}, C>0$ and $E r$ is the error term given by (3.58) below. The key fact is, that the trouble $O(1)$ terms in both estimates (1.37) and (1.39) cancel each other out and the higher order terms remain. This makes it possible to further upgrade the convergence rate of remainder.

Thus, combining estimates (1.37) and (1.39) altogether we lead to the refined estimate:

$$
\begin{align*}
& \sum_{k=1}^{K} \frac{\left|\beta_{k}(t)\right|^{2}}{\lambda_{k}^{2}}\|Q\|_{2}^{2}+\frac{D^{2}(t)}{(T-t)^{2}}+\sum_{k=1}^{K} \int_{t}^{T} \frac{\left\|R_{k}\right\|_{L^{2}}^{2}}{\lambda_{k}^{3}} d s \\
\leq & C\left((T-t)^{2}+\left|\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{2}}\right|+\left|\sum_{k=1}^{K} \int_{t}^{T} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k} d s\right|+\int_{t}^{T} \frac{D^{2}+\sum_{k=1}^{K}\left|M_{k}\right|}{(T-t)^{2}} d s\right) . \tag{1.40}
\end{align*}
$$

Note that, the localized mass $M_{k}$ arises in all the controls of functionals above. The main estimates of localized mass are collected in Theorem 3.1 below.

Step 3. Upgradation to the higher convergence rate.
Quite differently from the first upgradation step, the upgradation to the higher convergence rate requires delicate iteration arguments through different Gronwall type inequalities.

More precisely, two Gronwall type inequalities will be derived from the refined estimate (1.40), which enable to upgrade the convergence rate to the second order, i.e., $D(t)=O\left((T-t)^{2}\right)$.

For the further upgradation, we use the modified generalized energy introduced in [53]

$$
\begin{align*}
\mathscr{I}:= & \frac{1}{2} \int|\nabla R|^{2} d x+\frac{1}{2} \sum_{k=1}^{K} \int \frac{1}{\lambda_{k}^{2}}|R|^{2} \Phi_{k} d x-\operatorname{Re} \int F(u)-F(U)-f(U) \bar{R} d x \\
& +\sum_{k=1}^{K} \frac{\gamma_{k}}{2 \lambda_{k}} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x, \tag{1.41}
\end{align*}
$$

where $F(z):=\frac{d}{2 d+4}|z|^{2+\frac{4}{d}}, f(z):=|z|^{\frac{4}{z}} z, z \in \mathbb{C}$. Again, the corresponding monotonicity property allows to derive a new Gronwall type inequality, which enables us to upgrade the convergence rate to the much faster exponential decay rate, that is, $D(t)=O\left(e^{-\frac{\delta}{T-t}}\right)$ for some $\delta>0$. The exponential decay rate of geometrical parameters and modulation equations is then obtained as well.

Step 4. Proof of the main results.
By virtue of the exponential decay results in the previous step, the multi-bubble blow-up solutions indeed converge exponentially fast to the sum of pseudo-conformal blow-up solutions, which, via Theorem 1.1, yields the uniqueness in Theorem 1.3.

The proof of Theorems 1.5 and 1.7 is based on Theorem 1.3 and the pseudo-conformal invariance. In particular, the key observation in the proof of Theorem 1.7 is that, an improved convergence rate can be gained for the space time estimate of gradient by the monotonicity of virial functional (see Lemma 5.2 and Corollary 5.3 below). This is quite different from the proof of Theorem 1.5, which relies on the pointwise decay rate in $\Sigma$. The improved space time estimate is the key towards the verification of the weaker double average condition (1.23), which enables to enlarge the uniqueness class of multi-solitons with even lower convergence rate $(1 / t)^{\frac{1}{2}+}$.

The remainder of this paper is structured as follows. Section 2 contains the geometrical decomposition and preliminary controls of the remainder and modulations equations. Then, Sections 3 is dedicated to the controls of localized mass and energy and the key monotonicity properties of (modified) localized virial functionals. The crucial upgradation procedures are then performed in Section 4. Eventually, the main results are proved in Section 5. For simplicity, some tools and the proof of modulation equations are presented in the Appendix, i.e., Section 6.

## 2. Geometrical decomposition and modulation equations

The main result of this section is the geometrical decomposition of multi-bubble blow-up solutions for all time close to the blow-up time $T$.

Let us use the notations $\lambda=\left(\lambda_{k}\right)_{1 \leq k \leq K}, \gamma=\left(\gamma_{k}\right)_{1 \leq k \leq K}, \theta=\left(\theta_{k}\right)_{1 \leq k \leq K}$ for the vectors in $\mathbb{R}^{K}$, and $\alpha=\left(\alpha_{k}\right)_{1 \leq k \leq K}, \beta=\left(\beta_{k}\right)_{1 \leq k \leq K}$ for the vectors in $\mathbb{R}^{d k}$, where $\alpha_{k}=\left(\alpha_{k, i}\right)_{1 \leq i \leq K}, \beta_{k}=\left(\beta_{k, i}\right)_{1 \leq i \leq d} \in \mathbb{R}^{d}$, $1 \leq k \leq K$. Hence, we have $\mathcal{P}=(\lambda, \alpha, \beta, \gamma, \theta) \in \mathbb{X}:=\mathbb{R}^{K} \times \mathbb{R}^{d K} \times \mathbb{R}^{d K} \times \mathbb{R}^{K} \times \mathbb{R}^{K}$.

Let $|\lambda|:=\sum_{k=1}^{K}\left|\lambda_{k}\right|$. Similar notations also apply to the vectors $\gamma, \theta \in \mathbb{R}^{K}, \alpha, \beta \in \mathbb{R}^{d K}$ and $\mathcal{P} \in \mathbb{X}$.

### 2.1. Geometrical decomposition.

Theorem 2.1. (Geometrical decomposition) Let v be the multi-bubble blow-up solution to (NLS) satisfying (1.22). Then, for $T^{*}$ sufficiently close to $T$, there exist unique modulation parameters $\mathcal{P}=(\lambda, \alpha, \beta, \gamma, \theta) \in C^{1}\left(\left[T^{*}, T\right) ; \mathbb{X}\right)$, such that $v$ admits the geometrical decomposition

$$
\begin{equation*}
v(t, x)=\sum_{k=1}^{K} U_{k}(t, x)+R(t, x)(:=U(t, x)+R(t, x)), \quad t \in\left[T^{*}, T\right), x \in \mathbb{R}^{d}, \tag{2.1}
\end{equation*}
$$

where

$$
\begin{equation*}
U_{k}(t, x)=\lambda_{k}^{-\frac{d}{2}}(t) Q_{k}\left(t, \frac{x-\alpha_{k}(t)}{\lambda_{k}(t)}\right) e^{i \theta_{k}(t)} \text { with } Q_{k}(t, y)=Q(y) e^{i\left(\beta_{k}(t) \cdot y-\frac{1}{4} \gamma_{k}(t)|y|^{2}\right)} \tag{2.2}
\end{equation*}
$$

and the following orthogonality conditions hold on $\left[T_{*}, T\right)$ : for each $1 \leq k \leq K$,

$$
\begin{align*}
& \operatorname{Re} \int\left(x-\alpha_{k}\right) U_{k}(t) \bar{R}(t) d x=0, \quad \operatorname{Re} \int\left|x-\alpha_{k}\right|^{2} U_{k}(t) \bar{R}(t) d x=0  \tag{2.3}\\
& \operatorname{Im} \int \nabla U_{k}(t) \bar{R}(t) d x=0, \quad \operatorname{Im} \int \Lambda_{k} U_{k}(t) \bar{R}(t) d x=0, \quad \operatorname{Im} \int \varrho_{k}(t) \bar{R}(t) d x=0 .
\end{align*}
$$

Here,

$$
\begin{gather*}
\Lambda_{k}:=\frac{d}{2} I_{d}+\left(x-\alpha_{k}\right) \cdot \nabla,  \tag{2.4}\\
\varrho_{k}(t, x)=\lambda_{k}^{-\frac{d}{2}} \rho_{k}\left(t, \frac{x-\alpha_{k}}{\lambda_{n}}\right) e^{i \theta_{k}} \text { with } \rho_{k}(t, y):=\rho(y)^{i\left(\beta_{k}(t) \cdot y-\frac{1}{4} \gamma_{k}(t)|y|^{2}\right)}, \tag{2.5}
\end{gather*}
$$

and $\rho$ is given by (6.10) below.
Moreover, the following estimates hold for the geometrical parameters and remainder:

$$
\begin{align*}
& \sum_{k=1}^{K}\left(\left|\lambda_{k}(t)-\omega_{k}(T-t)\right|+\left|\alpha_{k}(t)-x_{k}\right|\right)=o(T-t)  \tag{2.6}\\
& \sum_{k=1}^{K}\left(\left|\beta_{k}(t)\right|+\left|\gamma_{k}(t)-\omega_{k}^{2}(T-t)\right|+\left|\theta_{k}(t)-\omega_{k}^{-2}(T-t)^{-1}-\vartheta_{k}\right|\right)=o(1),  \tag{2.7}\\
& \|R(t)\|_{L^{2}}+(T-t)\|\nabla R(t)\|_{L^{2}}=o(1) \tag{2.8}
\end{align*}
$$

Remark 2.2. (i) The geometrical decomposition (2.1) is valid for any time close to the blow-up time, and thus may give rise to the singularity of Jacobian matrix used in [53].
(ii) Estimate (2.6) gives the precise leading asymptotic order of $\lambda_{k}$, i.e.,

$$
\begin{equation*}
\lambda_{k}(t) \approx \omega_{k}(T-t), \text { for close to } T \tag{2.9}
\end{equation*}
$$

which particularly characterizes the blow-up rate. However, estimate (2.7) is insufficient to yield

$$
\begin{equation*}
\gamma_{k}(t) \approx \omega_{k}^{2}(T-t), \text { for t close to } T \tag{2.10}
\end{equation*}
$$

As a matter of fact, such precise estimate (2.10) will be derived after analyzing the localized virial functional (see Theorem 3.9 below). It should be also mentioned that, (2.10) is important in the derivation of refined energy estimate, which allows to upgrade the convergence rate of remainder to the second order (see Subsections 3.3 and 4.1 below).

In order to avoid the singularity of Jacobian matrix, inspired by [45], the idea here is to take, via (1.22), the solution $v$ as a small perturbation of $\sum_{k=1}^{K} S_{k}$. Hence, a new family of parameters $\widetilde{\mathcal{P}}_{k}=\left(\widetilde{\lambda}_{k}, \widetilde{\alpha}_{k}, \widetilde{\beta}_{k}, \widetilde{\gamma}_{k}, \widetilde{\theta}_{k}\right) \in \mathbb{X}$ is introduced to measure the small deformations, and a universal bound will be obtained for the corresponding determinant of Jacobian matrix.

More precisely, given any $L>0, \omega_{k}>0, x_{k} \in \mathbb{R}^{d}, \vartheta_{k} \in \mathbb{R}, 1 \leq k \leq K$, we set

$$
\begin{equation*}
S_{L}(x):=\sum_{k=1}^{K} S_{k, L}(x)=\sum_{k=1}^{K}\left(\omega_{k} L\right)^{-\frac{d}{2}} Q_{0, k, L}\left(\frac{x-x_{k}}{\omega_{k} L}\right) e^{i \theta_{0, k}} \tag{2.11}
\end{equation*}
$$

where

$$
\begin{equation*}
Q_{0, k, L}(y)=Q(y) e^{i\left(\beta_{0, k} \cdot y-\frac{1}{4} y_{0, k}|y|^{2}\right)} \tag{2.12}
\end{equation*}
$$

with $\beta_{0, k}=0, \gamma_{0, k}=\omega_{k}^{2} L$, and $\theta_{0, k}=\omega_{k}^{-2} L^{-1}+\vartheta_{k}, 1 \leq k \leq K$. Note that, in the case $L=T-t, S_{k, L}$ is exactly the pseudo-conformal blow-up solution $S_{k}$ given by (1.10), and thus $S_{L}=\sum_{k=1}^{K} S_{k}$.

The proof of Theorem 2.1 is based on Lemma 2.3 below.
Lemma 2.3. There exists a universal small constant $\delta_{*}>0$ such that the following holds. For any $0<\delta, L<\delta_{*}$ and for any $v$ satisfying $\left\|v-S_{L}\right\|_{L^{2}} \leq \delta$, there exist unique $C^{1}$ parameters $\widetilde{\mathcal{P}}(v)=(\widetilde{\lambda}, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\theta}) \in \mathbb{X}$ with respect to $v$, such that $v$ admits the decomposition

$$
\begin{equation*}
v=\sum_{k=1}^{K}\left(\widetilde{\lambda}_{k} \omega_{k} L\right)^{-\frac{d}{2}} Q_{k, L}\left(\frac{x-x_{k}-\widetilde{\alpha}_{k} \omega_{k} L}{\widetilde{\lambda}_{k} \omega_{k} L}\right) e^{i\left(\widetilde{\theta}_{k}+\omega_{k}^{-2} L^{-1}+\vartheta_{k}\right)}+R_{L}\left(=: \sum_{k=1}^{K} U_{k, L}+R_{L}\right) \tag{2.13}
\end{equation*}
$$

and the following orthogonality conditions hold: for each $1 \leq k \leq K$,

$$
\begin{align*}
& \operatorname{Re} \int\left(x-x_{k}-\widetilde{\alpha}_{k} \omega L\right) U_{k, L} \overline{R_{L}} d x=0, \operatorname{Re} \int\left|x-x_{k}-\widetilde{\alpha}_{k} \omega L\right|^{2} U_{k, L} \overline{R_{L}} d x=0, \\
& \operatorname{Im} \int \nabla U_{k, L} \overline{R_{L}} d x=0, \operatorname{Im} \int \Lambda_{k} U_{k, L} \overline{R_{L}} d x=0, \quad \operatorname{Im} \int \varrho_{k, L} \overline{R_{L}} d x=0 . \tag{2.14}
\end{align*}
$$

Here, for each $1 \leq k \leq K$,

$$
\begin{gather*}
Q_{k, L}(y)=Q(y) e^{i \beta_{k} \cdot y-i \frac{\gamma_{k}}{4}|y|^{2}},  \tag{2.15}\\
\varrho_{k, L}(x)=\lambda_{k}^{-\frac{d}{2}} \rho_{k}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) e^{i \theta_{k}} \text { with } \rho_{k}(y)=\rho(y) e^{i \beta_{k} \cdot y-i \frac{\gamma_{k}}{4}|y|^{2}}, \tag{2.16}
\end{gather*}
$$

where $\rho$ is given by (6.10) in Appendix, and the parameters $\left(\lambda_{k}, \alpha_{k}, \beta_{k}, \gamma_{k}, \theta_{k}\right)$ depend on $\left(\widetilde{\lambda}_{k}, \widetilde{\alpha}_{k}, \widetilde{\beta}_{k}, \widetilde{\gamma}_{k}, \widetilde{\theta}_{k}\right)$ and $L$ as follows

$$
\begin{align*}
& \lambda_{k}=\widetilde{\lambda}_{k} \omega_{k} L, \quad \alpha_{k}=x_{k}+\widetilde{\alpha}_{k} \omega_{k} L, \quad \beta_{k}=\widetilde{\beta}_{k},  \tag{2.17}\\
& \gamma_{k}=\widetilde{\gamma}_{k}+\omega_{k}^{2} L, \quad \theta_{k}=\widetilde{\theta}_{k}+\omega_{k}^{-2} L^{-1}+\vartheta_{k}, \quad 1 \leq k \leq K
\end{align*}
$$

Moreover, there exists a universal constant $C>0$ such that

$$
\begin{align*}
& \sum_{k=1}^{K}\left(\left|\widetilde{\lambda}_{k}-1\right|+\left|\widetilde{\alpha}_{k}\right|+\left|\widetilde{\beta}_{k}\right|+\left|\widetilde{\gamma}_{k}\right|+\left|\widetilde{\theta}_{k}\right|\right) \leq C\left\|v-S_{L}\right\|_{L^{2}},  \tag{2.18}\\
& \left\|R_{L}\right\|_{L^{2}} \leq C\left\|v-S_{L}\right\|_{L^{2}}  \tag{2.19}\\
& L\left\|\nabla R_{L}\right\|_{L^{2}} \leq C\left(\left\|v-S_{L}\right\|_{L^{2}}+L\left\|\nabla v-\nabla S_{L}\right\|_{L^{2}}\right) \tag{2.20}
\end{align*}
$$

Proof. Set-up of the notations. To simplify the notations, we write $U_{L}:=\sum_{k=1}^{K} U_{k, L}, R:=R_{L}$, and

$$
\begin{equation*}
\widetilde{R}:=v-S_{L} . \tag{2.21}
\end{equation*}
$$

Set $\mathbb{Y}:=\mathbb{R} \times \mathbb{R}^{d} \times \mathbb{R}^{d} \times \mathbb{R} \times \mathbb{R}, \widetilde{\mathcal{P}}_{0, j}:=(1,0,0,0,0) \in \mathbb{Y}$ and $\widetilde{\mathcal{P}}_{0}=\left(\widetilde{\mathcal{P}}_{0,1}, \cdots, \widetilde{\mathcal{P}}_{0, K}\right) \in \mathbb{Y}^{K}$. Similarly, let $\widetilde{\mathcal{P}}_{j}:=\left(\widetilde{\lambda}_{j}, \widetilde{\alpha}_{j}, \widetilde{\beta}_{j}, \widetilde{\gamma}_{j}, \widetilde{\theta}_{j}\right) \in \mathbb{Y}, \widetilde{\mathcal{P}}:=\left(\widetilde{\mathcal{P}}_{1}, \cdots, \widetilde{\mathcal{P}}_{K}\right) \in \mathbb{Y}^{K}$. Let $B_{\delta}\left(v_{0}, \widetilde{\mathcal{P}}_{0}\right)$ denote the closed ball centered at $\left(v_{0}, \widetilde{\mathcal{P}}_{0}\right)$ of radius $\delta$, i.e.,

$$
\begin{equation*}
B_{\delta}\left(v_{0}, \widetilde{\mathscr{P}}_{0}\right):=\left\{(v, \widetilde{\mathcal{P}}):\left\|v-v_{0}\right\|_{L^{2}} \leq \delta,\left|\widetilde{\mathscr{P}}-\widetilde{\mathcal{P}}_{0}\right| \leq \delta\right\} \tag{2.22}
\end{equation*}
$$

where $\delta$ is a small constant to be chosen later, and

$$
\begin{equation*}
\left|\widetilde{\mathcal{P}}-\widetilde{\mathcal{P}}_{0}\right|:=\sum_{j=1}^{K}\left|\widetilde{\mathcal{P}}_{j}-\widetilde{\mathcal{P}}_{0, j}\right|=\sum_{j=1}^{K}\left(\left|\widetilde{\lambda}_{j}-1\right|+\left|\widetilde{\alpha}_{j}\right|+\widetilde{\beta}_{j}\left|+\left|\widetilde{\gamma}_{j}\right|+\left|\widetilde{\theta}_{j}\right|\right) .\right. \tag{2.23}
\end{equation*}
$$

For every $1 \leq k \leq K$, define the functionals by

$$
\begin{aligned}
f_{1}^{k}(v, \widetilde{\mathcal{P}}) & :=\lambda_{k}^{-2} \operatorname{Re} \int\left|x-\alpha_{k}\right|^{2} U_{k, L} \bar{R} d x, \quad f_{2, i}^{k}(v, \widetilde{\mathcal{P}}):=\lambda_{k}^{-1} \operatorname{Re} \int\left(x_{i}-\alpha_{k, i}\right) U_{k, L} \bar{R} d x, \\
f_{3, i}^{k}(v, \widetilde{\mathcal{P}}) & :=\lambda_{k} \operatorname{Im} \int \partial_{i} U_{k, L} \bar{R} d x, f_{4}^{k}(v, \widetilde{\mathcal{P}}):=\operatorname{Im} \int \Lambda_{k} U_{k, L} \bar{R} d x, f_{5}^{k}(v, \widetilde{\mathscr{P}}):=\operatorname{Im} \int \varrho_{k, L} \bar{R} d x,
\end{aligned}
$$

where $1 \leq i \leq d$, and the parameters $\lambda_{k}, \alpha_{k}, \beta_{k}, \gamma_{k}, \theta_{k}$ are defined as in (2.17).
Let $F^{k}:=\left(f_{1}^{k}, f_{2,1}^{k}, f_{2,2}^{k}, \cdots, f_{5}^{k}\right)$ and $\frac{\partial F^{k}}{\partial \widetilde{\mathcal{P}}_{j}}$ denote the Jacobian matrix of $F^{k}$ with respect to $\widetilde{\mathcal{P}}_{j}$

$$
\frac{\partial F^{k}}{\partial \widetilde{\mathcal{P}}_{j}}:=\left(\begin{array}{ccc}
\frac{\partial f_{1}^{k}}{\partial \bar{\lambda}_{j}} & \cdots & \frac{\partial f_{1}^{k}}{\partial \bar{\theta}_{j}}  \tag{2.24}\\
\vdots & & \vdots \\
\frac{\partial f_{s}^{k}}{\partial \widetilde{\lambda}_{j}} & \cdots & \frac{\partial f_{5}^{k}}{\partial \widetilde{\theta}_{j}}
\end{array}\right), 1 \leq j, k \leq K
$$

Similarly, let $F:=\left(F^{1}, \cdots, F^{K}\right)$ and $\frac{\partial F}{\partial \widetilde{\mathscr{P}}}:=\left(\frac{\partial F^{k}}{\partial \widetilde{\mathcal{P}}_{j}}\right)_{1 \leq j, k \leq K}$ be the corresponding Jacobian matrix.
Note that, for every $(v, \widetilde{\mathcal{P}}) \in B_{\delta}\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right)$, since

$$
R=v-U_{L}=\widetilde{R}+S_{L}-U_{L}
$$

we have

$$
\begin{equation*}
\|R\|_{L^{2}} \leq\|\widetilde{R}\|_{L^{2}}+\left\|S_{L}-U_{L}\right\|_{L^{2}} \tag{2.25}
\end{equation*}
$$

Using the explicit expressions of $S_{L}$ and $U_{L}$ in (2.11) and (2.13), respectively, we compute that for some $C>0$ (see the proof of [52, Theorem 2.15] and [53, Theorem 2.14]),

$$
\begin{align*}
\left\|S_{L}-U_{L}\right\|_{L^{2}} & \leq C \sum_{j=1}^{K}\left(\left|\frac{\lambda_{j}^{\frac{d}{2}}-\left(\omega_{j} L\right)^{\frac{d}{2}}}{\left(\omega_{j} L\right)^{\frac{d}{2}}}\right|+\left|\frac{\alpha_{j}-x_{j}}{\lambda_{j}}\right|+\left(\frac{\lambda_{j}}{\omega_{j} L}\right)^{\frac{d}{2}}\left|\widetilde{\beta}_{j}\right|+\left(\frac{\lambda_{j}}{\omega_{j} L}\right)^{\frac{d}{2}}\left|\widetilde{\gamma}_{j}\right|+\left|\widetilde{\theta}_{j}\right|+\left|\frac{\omega_{j} L}{\lambda_{j}}-1\right|\right) \\
& \leq C\left|\widetilde{\mathcal{P}}-\widetilde{\mathcal{P}}_{0}\right| . \tag{2.26}
\end{align*}
$$

Thus, we infer that there exists a universal constant $\widetilde{C}>0$ such that

$$
\begin{equation*}
\|R\|_{L^{2}} \leq \widetilde{C}\left(\|\widetilde{R}\|_{L^{2}}+\left|\widetilde{\mathcal{P}}-\widetilde{\mathcal{P}}_{0}\right|\right) \leq 2 \widetilde{C} \delta, \quad \forall(v, \widetilde{\mathcal{P}}) \in B_{\delta}\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right) \tag{2.27}
\end{equation*}
$$

Step 1. Nondegeneracy of Jacobian. We claim that, there exist small constants $\delta_{*}, c_{1}, c_{2}>0$ such that for any $0<\delta, L \leq \delta_{*}$,

$$
\begin{equation*}
0<c_{1} \leq\left|\operatorname{det} \frac{\partial F}{\partial \widetilde{\mathcal{P}}}(v, \widetilde{\mathcal{P}})\right| \leq c_{2}<\infty, \quad \forall(v, \widetilde{\mathcal{P}}) \in B_{\delta}\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right) . \tag{2.28}
\end{equation*}
$$

In order to prove (2.28), using the results in the proof of [52, Lemma 4.2] and the chain rule we compute that for any $(v, \widetilde{\mathcal{P}}) \in B_{\delta}\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right)$ and for any $1 \leq k \leq K, 1 \leq i, h \leq d$,

$$
\begin{align*}
& \partial_{\widetilde{\lambda}_{k}} f_{1}^{k}=-\frac{1}{\widetilde{\lambda}_{k}}\|x Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right), \quad \partial_{\widetilde{\alpha}_{k, h}} f_{2, i}^{k}=-\frac{\delta_{i h}}{2 \widetilde{\lambda}_{k}}\|Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right),  \tag{2.29}\\
& \partial_{\widetilde{\lambda}_{k}} f_{3, i}^{k}=\frac{\beta_{k, i}}{2 \widetilde{\lambda}_{k}}\|Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right), \quad \partial_{\widetilde{\beta}_{k, h}} f_{3, i}^{k}=-\frac{\delta_{i h}}{2}\|Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right),  \tag{2.30}\\
& \partial_{\widetilde{\alpha}_{k, h}} f_{4}^{k}=-\frac{\beta_{k, h}}{2 \widetilde{\lambda}_{k}}\|Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right), \quad \partial_{\widetilde{\gamma}_{k}} f_{4}^{k}=\frac{1}{4}\|x Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right),  \tag{2.31}\\
& \left.\partial_{\widetilde{\lambda}_{k}} f_{5}^{k}=\left.\frac{\gamma_{k}}{2 \widetilde{\lambda}_{k}}\langle\rho,| x\right|^{2} Q\right\rangle+O\left(\|R\|_{L^{2}}\right), \quad \partial_{\widetilde{\alpha}_{k, h}} f_{5}^{k}=\frac{\beta_{k, h}}{2 \widetilde{\lambda}_{k}}\|x Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right), \tag{2.32}
\end{align*}
$$

$$
\begin{equation*}
\left.\partial_{\bar{\gamma}_{k}} f_{5}^{k}=-\left.\frac{1}{4}\langle\rho,| x\right|^{2} Q\right\rangle+O\left(\|R\|_{L^{2}}\right), \quad \partial_{\widehat{\theta}_{k}} f_{5}^{k}=-\frac{1}{2}\|x Q\|_{L^{2}}^{2}+O\left(\|R\|_{L^{2}}\right) \tag{2.33}
\end{equation*}
$$

Moreover, using the exponential decay of $Q$ and $\rho$ in (6.8) and (6.11), respectively, we infer that, there exists $\mu>0$ such that for any $1 \leq j \neq k \leq K$,

$$
\begin{equation*}
\left|\partial_{\widetilde{\lambda}_{j}} F^{k}\right|+\left|\partial_{\widetilde{\alpha}_{j, h}} F^{k}\right|+\left|\partial_{\widetilde{\beta}_{j, h}} F^{k}\right|+\left|\partial_{\widetilde{\gamma}_{j}} F^{k}\right|+\left|\partial_{\widetilde{\theta}_{j}} F^{k}\right|=O\left(e^{-\frac{\mu\left|x_{j}-x_{k}\right|}{L}}\right) . \tag{2.34}
\end{equation*}
$$

Thus, we conclude that for any $(v, \widetilde{\mathcal{P}}) \in B_{\delta}\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right)$,

$$
\begin{align*}
& \left|\operatorname{det} \frac{\partial F^{k}}{\partial \widetilde{\mathcal{P}}_{k}}(v, \widetilde{\mathcal{P}})\right|=2^{-(3+2 d)} \widetilde{\lambda}_{k}^{-(1+d)}\|Q\|_{L^{2}}^{4 d}\|x Q\|_{L^{2}}^{6}+O\left(\|R\|_{L^{2}}\right), \quad 1 \leq k \leq K,  \tag{2.35}\\
& \left|\operatorname{det} \frac{\partial F^{k}}{\partial \widetilde{\mathcal{P}}_{j}}(v, \widetilde{\mathcal{P}})\right|=O\left(e^{-\frac{\mu\left(x_{j}-x_{k} \mid\right.}{L}}\right), \quad j \neq k . \tag{2.36}
\end{align*}
$$

Taking into account (2.27), we obtain that for $\delta, L$ small enough and for any $(v, \widetilde{\mathcal{P}}) \in B_{\delta}\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right)$,

$$
\begin{align*}
\left|\operatorname{det} \frac{\partial F}{\partial \widetilde{\mathcal{P}}}(v, \widetilde{\mathcal{P}})\right| & =2^{-(3+2 d) K} \prod_{k=1}^{K} \widetilde{\lambda}_{k}^{-(1+d)}\|Q\|_{L^{2}}^{4 d K}\|x Q\|_{L^{2}}^{6 K}+O\left(\|R\|_{L^{2}}+e^{-\frac{\langle\widetilde{\widetilde{\sigma}}}{L}}\right) \\
& =2^{-(3+2 d) K} \prod_{k=1}^{K} \widetilde{\lambda}_{k}^{-(1+d)}\|Q\|_{L^{2}}^{4 d K}\|x Q\|_{L^{2}}^{6 K}+O\left(\delta+e^{-\frac{-\widetilde{\widetilde{T}}}{L}}\right), \tag{2.37}
\end{align*}
$$

where $\mu>0$ is a universal small constant, $\widetilde{\sigma}:=\min _{j \neq k}\left\{\left|x_{j}-x_{k}\right|\right\}>0$. This yields (2.28), as claimed.
Step 2. Estimate of modulation parameters. Below we prove that there exists a universal constant $C_{*}(\geq 1)$ such that, for any $0<\delta, L \leq \delta_{*}$ and for any $\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{1}\right)\right),\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right) \in B_{\delta}\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right)$, if $F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{1}\right)\right)=F\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)=0$, then

$$
\begin{equation*}
\left|\widetilde{\mathcal{P}}\left(v_{1}\right)-\widetilde{\mathcal{P}}\left(v_{2}\right)\right| \leq C_{*}\left\|v_{1}-v_{2}\right\|_{L^{2}} . \tag{2.38}
\end{equation*}
$$

For this purpose, we note that

$$
\begin{aligned}
0 & =F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{1}\right)\right)-F\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right) \\
& =\left(F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{1}\right)\right)-F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)\right)+\left(F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)-F\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)\right),
\end{aligned}
$$

which yields immediately that

$$
\begin{equation*}
F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{1}\right)\right)-F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)=F\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)-F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right) \tag{2.39}
\end{equation*}
$$

By the differential mean value theorem, this yields that

$$
\begin{equation*}
\left(\frac{\partial F}{\partial \widetilde{\mathcal{P}}}\left(v_{1}, \widetilde{\mathcal{P}}_{r}\right)\right)\left(\widetilde{\mathcal{P}}\left(v_{1}\right)-\widetilde{\mathcal{P}}\left(v_{2}\right)\right)^{t}=\left(F\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)-F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)\right)^{t} \tag{2.40}
\end{equation*}
$$

where $\widetilde{\mathcal{P}}_{r}=r \widetilde{\mathcal{P}}\left(v_{1}\right)+(1-r) \widetilde{\mathcal{P}}\left(v_{2}\right)$ for some $0<r<1$, and the superscript $t$ means the transpose of matrices.

In view of (2.28), the Jacobian matrix $\frac{\partial F}{\partial \widetilde{\mathcal{P}}}\left(v_{1}, \widetilde{\mathcal{P}}_{r}\right)$ is invertible, and thus we may reformulate equation (2.40) as follows

$$
\begin{equation*}
\left(\widetilde{\mathcal{P}}\left(v_{1}\right)-\widetilde{\mathcal{P}}\left(v_{2}\right)\right)^{t}=\left(\frac{\partial F}{\partial \widetilde{\mathscr{P}}}\left(v_{1}, \widetilde{\mathscr{P}}_{r}\right)\right)^{-1}\left(F\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)-F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)\right)^{t} \tag{2.41}
\end{equation*}
$$

Then, using (2.29)-(2.34) we get that for a universal constant $C>0$,

$$
\begin{equation*}
\left\|\left(\frac{\partial F}{\partial \widetilde{\mathcal{P}}}\left(v_{1}, \widetilde{\mathcal{P}}_{r}\right)\right)^{-1}\right\| \leq C \tag{2.42}
\end{equation*}
$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm of matrices.
Moreover, by the exponential decay of $Q$ and $\rho$ in (6.8) and (6.11), respectively,

$$
\begin{equation*}
\left|F\left(v_{2}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)-F\left(v_{1}, \widetilde{\mathcal{P}}\left(v_{2}\right)\right)\right| \leq C\left\|v_{2}-v_{1}\right\|_{L^{2}} \tag{2.43}
\end{equation*}
$$

Thus, combining (2.41), (2.42) and (2.43) altogether we prove (2.38).
Step 3. Proof of main results. Let $\delta_{*}, C_{*}$ be the universal constants as in Step 1 and Step 2, respectively. Define the set $B$ in the ball $B_{\frac{\delta_{*}}{C_{*}}}\left(S_{L}\right)$ by

$$
\begin{equation*}
B:=\left\{v \in B_{\frac{\delta_{*}}{C_{*}}}\left(S_{L}\right): \exists \widetilde{\mathcal{P}} \in B_{\delta_{*}}\left(\widetilde{\mathcal{P}}_{0}\right), \text { such that } F(v, \widetilde{\mathcal{P}})=0\right\} . \tag{2.44}
\end{equation*}
$$

In order to prove the geometrical decomposition (2.13), it suffices to prove that

$$
\begin{equation*}
B=B_{\frac{\delta_{x}}{C_{*}}}\left(S_{L}\right) \tag{2.45}
\end{equation*}
$$

For this purpose, since $B_{\frac{\delta_{ \pm}}{C}}\left(S_{L}\right)$ is connected and $S_{L} \in B$ due to the fact $F\left(S_{L}, \widetilde{\mathcal{P}}_{0}\right)=0$, we only need to prove that $B$ is both open and closed in $B_{\frac{\delta_{t}}{C_{*}}}\left(S_{L}\right)$.

To this end, suppose that $v \in B$, then by definition there exists $\widetilde{\mathcal{P}}(v) \in B_{\delta_{*}}\left(\widetilde{\mathcal{P}}_{0}\right)$ such that $F(v, \widetilde{\mathcal{P}}(v))=0$. By (2.28), the corresponding Jacobian matrix at ( $v, \widetilde{\mathcal{P}}(v)$ ) is non-degenerate. Thus, the implicit function theorem yields the existence of a small open neighborhood $\mathcal{U}(v)$ of $v$ in the ball $B_{\frac{\delta_{*}}{C_{*}}}\left(S_{L}\right)$ such that $\mathcal{U}(v) \subseteq B$, and so the set $B$ is open in $B_{\frac{\delta_{t}}{C_{ \pm}}}\left(S_{L}\right)$.

It remains to show that $B$ is also closed in $B_{\frac{\delta_{*}}{c_{*}}}\left(S_{L}\right)$.
To this end, take any sequence $\left\{v_{n}\right\} \subseteq B$ such that $v_{n} \rightarrow v_{*}$ in $L^{2}$ for some $v_{*} \in B_{\frac{\sigma_{*}}{C_{*}}}\left(S_{L}\right)$. Then, by definition there exist modulation parameters $\widetilde{\mathcal{P}}\left(v_{n}\right) \in B_{\delta_{*}}\left(\widetilde{\mathcal{P}}_{0}\right), n \in \mathbb{N}$, such that $F\left(v_{n}, \widetilde{\mathcal{P}}\left(v_{n}\right)\right)=0$.

In particular, $\left\{\widetilde{\mathcal{P}}\left(v_{n}\right)\right\}$ is uniformly bounded in the finite dimensional space $\widetilde{Y}^{K}$. This yields that along a subsequence (still denoted by $\{n\}), \widetilde{\mathcal{P}}\left(v_{n}\right) \rightarrow \widetilde{\mathcal{P}}_{*}\left(\in B_{\delta_{*}}\left(\widetilde{\mathcal{P}}_{0}\right)\right.$ ) for some $\widetilde{\mathcal{P}}_{*} \in \mathbb{Y}^{K}$.

Let $U_{k, L, \widetilde{\mathcal{P}}\left(v_{n}\right)}$ and $U_{k, L, \widetilde{\mathcal{P}}_{*}}$ be the $k$-th blow-up profiles corresponding to $\widetilde{\mathcal{P}}\left(v_{n}\right)$ and $\widetilde{\mathcal{P}}_{*}$, respectively. Similarly denote $\varrho_{k, L, \widetilde{\mathcal{P}}\left(v_{n}\right)}$ and $\varrho_{k, L, \widetilde{\mathcal{P}}_{*}}$. Then, using the explicit expressions (2.13) and (2.16) we infer that $\langle x\rangle^{2} U_{k, L, \widetilde{\mathcal{P}}\left(v_{n}\right)} \rightarrow\langle x\rangle^{2} U_{k, L, \widetilde{\mathcal{P}}_{*}}$ in $H^{1}, \varrho_{k, L, \widetilde{\mathcal{P}}\left(v_{n}\right)} \rightarrow \varrho_{k, L, \widetilde{\mathcal{P}}_{*}}$ in $L^{2}$, and $v_{n}-\sum_{k=1}^{K} U_{k, L, \widetilde{\mathcal{P}}\left(v_{n}\right)} \rightarrow$ $v_{*}-\sum_{k=1}^{K} U_{k, L, \widetilde{\mathcal{P}}_{*}}$ in $L^{2}$. Since $F\left(v_{n}, \widetilde{\mathcal{P}}\left(v_{n}\right)\right)=0$, letting $n \rightarrow \infty$ we thus obtain $F\left(v_{*}, \widetilde{\mathcal{P}}_{*}\right)=0$. In particular, this yields that $v_{*} \in B$. Thus, $B$ is a closed set in $B_{\frac{\delta_{*}}{C_{*}}}\left(S_{L}\right)$ and (2.45) is proved.

Therefore, we obtain the geometrical decomposition (2.13) satisfying the orthogonality conditions in (2.14). Moreover, estimate (2.18) follows from (2.38) by taking $v_{1}=v$ and $v_{2}=S_{L}$. Estimate (2.19) then follows from (2.18), (2.25) and (2.26).

Regarding estimate (2.20), similarly to (2.26), we have

$$
\begin{align*}
& \left\|\nabla U_{L}-\nabla S_{L}\right\|_{L^{2}} \leq C \sum_{k=1}^{K}\left(\frac{1}{\omega_{k} L}\left|\frac{\omega_{k} L}{\lambda_{k}}-1\right|+\left|\frac{\alpha_{k}-x_{k}}{\omega_{k} L \lambda_{k}}\right|+\left|\frac{\beta_{k}}{\omega_{k} L}\right|+\left|\frac{\gamma_{k}-\omega_{k}^{2} L}{\omega_{k} L}\right|\right. \\
& \left.+\left|\frac{\lambda_{k}^{1+\frac{d}{2}}-\left(\omega_{k} L\right)^{1+\frac{d}{2}}}{\lambda_{k}\left(\omega_{k} L\right)^{1+\frac{d}{2}}}\right|+\left|\frac{\theta_{k}-\omega_{k}^{-2} L^{-1}-\vartheta_{k}}{\lambda_{k}}\right|\right) \\
& \leq C L^{-1}\left|\widetilde{\mathcal{P}}-\widetilde{\mathcal{P}}_{0}\right|, \tag{2.46}
\end{align*}
$$

which along with (2.18) yields that

$$
\begin{equation*}
\left\|\nabla R_{L}\right\|_{L^{2}} \leq\left\|\nabla v-\nabla S_{L}\right\|_{L^{2}}+\left\|\nabla S_{L}-\nabla U_{L}\right\|_{L^{2}} \leq C\left(\left\|\nabla v-\nabla S_{L}\right\|_{L^{2}}+L^{-1}\left\|v-S_{L}\right\|_{L^{2}}\right) \tag{2.47}
\end{equation*}
$$

thereby proving (2.20). Therefore, the proof of Lemma 2.3 is complete.
Proof of Theorem 2.1. We take $L=T-t$ in Lemma 2.3 and get $S_{L}=\sum_{k=1}^{K} S_{k}$, where $\left\{S_{k}\right\}$ are given by (1.10). By (1.22), $\left\|v-S_{L}\right\|_{L^{2}}=o(1)$. This yields that $\left\|v-S_{L}\right\|_{L^{2}} \leq \delta_{*}$ for any $t$ close to $T$, where $\delta_{*}$ is as in Lemma 2.3. Thus, Lemma 2.3 gives the existence of geometrical parameters $\mathcal{P}=$ $(\lambda, \alpha, \beta, \gamma, \theta) \in \mathbb{X}$, such that the geometrical decomposition (2.1) and the orthogonality conditions in (2.3) hold. Moreover, the estimates (2.6)-(2.8) follow directly from (2.17)-(2.20).

The $C^{1}$-regularity of $\mathcal{P}$ can be proved by using the regularization arguments as in [37]. More precisely, we may take a sequence $v_{n}\left(T^{*}\right) \in H^{3}, n \geq 1$, such that $v_{n}\left(T^{*}\right) \rightarrow v\left(T^{*}\right)$ in $H^{1}$ as $n \rightarrow \infty$. Let $v_{n}$ be the solutions to (NLS) corresponding to $v_{n}\left(T^{*}\right), n \geq 1$. Then, by the local well-posedness theory, for any $\tilde{t} \in\left[T^{*}, T\right)$, we have that, for $n$ large enough, $v_{n} \in C\left(\left[T^{*}, \tilde{t}\right] ; H^{3}\right) \cap C^{1}\left(\left[T^{*}, \tilde{t}\right] ; H^{1}\right)$ and $v_{n} \rightarrow v$ in $C\left(\left[T^{*}, \tilde{t}\right] ; H^{1}\right)$ as $n \rightarrow \infty$. Let $\mathcal{P}_{n}$ and $R_{n}$ be the geometrical parameter and remainder corresponding to $v_{n}, n \geq 1$. Using Lemma 2.3 and (2.38), we infer that for $n$ large enough, $\mathcal{P}_{n} \in$ $C^{1}\left(\left[T^{*}, \tilde{t}\right] ; \mathbb{X}\right)$ and $\mathcal{P}_{n} \rightarrow \mathcal{P}$ in $C\left(\left[T^{*}, \tilde{t}\right] ; \mathbb{X}\right)$ as $n \rightarrow \infty$. Moreover, the computations of modulation equations in Subsection 2.2 below show that $\dot{\mathcal{P}}_{n}$ can be expressed in terms of $\mathcal{P}_{n}$ and the remainder $R_{n}$, and thus $\dot{\mathcal{P}}_{n}$ converges in $C\left(\left[T^{*}, \tilde{t}\right] ; \mathbb{X}\right)$. This yields that $\mathcal{P} \in C^{1}\left(\left[T^{*}, \tilde{t}\right] ; \mathbb{X}\right)$. Thus, the $C^{1}-$ regularity of $\mathcal{P}$ on $\left[T^{*}, T\right)$ follows, due to the arbitrariness of $\tilde{t}$.

Therefore, the proof of Theorem 2.1 is complete.
Below we introduce the localized remainder $R_{k}$ and the localization function $\Phi_{k}$ as in [53], $1 \leq k \leq K$. We also refer the readers to [11, 35, 41], where different types of localization functions are introduced in the setting of multi-solitons.

Because (NLS) is invariant under the orthogonal transform, we may take an orthonormal basis $\left\{\mathbf{v}_{j}\right\}_{j=1}^{d}$ of $\mathbb{R}^{d}$ as in [39], such that $\left(x_{j}-x_{k}\right) \cdot \mathbf{v}_{1} \neq 0$ for any $j \neq k$. Without loss of generality, we may assume that $x_{1} \cdot \mathbf{v}_{1}<x_{2} \cdot \mathbf{v}_{1}<\cdots<x_{K} \cdot \mathbf{v}_{1}$. Then,

$$
\begin{equation*}
\sigma:=\frac{1}{12} \min _{1 \leq k \leq K-1}\left\{\left(x_{k+1}-x_{k}\right) \cdot \mathbf{v}_{1}\right\}>0 . \tag{2.48}
\end{equation*}
$$

Let $\Phi(x)$ be a smooth function on $\mathbb{R}^{d}$ such that $0 \leq \Phi(x) \leq 1, \Phi(x)=1$ for $x \cdot \mathbf{v}_{1} \leq 4 \sigma, \Phi(x)=0$ for $x \cdot \mathbf{v}_{1} \geq 8 \sigma$ and $\|\nabla \Phi\|_{L^{\infty}} \leq C \sigma^{-1}$. The localization functions $\Phi_{k}$ are defined by

$$
\begin{align*}
& \Phi_{1}(x):=\Phi\left(x-x_{1}\right), \quad \Phi_{K}(x):=1-\Phi\left(x-x_{K-1}\right), \\
& \Phi_{k}(x):=\Phi\left(x-x_{k}\right)-\Phi\left(x-x_{k-1}\right), \quad 2 \leq k \leq K-1 . \tag{2.49}
\end{align*}
$$

In particular, we have the partition of unity $1=\sum_{k=1}^{K} \Phi_{k}$ and the decomposition

$$
\begin{equation*}
R=\sum_{k=1}^{K} R_{k}, \text { with } R_{k}:=R \Phi_{k} . \tag{2.50}
\end{equation*}
$$

As a consequence of Theorem 2.1 and the decoupling Lemma 6.2 in Appendix, the following almost orthogonality holds for the localized remainders.

Lemma 2.4. (Almost orthogonality) There exist $C, \delta>0$ such that for $1 \leq k \leq K$ and $t$ close to $T$,

$$
\begin{align*}
& \left|\operatorname{Re} \int\left(x-\alpha_{k}\right) U_{k} \overline{R_{k}} d x\right|+\left|\operatorname{Re} \int\right| x-\left.\alpha_{k}\right|^{2} U_{k} \overline{R_{k}} d x \left\lvert\, \leq C e^{\left.-\frac{\delta}{T-t} \right\rvert\,}\|R\|_{L^{2}}\right.  \tag{2.51}\\
& \left|\operatorname{Im} \int \nabla U_{k} \overline{R_{k}} d x\right|+\left|\operatorname{Im} \int \Lambda_{k} U_{k} \overline{R_{k}} d x\right|+\left|\operatorname{Im} \int \varrho_{k} \bar{R} d x\right| \leq C e^{-\frac{\delta}{T-i}\|R\|_{L^{2}}}
\end{align*}
$$

2.2. Modulation equations. The leading order of the dynamics of geometrical parameters are characterized by the vector of modulation equations, defined by

$$
\operatorname{Mod}_{k}:=\left|\lambda_{k} \dot{\lambda}_{k}+\gamma_{k}\right|+\left|\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right|+\left|\lambda_{k} \dot{\alpha}_{k}-2 \beta_{k}\right|+\left|\lambda_{k}^{2} \dot{\beta}_{k}+\gamma_{k} \beta_{k}\right|+\left|\lambda_{k}^{2} \dot{\theta}_{k}-1-\left|\beta_{k}\right|^{2}\right|
$$

where $\dot{\lambda}_{k}:=\frac{d}{d t} \lambda_{k}$ and similar notations apply to the remaining four geometrical parameters. Set

$$
\operatorname{Mod}:=\sum_{k=1}^{K} \operatorname{Mod}_{k} .
$$

In particular, for the pseudo-conformal blow-up solution $S_{k}$ given by (1.10), we have $\operatorname{Mod}=0$.
We also use the notation

$$
\begin{equation*}
P:=\sum_{k=1}^{K}\left(\left|\lambda_{k}\right|+\left|\alpha_{k}-x_{k}\right|+\left|\beta_{k}\right|+\left|\gamma_{k}\right|\right) . \tag{2.52}
\end{equation*}
$$

The main control of modulation equations is presented below.
Theorem 2.5. (Control of modulation equations) There exists $C>0$ such that for $t$ close to $T$,

$$
\begin{equation*}
\operatorname{Mod}(t) \leq C\left(\sum_{k=1}^{K}\left|M_{k}(t)\right|+P^{2} D(t)+D^{2}(t)+e^{-\frac{\delta}{T-t}}\right) \tag{2.53}
\end{equation*}
$$

where the quantity

$$
\begin{equation*}
D(t):=\|R(t)\|_{L^{2}}+(T-t)\|\nabla R(t)\|_{L^{2}} \tag{2.54}
\end{equation*}
$$

and for every $1 \leq k \leq K$,

$$
\begin{equation*}
M_{k}(t):=2 \operatorname{Re}\left\langle R_{k}(t), U_{k}(t)\right\rangle+\int|R(t)|^{2} \Phi_{k} d x \text {, with } R_{k}=R \Phi_{k} . \tag{2.55}
\end{equation*}
$$

Moreover, the following identity holds

$$
\begin{align*}
\frac{1}{4}\|y Q\|_{2}^{2}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right)= & M_{k}-\int|R|^{2} \Phi_{k} d x+\operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{d}}\left|\varepsilon_{k}\right|^{2}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{d}-2} Q_{k}^{2}{\overline{\varepsilon_{k}}}^{2} d y \\
& +\operatorname{Re} \int\left(y \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y \\
& +O\left(\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right) M o d+P^{2}\|R\|_{L^{2}}+D^{3}+e^{-\frac{\delta}{T-t}}\right), \tag{2.56}
\end{align*}
$$

where the renormalized remainder $\varepsilon_{k}$ is defined by

$$
\begin{equation*}
R_{k}(t, x)=\lambda_{k}^{-\frac{d}{2}} \varepsilon_{k}\left(t, \frac{x-\alpha_{k}}{\lambda_{k}}\right) e^{i \theta_{k}} \tag{2.57}
\end{equation*}
$$

and

$$
\begin{equation*}
f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}:=\frac{2}{d}\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{d}-2} Q_{k}\left|\varepsilon_{k}\right|^{2}+\frac{1}{d}\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{d}-2} \overline{Q_{k}} \varepsilon_{k}^{2}+\frac{1}{d}\left(\frac{2}{d}-1\right)\left|Q_{k}\right|^{\frac{4}{d}-4} Q_{k}^{3} \bar{\varepsilon}_{k}^{2} . \tag{2.58}
\end{equation*}
$$

Remark 2.6. The identity (2.56) is important in the derivation of the monotonicity of (modified) localized virial functionals (See Theorems 3.8 and 3.13). In fact, this specific algebraic identity provides the necessary quadratic terms to perform the localized coercivity of linearized operators.

The proof of Theorem 2.5 is similar to that of [53, Proposition 4.3] and thus is postponed to the Appendix for the simplicity of exposition.

We end this section with the preliminary estimates of remainder related to the quantity $D$.

Lemma 2.7. (Preliminary estimates of remainder) We have that for t close to $T$,

$$
\begin{equation*}
\|R(t)\|_{H^{1}} \leq C \frac{D(t)}{T-t}, \quad\|R(t)\|_{L^{2}}\|\nabla R(t)\|_{L^{2}} \leq \frac{D^{2}(t)}{T-t}, \tag{2.59}
\end{equation*}
$$

and

$$
\begin{equation*}
\|R(t)\|_{L^{p}}^{p} \leq C(T-t)^{-d\left(\frac{p}{2}-1\right)} D^{p}(t), \quad 2 \leq p<\infty . \tag{2.60}
\end{equation*}
$$

Moreover, we have

$$
\begin{equation*}
D(t)=o(1), \text { for } t \text { close to } T . \tag{2.61}
\end{equation*}
$$

Proof. Estimate (2.59) follows directly from the definition of $D$, and (2.60) follows from the Gagliardo-Nirenberg inequality (6.1) below. Estimate (2.61) is a consequence of the preliminary estimate of remainder in (2.8).

## 3. Localized mass, energy and localized virial functionals

This section mainly contains the crucial controls of localized mass, energy and (modified) localized virial functionals.

We first deal with the sharp estimate of localized mass and its relationship with the remainder in Subsection 3.1. The coercivity type control of energy will be also proved there, which allows to upgrade the convergence rate of remainder to the first order, i.e., $D=O(T-t)$. Then, Subsection 3.2 contains the key monotonicity of (modified) localized virial functionals, which yield the refined space time estimate of remainder and the precise asymptotic order of parameter $\gamma$. By virtue of these estimates, in Subsection 3.3 we are able to prove the $H^{1}$ dispersion of remainder along a sequence and obtain the energy quantization, which is the key towards the derivation of refined energy estimate.
3.1. Localized mass and energy. Recall from (2.55) that

$$
\begin{equation*}
M_{k}=2 \operatorname{Re}\left\langle R_{k}, U_{k}\right\rangle+\int|R|^{2} \Phi_{k} d x \tag{3.1}
\end{equation*}
$$

where $R_{k}=R \Phi_{k}$ and $\left\{\Phi_{k}\right\}$ are the localization functions given by (2.49). Let $\widetilde{R}:=v-\sum_{k=1}^{K} S_{k}$ with $S_{k}$ given by (1.10), $1 \leq k \leq K$.

The main estimates of localized mass are formulated in Theorem 3.1 below.
Theorem 3.1. (Control of localized mass) For every $1 \leq k \leq K$ and for any t close to $T$, we have

$$
\begin{align*}
& \left|M_{k}(t)\right| \leq C \int_{t}^{T} \int_{s}^{T}\|\widetilde{R}\|_{H^{1}}^{2} d r d s+C e^{-\frac{\delta}{T-t}} \leq C(T-t)^{2+\zeta},  \tag{3.2}\\
& \left|M_{k}(t)\right| \leq C\|\nabla \Phi\|_{L^{\infty}} \int_{t}^{T} \frac{D^{2}(s)}{T-s} d s+C e^{-\frac{\delta}{T-t}} . \tag{3.3}
\end{align*}
$$

Moreover, for any $\varepsilon>0$ as in Case (I) and Case (II), we have that for $t$ close to $T$

$$
\begin{equation*}
\left|\sum_{k=1}^{K} \frac{M_{k}(t)}{\lambda_{k}^{2}(t)}\right| \leq \frac{C \varepsilon}{(T-t)^{2}} \int_{t}^{T} \frac{D^{2}(s)}{T-s} d s+C e^{-\frac{\delta}{T-t}}, \tag{3.4}
\end{equation*}
$$

where $C, \delta>0$ are independent of $\varepsilon$ and $t$.
Assume additionally that for $t$ close to $T$

$$
\begin{equation*}
\gamma_{k}(t)=\omega_{k}^{2}(T-t)+o(T-t), \tag{3.5}
\end{equation*}
$$

then we also have

$$
\begin{equation*}
\left|\sum_{k=1}^{K} \frac{\gamma_{k}(t)}{\lambda_{k}^{4}(t)} M_{k}(t)\right| \leq \frac{C \varepsilon}{(T-t)^{3}} \int_{t}^{T} \frac{D^{2}(s)}{T-s} d s+C e^{-\frac{\delta}{T-t}} \tag{3.6}
\end{equation*}
$$

Remark 3.2. One advantage of the estimate of localized mass is that it allows to control the unstable direction $Q$ in the scalar (6.14) below, while the (almost) orthogonality in the previous section controls the remaining five unstable directions.

Remark 3.3. Estimate (2.8) and the simple inequality $\left|M_{k}\right| \leq C\left(\|R\|_{L^{2}}+\|R\|_{L^{2}}^{2}\right)$ merely give the crude estimate $M_{k}(t)=o(1)$, which is insufficient to run the upgradation procedure.

In order to explore higher convergence rate, the keypoint here is that two more orders ( $T$ $t)^{2+}$ can be gained by the local virial identities. This convergence rate is indeed almost sharp to justify the integrability in (3.2) below. Moreover, estimates (3.3)-(3.6) relate the localized mass and the remainder together and enable us to upgrade their estimates simultaneously by establishing Gronwall type inequalities.

Proof of Theorem 3.1. Using the geometrical decomposition (2.1) and Lemma 6.2 in Appendix we have that for $1 \leq k \leq K$,

$$
\begin{aligned}
\int|v|^{2} \Phi_{k} d x & =\int|U|^{2} \Phi_{k} d x+\int|R|^{2} \Phi_{k} d x+2 \operatorname{Re} \int \bar{U} R \Phi_{k} d x \\
& =\int|U|^{2} \Phi_{k} d x+M_{k}+O\left(e^{-\frac{\delta}{T-t}\|R\|_{L^{2}}}\right)
\end{aligned}
$$

where $\delta>0$. This yields that

$$
\begin{equation*}
M_{k}=\int|v|^{2} \Phi_{k} d x-\int|U|^{2} \Phi_{k} d x+O\left(e^{\left.-\frac{\delta}{T-t}\|R\|_{L^{2}}\right) . . . . .}\right. \tag{3.7}
\end{equation*}
$$

Note that, by (1.23), there exists a sequence $\left\{t_{n}\right\}$ to $T$ such that

$$
\begin{equation*}
\lim _{t_{n} \rightarrow T}\left\|\widetilde{R}\left(t_{n}\right)\right\|_{H^{1}}=0 \tag{3.8}
\end{equation*}
$$

Then, we split

$$
\begin{align*}
& \int|v(t)|^{2} \Phi_{k} d x-\int|U(t)|^{2} \Phi_{k} d x \\
= & \left(\int|v(t)|^{2} \Phi_{k} d x-\int\left|v\left(t_{n}\right)\right|^{2} \Phi_{k} d x\right)+\left(\int\left|v\left(t_{n}\right)\right|^{2} \Phi_{k} d x-\int\left|S\left(t_{n}\right)\right|^{2} \Phi_{k} d x\right) \\
& +\left(\int\left|S\left(t_{n}\right)\right|^{2} \Phi_{k} d x-\int|U(t)|^{2} \Phi_{k} d x\right), \tag{3.9}
\end{align*}
$$

where $S=\sum_{k=1}^{K} S_{k}$. Note that, by the asymptotic (1.22) and the conservation law of mass,

$$
\begin{align*}
\left.\lim _{t_{n} \rightarrow T}\left|\int\right| v\left(t_{n}\right)\right|^{2} \Phi_{k} d x-\int\left|S\left(t_{n}\right)\right|^{2} \Phi_{k} d x \mid & \leq C \lim _{t_{n} \rightarrow T}\left\|v\left(t_{n}\right)-S\left(t_{n}\right)\right\|_{L^{2}}\left(\left\|v\left(t_{n}\right)\right\|_{L^{2}}+\sum_{k=1}^{K}\left\|S_{k}\left(t_{n}\right)\right\|_{L^{2}}\right) \\
& \leq C \lim _{t_{n} \rightarrow T}\left\|v\left(t_{n}\right)-S\left(t_{n}\right)\right\|_{L^{2}}=0, \tag{3.10}
\end{align*}
$$

and by Lemma 6.2 in Appendix below,

$$
\begin{equation*}
\lim _{t_{n} \rightarrow T} \int\left|S\left(t_{n}\right)\right|^{2} \Phi_{k} d x=\|Q\|_{L^{2}}^{2}, \quad \int|U(t)|^{2} \Phi_{k} d x=\|Q\|_{L^{2}}^{2}+O\left(e^{-\frac{\delta}{T-t}}\right) \tag{3.11}
\end{equation*}
$$

which yield that

$$
\begin{equation*}
\left.\limsup _{t_{n} \rightarrow T}\left|\int\right| S\left(t_{n}\right)\right|^{2} \Phi_{k} d x-\int|U(t)|^{2} \Phi_{k} d x \left\lvert\,=O\left(e^{-\frac{\delta}{T-t}}\right) .\right. \tag{3.12}
\end{equation*}
$$

Thus, plugging (3.9), (3.10) and (3.12) into (3.7) we obtain

$$
\begin{equation*}
\left|M_{k}(t)\right| \leq\left.\limsup _{t_{n} \rightarrow T}\left|\int\right| v(t)\right|^{2} \Phi_{k} d x-\int\left|v\left(t_{n}\right)\right|^{2} \Phi_{k} d x \left\lvert\,+C e^{-\frac{\delta}{T-t}} .\right. \tag{3.13}
\end{equation*}
$$

To estimate the R.H.S. above, we use the local virial identities (see [44, Lemma 3.6])

$$
\begin{align*}
\frac{d}{d t} \int|v|^{2} \Phi_{k} d x & =2 \operatorname{Im} \int \bar{v} \nabla v \cdot \nabla \Phi_{k} d x  \tag{3.14}\\
\frac{d}{d t}\left(\operatorname{Im} \int \bar{v} \nabla v \cdot \nabla \Phi_{k} d x\right) & =2 \operatorname{Re} \int \nabla^{2} \Phi_{k}(\nabla v, \nabla \bar{v}) d x-\frac{2}{2+d} \int \Delta \Phi_{k}|v|^{2+\frac{4}{d}} d x-\frac{1}{2} \int \Delta^{2} \Phi_{k}|v|^{2} d x \tag{3.15}
\end{align*}
$$

Since by the Gagliardo-Nirenberg inequality (6.1) below and the conservation law of mass,

$$
\|\widetilde{R}\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \leq C\|\widetilde{R}\|_{L^{2}}^{\frac{4}{d}}\|\nabla \widetilde{R}\|_{L^{2}}^{2} \leq C\left(\|v\|_{L^{2}}^{\frac{4}{d}}+\|S\|_{L^{2}}^{\frac{4}{d}}\right)\|\nabla \widetilde{R}\|_{L^{2}}^{2} \leq C\|\nabla \widetilde{R}\|_{L^{2}}^{2} .
$$

Taking into account the exponential decay of $S$ on the support of $\nabla \Phi_{k}$ we get from (3.15)

$$
\begin{align*}
\left|\frac{d}{d t}\left(\operatorname{Im} \int \bar{v} \nabla v \cdot \nabla \Phi_{k} d x\right)\right| & \leq C \int_{\left|x-x_{k}\right| \geq 4 \sigma, 1 \leq k \leq K}|\nabla S+\nabla \widetilde{R}|^{2}+|S+\widetilde{R}|^{2+\frac{4}{d}}+|S+\widetilde{R}|^{2} d x \\
& \leq C\left(\|\widetilde{R}\|_{H^{1}}^{2}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.16}
\end{align*}
$$

Similarly,

$$
\begin{align*}
\lim _{t_{n} \rightarrow T}\left|\operatorname{Im} \int \bar{v}\left(t_{n}\right) \nabla v\left(t_{n}\right) \cdot \nabla \Phi_{k} d x\right| & \leq \lim _{t_{n} \rightarrow T} \int_{\left|x-x_{k}\right| \geq 4 \sigma, 1 \leq k \leq K}\left|S\left(t_{n}\right)+\widetilde{R}\left(t_{n}\right) \| \nabla S\left(t_{n}\right)+\nabla \widetilde{R}\left(t_{n}\right)\right| d x \\
& \leq C \lim _{t_{n} \rightarrow T}\left(\left\|\widetilde{R}\left(t_{n}\right)\right\|_{H^{1}}^{2}+e^{-\frac{\delta}{T-t_{n}}}\right)=0 . \tag{3.17}
\end{align*}
$$

Thus, integrating (3.15) from $t$ to $t_{n}$ and using (3.16) and the boundary estimate (3.17) we obtain

$$
\begin{equation*}
\left|\operatorname{Im} \int \bar{v}(t) \nabla v(t) \cdot \nabla \Phi_{k} d x\right| \leq C\left(\int_{t}^{T}\|\widetilde{R}\|_{H^{1}}^{2} d s+e^{-\frac{\delta}{T-t}}\right) \tag{3.18}
\end{equation*}
$$

Plugging this into (3.14) and then integrating both sides again we obtain

$$
\begin{equation*}
\left.\left|\int\right| v\left(t_{n}\right)\right|^{2} \Phi_{k} d x-\int|v(t)|^{2} \Phi_{k} d x \left\lvert\, \leq C\left(\int_{t}^{T} \int_{s}^{T}\|\widetilde{R}\|_{H^{1}}^{2} d r d s+e^{-\frac{\delta}{T-t}}\right)\right. \tag{3.19}
\end{equation*}
$$

which, via (3.13), yields the first inequality in (3.2). The second inequality follows from (1.23).
Using (2.1), (3.13), (3.14) and the integration by parts formula to move the derivative to $U$ yields

$$
\begin{align*}
\left|M_{k}(t)\right| & \leq\left|\int_{t}^{T} 2 \operatorname{Im} \int \bar{v} \nabla v \cdot \nabla \Phi_{k} d x d s\right|+C e^{-\frac{\delta}{T-t}} \\
& \leq C\|\nabla \Phi\|_{L^{\infty}} \int_{t}^{T} \int_{\left|x-x_{k}\right| \geq 4 \sigma, 1 \leq k \leq K}|\nabla U \| U|+(|\nabla U|+|U|)|R|+|R||\nabla R| d x d s+C e^{-\frac{\delta}{T-t}} . \tag{3.20}
\end{align*}
$$

Then, using (2.59) and the exponential decay of $Q$ in (6.8) we obtain

$$
\left|M_{k}(t)\right| \leq C\|\nabla \Phi\|_{L^{\infty}} \int_{t}^{T}\|R\|_{L^{2}}\|\nabla R\|_{L^{2}} d s+C e^{-\frac{\delta}{T-t}}
$$

$$
\begin{equation*}
\leq C\|\nabla \Phi\|_{L^{\infty}} \int_{t}^{T} \frac{D^{2}(s)}{T-s} d s+C e^{-\frac{\delta}{T-t}} \tag{3.21}
\end{equation*}
$$

which yields (3.3).
Next, we prove estimate (3.4). In Case (II), estimate (3.4) follows immediately from (2.6) and (3.3), as $\|\nabla \Phi\|_{L^{\infty}} \leq C \sigma^{-1} \leq C \varepsilon$. Below we deal with Case (I).

The observation here is that, a refined exponential estimate can be gained for the sum of $M_{k}$, i.e.,

$$
\begin{equation*}
\sum_{k=1}^{K} M_{k}(t)=O\left(e^{-\frac{\delta}{T-t}}\right), \text { for } t \text { close to } T \tag{3.22}
\end{equation*}
$$

Actually, on one hand, by the conservation law of mass, (3.8) and Lemma 6.2 below,

$$
\begin{equation*}
\|v(t)\|_{L^{2}}^{2}=\lim _{t_{n} \rightarrow T}\left\|v\left(t_{n}\right)\right\|_{L^{2}}^{2}=\lim _{t_{n} \rightarrow T} \sum_{k=1}^{K}\left\|S_{k}\left(t_{n}\right)\right\|_{L^{2}}^{2}=K\|Q\|_{L^{2}}^{2} \tag{3.23}
\end{equation*}
$$

On the other hand, using (2.1) to expand the mass $\|v\|_{L^{2}}^{2}$ around the blow-up profile $U$ we get

$$
\begin{equation*}
\|v\|_{L^{2}}^{2}=\|U\|_{L^{2}}^{2}+2 \operatorname{Re}\langle U, R\rangle+\|R\|_{L^{2}}^{2}=K\|Q\|_{L^{2}}^{2}+\sum_{k=1}^{K} M_{k}+O\left(e^{-\frac{\delta}{T-t}}\right), \tag{3.24}
\end{equation*}
$$

where we also used Lemma 6.2 to decouple different bubbles in the last step. Thus, combining the two identities (3.23) and (3.24) we obtain (3.22), as claimed.

Then, since $\lambda_{k}=\widetilde{\lambda}_{k} \omega_{k}(T-t)$, due to (2.17) and (2.18) with $L$ replaced by $T-t$, we see that

$$
\begin{equation*}
\left|\frac{1}{\lambda_{k}^{2}}-\frac{1}{\omega^{2}(T-t)^{2}}\right|=\left|\frac{\left(\omega^{2}-\omega_{k}^{2}\right)+\left(1-\widetilde{\lambda}_{k}^{2}\right) \omega_{k}^{2}}{\widetilde{\lambda}_{k}^{2} \omega_{k}^{2} \omega^{2}(T-t)^{2}}\right| \leq C \frac{\left|\omega-\omega_{k}\right|+\left|\widetilde{\lambda}_{k}-1\right|}{(T-t)^{2}}, \tag{3.25}
\end{equation*}
$$

where $\omega$ is as in Case (I). Taking into account $\left|\omega_{k}-\omega\right| \leq \varepsilon$ in Case (I) and the estimate of $\widetilde{\lambda}_{k}$ in (2.18) with $S_{L}$ replaced by $S$ we get

$$
\begin{equation*}
\left|\frac{1}{\lambda_{k}^{2}}-\frac{1}{\omega^{2}(T-t)^{2}}\right| \leq C \frac{\varepsilon+\|v-S\|_{L^{2}}}{(T-t)^{2}} . \tag{3.26}
\end{equation*}
$$

Thus, combining (3.22) and (3.26) together we arrive that

$$
\begin{align*}
\left|\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{2}}\right| & =\left|\sum_{k=1}^{K}\left(\frac{1}{\lambda_{k}^{2}}-\frac{1}{\omega^{2}(T-t)^{2}}\right) M_{k}+\frac{1}{\omega^{2}(T-t)^{2}} \sum_{k=1}^{K} M_{k}\right| \\
& \leq C \frac{\varepsilon+\|v-S\|_{L^{2}}}{(T-t)^{2}} \sum_{k=1}^{K}\left|M_{k}\right|+C e^{-\frac{\delta}{T-t}} \\
& \leq C \frac{\varepsilon+\|v-S\|_{L^{2}}}{(T-t)^{2}} \int_{t}^{T} \frac{D^{2}(s)}{T-s} d s+C e^{-\frac{\delta}{T-t}} \tag{3.27}
\end{align*}
$$

where we also used (3.3) in the last step, and $C, \delta>0$ are independent of $\varepsilon$. Thus, taking $t$ close to $T$ such that $\|v-S\|_{L^{2}} \leq \varepsilon$ we obtain (3.4).

It remains to prove (3.6). In Case (II), using (2.6), (3.3) and (3.5) we have

$$
\left|\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k}\right| \leq \frac{C}{(T-t)^{3}} \sum_{k=1}^{K}\left|M_{k}\right| \leq \frac{C\|\nabla \Phi\|_{L^{\infty}}}{(T-t)^{3}} \sum_{k=1}^{K} \int_{t}^{T} \frac{D^{2}(s)}{T-s} d s+C e^{-\frac{\delta}{T-t}} .
$$

Then, taking into account $\|\nabla \Phi\|_{L^{\infty}} \leq C \sigma^{-1} \leq C \varepsilon$ we obtain (3.6).

Regarding Case (I) we use similar arguments as in the proof of (3.4). For simplicity, we set $\lambda_{0}=\omega(T-t)$ and $\gamma_{0}=\omega^{2}(T-t)$, where $\omega$ is as in Case (I). By (2.6) and (3.5),

$$
\begin{align*}
& \lambda_{k}=\widetilde{\lambda}_{k} \omega_{k}(T-t), \text { with } \widetilde{\lambda}_{k}=1+o(1),  \tag{3.28}\\
& \gamma_{k}=\widetilde{\gamma}_{k}+\omega_{k}^{2}(T-t), \text { with } \widetilde{\gamma}_{k}=o(T-t) . \tag{3.29}
\end{align*}
$$

Then, in view of (3.22), we infer that

$$
\begin{equation*}
\left|\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k}\right|=\left|\sum_{k=1}^{K}\left(\frac{\gamma_{k}}{\lambda_{k}^{4}}-\frac{\gamma_{0}}{\lambda_{0}^{4}}\right) M_{k}+\frac{\gamma_{0}}{\lambda_{0}^{4}} \sum_{k=1}^{K} M_{k}\right| \leq C\left(\sum_{k=1}^{K}\left|\frac{\gamma_{k}}{\lambda_{k}^{4}}-\frac{\gamma_{0}}{\lambda_{0}^{4}}\right|\left|M_{k}\right|+e^{-\frac{\delta}{T-t}}\right) . \tag{3.30}
\end{equation*}
$$

Note that, by the straightforward computations,

$$
\left|\frac{\gamma_{k}}{\lambda_{k}^{4}}-\frac{\gamma_{0}}{\lambda_{0}^{4}}\right|=(T-t)^{-3}\left(\widetilde{\lambda}_{k}^{-4} \omega_{k}^{-4} \omega^{-4}\right)\left|\frac{\widetilde{\gamma}_{k}}{T-t} \omega^{4}+\omega_{k}^{2} \omega^{4}-\omega^{2} \widetilde{\lambda}_{k}^{4} \omega_{k}^{4}\right|,
$$

and

$$
\omega_{k}^{2} \omega^{4}-\omega^{2} \widetilde{\lambda}_{k}^{4} \omega_{k}^{4}=\omega_{k}^{2} \omega^{2}\left(\left(1-\widetilde{\lambda}_{k}\right)\left(1+\widetilde{\lambda}_{k}\right)\left(1+\widetilde{\lambda}_{k}^{2}\right) \omega^{2}+\left(\omega-\omega_{k}\right)\left(\omega+\omega_{k}\right) \widetilde{\lambda}_{k}^{4}\right) .
$$

Then, using (3.28) and (3.29) and taking $t$ close to $T$ such that $\left|\frac{\widetilde{\gamma}_{k}}{T-t}\right|+\left|1-\widetilde{\lambda}_{k}\right| \leq \varepsilon$ we obtain

$$
\left|\frac{\gamma_{k}}{\lambda_{k}^{4}}-\frac{\gamma_{0}}{\lambda_{0}^{4}}\right| \leq \frac{C \varepsilon}{(T-t)^{3}},
$$

where $C>0$ is independent of $\varepsilon$.
Therefore, inserting this into (3.30) and using (3.3) we obtain (3.6) and finish the proof.
Theorem 3.4 below contains the coercivity type control of energy.
Theorem 3.4. (Coercivity of energy) There exist $C_{1}, C_{2}>0$ such that for t close to $T$,

$$
\begin{align*}
& \sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}}\|Q\|_{2}^{2}+\sum_{k=1}^{K} \frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|y Q\|_{L^{2}}^{2}+C_{1} \frac{D^{2}}{(T-t)^{2}} \\
\leq & E(v)+\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}+\sum_{k=1}^{K} \frac{M_{k}}{2 \lambda_{k}^{2}}+C_{2}\left(\sum_{k=1}^{K} \frac{M_{k}^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.31}
\end{align*}
$$

Remark 3.5. At this stage, the exact value of energy $E(v)$ is unclear, due to the low convergence rate in (1.22). The precise identification of $E(v)$ will be proved in Subsection 3.3 below. As we shall see, the energy indeed admits the quantization into the sum of the energies of pseudo-conformal blowup solutions. The key ingredients for this fact are the $H^{1}$ dispersion of remainder and the refined estimates of geometrical parameters along a sequence, which in turn rely on the monotonicity of localized virial functional in Subsection 3.2 below.

Proof. The arguments presented below follow the lines as in the proof of [53, Lemma 5.14] and give more precise estimates of errors.

According to (3.1), for $1 \leq k \leq K$,

$$
\frac{1}{\lambda_{k}^{2}} \operatorname{Re} \int \overline{U_{k}} R_{k}+\frac{1}{2}|R|^{2} \Phi_{k} d x-\frac{1}{2 \lambda_{k}^{2}} M_{k}=0 .
$$

Thus we first reformulate the energy

$$
\begin{equation*}
E(v)=E(v)+\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}} \operatorname{Re} \int \overline{U_{k}} R_{k}+\frac{1}{2}|R|^{2} \Phi_{k} d x-\sum_{k=1}^{K} \frac{1}{2 \lambda_{k}^{2}} M_{k} . \tag{3.32}
\end{equation*}
$$

Then, expanding the energy $E(v)$ around the main blow-up profile $U$ up to the second order of remainder we obtain that, similarly to [53, (5.76)],

$$
\begin{align*}
E(v)= & \sum_{k=1}^{K}\left(\frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}}\|Q\|_{L^{2}}^{2}+\frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|y Q\|_{L^{2}}^{2}\right) \\
& -\sum_{k=1}^{K} \operatorname{Re} \int\left(\Delta U_{k}-\frac{1}{\lambda_{k}^{2}} U_{k}+\left|U_{k}\right|^{\frac{4}{d}} U_{k}\right) \overline{R_{k}} d x \\
& +\frac{1}{2} \operatorname{Re} \int|\nabla R|^{2}+\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}}|R|^{2} \Phi_{k}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}^{2} d x \\
& -\sum_{k=1}^{K} \frac{M_{k}}{2 \lambda_{k}^{2}}+O\left(\sum_{l=3}^{2+\frac{4}{d}} \int|U|^{2+\frac{4}{d}-l}|R|^{l} d x+e^{-\frac{\delta}{T-t}}\right) \\
= & \sum_{j=0}^{2} E_{j}-\sum_{k=1}^{K} \frac{M_{k}}{2 \lambda_{k}^{2}}+O\left(\sum_{l=3}^{2+\frac{4}{d}} \int|U|^{2+\frac{4}{d}-l}|R|^{l} d x+e^{-\frac{\delta}{T-t}}\right), \tag{3.33}
\end{align*}
$$

where the exponential decay error arises from the interactions between different bubbles due to Lemma 6.2, while the remaining errors come from the remainders of orders higher than two in the Taylor expansion of nonlinearity. Note that, this error can be bounded easily by using (2.8), (2.60) and the fact that $\|U\|_{L^{\infty}} \leq C(T-t)^{-\frac{d}{2}}$ :

$$
\begin{align*}
\sum_{l=3}^{2+\frac{4}{d}} \int|U|^{2+\frac{4}{d}-l}|R|^{l} d x & \leq C \sum_{l=3}^{2+\frac{4}{d}}(T-t)^{-\frac{d}{2}\left(2+\frac{4}{d}-l\right)}\|R\|_{L^{l}}^{l} \\
& \leq C \sum_{l=3}^{2+\frac{4}{d}}(T-t)^{-2} D^{l}(t) \leq C(T-t)^{-2} D^{3}(t) \tag{3.34}
\end{align*}
$$

For the linear terms of $R$ in $E_{1}$, we infer from (6.25) and the renormalized variable $\varepsilon_{k}$ in (2.57) that

$$
\begin{equation*}
E_{1}=-\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}} \operatorname{Im} \int\left(\gamma_{k} \Lambda Q_{k}-2 \beta_{k} \cdot \nabla Q_{k}\right) \overline{\varepsilon_{k}} d y-\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}} \operatorname{Re} \int\left|\beta_{k}-\frac{\gamma_{k}}{2} y\right|^{2} Q_{k} \overline{\varepsilon_{k}} d y . \tag{3.35}
\end{equation*}
$$

Note that, by the almost orthogonality in Lemma 2.4,

$$
\begin{equation*}
\frac{1}{\lambda_{k}^{2}} \operatorname{Im} \int\left(\gamma_{k} \Lambda Q_{k}-2 \beta_{k} \cdot \nabla Q_{k}\right) \overline{\varepsilon_{k}} d x=O\left(e^{\left.-\frac{\delta}{T-t}\|R\|_{L^{2}}\right) . . . . . . .}\right. \tag{3.36}
\end{equation*}
$$

Moreover, using Lemma 2.4 again and (3.1) we have

$$
\begin{align*}
\frac{1}{\lambda_{k}^{2}} \operatorname{Re} \int\left|\beta_{k}-\frac{\gamma_{k}}{2} y\right|^{2} Q_{k} \overline{\varepsilon_{k}} d y & =\frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}} \operatorname{Re} \int Q_{k} \overline{\varepsilon_{k}} d y+O\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right) \\
& =\frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}+O\left(\frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}\|R\|_{L^{2}}^{2}+e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right) . \tag{3.37}
\end{align*}
$$

Inserting (3.36) and (3.37) into (3.35) we obtain that

$$
\begin{equation*}
E_{1}=-\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}+O\left(\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}\|R\|_{L^{2}}^{2}+e^{-\frac{\delta}{T-t}\|R\|_{L^{2}}}\right) . \tag{3.38}
\end{equation*}
$$

Regarding the quadratic terms of $R$ in $E_{2}$, we claim that for some $C>0$,

$$
\begin{equation*}
E_{2} \geq C \frac{D^{2}(t)}{(T-t)^{2}}+O\left(\sum_{k=1}^{K} \frac{M_{k}^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) \tag{3.39}
\end{equation*}
$$

In order to prove (3.39), as in $[53,(5.80)]$, using the renormalized variable $\widetilde{\varepsilon}_{k}$ defined by

$$
\begin{equation*}
R(t, x)=\lambda_{k}^{-\frac{d}{2}} \widetilde{\varepsilon}_{k}\left(t, \frac{x-\alpha_{k}}{\lambda_{k}}\right) e^{i \theta_{k}}, \tag{3.40}
\end{equation*}
$$

we have

$$
\begin{align*}
E_{2}= & \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}} \operatorname{Re} \int\left(\left|\nabla \widetilde{\varepsilon}_{k}\right|^{2}+\left|\widetilde{\varepsilon}_{k}\right|^{2}\right) \phi_{A}-\left(1+\frac{2}{d}\right) Q^{\frac{4}{d}}\left|\widetilde{\varepsilon}_{k}\right|^{2}-\frac{2}{d} Q^{\frac{4}{d}-2}{\overline{Q_{k}}}^{2} \widetilde{\varepsilon}_{k}^{2} d y \\
& +\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}} \int\left(\left|\nabla \widetilde{\varepsilon}_{k}\right|^{2}+\left|\widetilde{\varepsilon}_{k}\right|^{2}\right)\left(\Phi_{k}\left(\lambda_{k} y+\alpha_{k}\right)-\phi_{A}(y)\right) d y+O\left(e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}\right) \\
= & : \sum_{k=1}^{K} E_{21, k}+\sum_{k=1}^{K} E_{22, k}+O\left(e^{\left.-\frac{\delta}{T-t}\|R\|_{L^{2}}^{2}\right),}\right. \tag{3.41}
\end{align*}
$$

where $\phi_{A}$ is the localized function defined as in Lemma 6.3 below.
Since by (2.2),

$$
\begin{equation*}
Q_{k}=Q+O\left(P\langle y\rangle^{4} Q\right) \tag{3.42}
\end{equation*}
$$

where $P$ is given by (2.52) and satisfies $P=o(1)$, due to (2.6) and (2.7), we compute

$$
\begin{equation*}
E_{21, k} \geq \frac{1}{\lambda_{k}^{2}} \operatorname{Re} \int\left(\left|\nabla \widetilde{\varepsilon}_{k}\right|^{2}+\left|\widetilde{\varepsilon}_{k}\right|^{2}\right) \phi_{A}-\left.\left(1+\frac{2}{d}\right) Q^{\frac{4}{d}} \widetilde{\varepsilon}_{k}\right|^{2}-\frac{2}{d} Q^{\frac{4}{d}} \widetilde{\varepsilon}_{k}^{2} d y+O\left(\frac{P}{\lambda_{k}^{2}}\|R\|_{L^{2}}^{2}\right) \tag{3.43}
\end{equation*}
$$

which along with the local coercivity of linearized operators in Lemma 6.3 yields that

$$
\begin{equation*}
E_{21, k} \geq \widetilde{c}_{k} \int\left(|\nabla R|^{2}+\frac{1}{\lambda_{k}^{2}}|R|^{2}\right) \phi_{A, k} d x-C \frac{1}{\lambda_{k}^{2}} \operatorname{Scal}\left(\widetilde{\varepsilon}_{k}\right)-\frac{C \varepsilon D^{2}}{(T-t)^{2}}, \tag{3.44}
\end{equation*}
$$

where $\phi_{A, k}(x):=\phi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right), \widetilde{c}_{k}>0$, we also used (2.9) and $P(t) \leq \varepsilon$ for $t$ close to $T$ in the last step.
In order to estimate the scalar $\operatorname{Scal}\left(\widetilde{\varepsilon}_{k}\right)$, we infer from (6.14), (3.40) and (3.42) and the almost orthogonality in Lemma 2.4 that

$$
\begin{equation*}
\frac{1}{\lambda_{k}^{2}} \operatorname{Scal}\left(\widetilde{\varepsilon}_{k}\right)=\frac{1}{\lambda_{k}^{2}}\left\langle\widetilde{\varepsilon}_{k, 1}, Q\right\rangle^{2}+O\left(e^{-\frac{\delta}{T-i}}\|R\|_{L^{2}}^{2}+\frac{P^{2}}{\lambda_{k}^{2}}\|R\|_{L^{2}}^{2}\right), \tag{3.45}
\end{equation*}
$$

where $\widetilde{\varepsilon}_{k, 1}=\operatorname{Re} \widetilde{\varepsilon}_{k}$. Note that, by (3.42),

$$
\begin{aligned}
\left\langle\widetilde{\varepsilon}_{k, 1}, Q\right\rangle & =\operatorname{Re}\left\langle\widetilde{\varepsilon}_{k}, Q_{k}\right\rangle+O\left(P\|R\|_{L^{2}}\right)=\operatorname{Re}\left\langle R, U_{k}\right\rangle+O\left(P\|R\|_{L^{2}}\right) \\
& =\frac{1}{2} M_{k}-\frac{1}{2} \int|R|^{2} \Phi_{k} d x+O\left(P\|R\|_{L^{2}}+e^{\left.-\frac{\delta}{T-t}\|R\|_{L^{2}}\right) .}\right.
\end{aligned}
$$

This yields that

$$
\begin{equation*}
\left\langle\widetilde{\varepsilon}_{k, 1}, Q\right\rangle^{2}=O\left(M_{k}^{2}+\|R\|_{L^{2}}^{4}+P^{2}\|R\|_{L^{2}}^{2}+e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}\right) . \tag{3.46}
\end{equation*}
$$

Hence, we may take $t$ even closer to $T$ such that $\|R(t)\|_{L^{2}}^{2} \leq \varepsilon$ to get

$$
\begin{equation*}
\frac{1}{\lambda_{k}^{2}}\left\langle\widetilde{\varepsilon}_{k, 1}, Q\right\rangle^{2}=O\left(\frac{M_{k}^{2}}{(T-t)^{2}}+\frac{\varepsilon D^{2}}{(T-t)^{2}}+e^{\left.-\frac{\delta}{T-t}\|R\|_{L^{2}}^{2}\right) . . . . . . .}\right. \tag{3.47}
\end{equation*}
$$

Plugging (3.45) and (3.47) into (3.44) we obtain

$$
\begin{equation*}
E_{21, k} \geq \widetilde{c}_{k} \int\left(|\nabla R|^{2}+\frac{|R|^{2}}{\lambda_{k}^{2}}\right) \phi_{A, k} d x+O\left(\frac{M_{k}^{2}}{(T-t)^{2}}+\frac{\varepsilon D^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-1}\|R\|_{L^{2}}^{2}}\right) \tag{3.48}
\end{equation*}
$$

Regarding $E_{22, k}$, we have from [53, (5.82)] that

$$
\begin{equation*}
E_{22, k} \geq \widetilde{c} \int\left(|\nabla R|^{2}+\frac{|R|^{2}}{\lambda_{k}^{2}}\right)\left(\Phi_{k}-\phi_{A, k}\right) d x+O\left(e^{\left.-\frac{\delta}{T-i}\|R\|_{H^{1}}^{2}\right), ~}\right. \tag{3.49}
\end{equation*}
$$

where $\widetilde{c}=\min \left\{\frac{1}{2}, \widetilde{c}_{k}, 1 \leq k \leq K\right\}>0$.
Thus, plugging (3.48) and (3.49) into (3.41) and taking $\varepsilon$ small enough we obtain (3.39), as claimed.

Therefore, combining (3.33), (3.34), (3.38) and (3.39) altogether we conclude that

$$
\begin{align*}
& \sum_{k=1}^{K}\left(\frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}}\|Q\|_{2}^{2}+\frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|y Q\|_{L^{2}}^{2}\right)+C_{1} \frac{D^{2}}{(T-t)^{2}} \\
\leq & E(v)+\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}+\sum_{k=1}^{K} \frac{M_{k}}{2 \lambda_{k}^{2}} \\
& +C_{2}\left(\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}\|R\|_{L^{2}}^{2}+\frac{D^{3}}{(T-t)^{2}}+\sum_{k=1}^{K} \frac{M_{k}^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right), \tag{3.50}
\end{align*}
$$

where $C_{1}, C_{2}>0$. Taking into account (2.7), (2.9) and (2.61) we may take $t$ closer to $T$ such that

$$
C_{2}\left(\frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}\|R\|_{L^{2}}^{2}+\frac{D^{3}}{(T-t)^{2}}\right) \leq \frac{C_{1}}{2} \frac{D^{2}}{(T-t)^{2}} .
$$

Plugging this into (3.50) we obtain (3.31) and finish the proof.
In particular, because $E(v)$ is a constant independent of $t$ due to the conservation law of energy, estimate (2.6) and Theorems 2.5, 3.1 and 3.4 yield the improved estimates of remainder and geometrical parameters $\beta$ and $\gamma$, which improve the previous ones in (2.7) and (2.8).

Corollary 3.6. (First upgradation of estimates) For t close to $T$, we have

$$
\begin{align*}
& |\beta(t)|+|\gamma(t)|+D(t) \leq C(T-t),  \tag{3.51}\\
& \operatorname{Mod}(t) \leq C(T-t)^{2} . \tag{3.52}
\end{align*}
$$

In particular,

$$
\begin{align*}
& P(t) \leq C(T-t)  \tag{3.53}\\
& \|R(t)\|_{L^{2}} \leq C(T-t), \quad\|\nabla R(t)\|_{L^{2}} \leq C . \tag{3.54}
\end{align*}
$$

3.2. (Modified) localized virial functionals. In this subsection, two types of localized virial functionals adapted to the multi-bubble case are introduced. The main results are Theorems 3.8 and 3.13 below, containing the key monotonicity properties of these functionals.

We first analyze the localized virial functional defined by (3.56) to obtain the refined space time estimate of remainder. This allows to obtain the precise asymptotic order and positivity of parameter $\gamma$. More importantly, it reveals the $H^{1}$ dispersion of remainder along a sequence, which is the key towards the derivation of energy quantization in Subsection 3.3 below.

Then, we prove the monotonicity of another modified localized virial functional defined by (3.103). This enables us to cancel the bad $O(1)$ terms in the energy estimate and thus leads to an improved estimate (4.4) in Subsection 4.1 below.

Let us first consider the localized virial functional. Let $\chi(x)=\psi(|x|)$ be a smooth radial function on $\mathbb{R}^{d}$, where $\psi$ satisfies $\psi^{\prime}(r)=r$ if $r \leq 1, \psi^{\prime}(r)=2-e^{-r}$ if $r \geq 2$, and

$$
\begin{equation*}
\left|\frac{\psi^{\prime \prime \prime}(r)}{\psi^{\prime \prime}(r)}\right| \leq C, \quad \frac{\psi^{\prime}(r)}{r}-\psi^{\prime \prime}(r) \geq 0 . \tag{3.55}
\end{equation*}
$$

Let $\chi_{A}(x):=A^{2} \chi\left(\frac{x}{A}\right), A>0$. The localized virial functional is defined by

$$
\begin{equation*}
\mathcal{L}:=\sum_{k=1}^{K} \frac{1}{2} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x-\sum_{k=1}^{K} \frac{\gamma_{k}}{4 \lambda_{k}}\|x Q\|_{L^{2}}^{2} . \tag{3.56}
\end{equation*}
$$

Remark 3.7. The localization functions $\left\{\Phi_{K}\right\}$ are introduced in (3.56) in this particular way mainly to ensure the key monotonicity property in (3.57) below.

Theorem 3.8. (Monotonicity of localized virial functional) We have that for t close to $T$,

$$
\begin{equation*}
\frac{d \mathcal{L}}{d t} \geq C \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \int\left|\nabla \varepsilon_{k}\right|^{2} e^{-\frac{W}{A}}+\left|\varepsilon_{k}\right|^{2} d y-\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{3}}+O(E r) \tag{3.57}
\end{equation*}
$$

where $C>0$ is independent of $t, \varepsilon_{k}$ is the renormalized remainder given by (2.57) and

$$
\begin{equation*}
E r(t)=(T-t)+(T-t)^{-2}\left(D^{2}(t)+\sum_{k=1}^{K}\left|M_{k}(t)\right|\right)+e^{-\frac{\delta}{T-t}} . \tag{3.58}
\end{equation*}
$$

Proof. The Morawetz type functional on the R.H.S. of (3.56) can be estimated similarly as in the proof of [53, Lemma 5.12].

To be precise, straightforward computations show that

$$
\begin{align*}
& \frac{d}{d t}\left(\sum_{k=1}^{K} \frac{1}{2} \operatorname{Im} \int \nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x\right) \\
= & \sum_{k=1}^{K} \frac{1}{2} \operatorname{Im}\left\langle\partial_{t}\left(\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\right) \cdot \nabla R, R_{k}\right\rangle+\sum_{k=1}^{K} \frac{1}{2 \lambda_{k}} \operatorname{Im}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}, \partial_{t} R\right\rangle \\
& +\sum_{k=1}^{K} \frac{1}{2} \operatorname{Im}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot\left(\nabla R_{k}+\nabla R \Phi_{k}\right), \partial_{t} R\right\rangle \\
= & \sum_{k=1}^{K} \mathcal{L}_{t, k 1}+\mathcal{L}_{t, k 2}+\mathcal{L}_{t, k 3} . \tag{3.59}
\end{align*}
$$

Using direct computations, (3.52) and (3.53) we have that, similarly to [53, (5.49)],

$$
\begin{equation*}
\mathcal{L}_{t, k 1} \leq C \lambda_{k}^{-2}\left(M o d_{k}+P\right)\|\nabla R\|_{L^{2}}\|R\|_{L^{2}} \leq C \frac{D^{2}}{(T-t)^{2}} \tag{3.60}
\end{equation*}
$$

Moreover, in order to treat $\mathcal{L}_{t, k 2}$, we infer from equation (NLS) that

$$
\begin{equation*}
i \partial_{t} R+\Delta R+|v|^{\frac{4}{d}} v-|U|^{\frac{4}{d}} U=-\eta \tag{3.61}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta=i \partial_{t} U+\Delta U+|U|^{\frac{4}{d}} U . \tag{3.62}
\end{equation*}
$$

Note that, by equation (6.24) and Lemma 6.2,

$$
\begin{equation*}
\|\eta\|_{L^{2}} \leq C\left(\frac{M o d}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right), \quad\|\nabla \eta\|_{L^{2}} \leq C\left(\frac{\operatorname{Mod}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) \tag{3.63}
\end{equation*}
$$

Then, in view of the expansion (6.19), equation (3.61) yields that

$$
\begin{equation*}
\mathcal{L}_{t, k 2}=-\frac{1}{2 \lambda_{k}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}, \Delta R+f^{\prime}(U) \cdot R+f^{\prime \prime}(U, R) \cdot R^{2}+\eta\right\rangle, \tag{3.64}
\end{equation*}
$$

where $f^{\prime}(U) \cdot R$ and $f^{\prime \prime}(U, R) \cdot R^{2}$ are given by (6.16) and (6.18) below, respectively. Note that, similarly to [53, (5.56)], we have

$$
\begin{align*}
& -\frac{1}{2 \lambda_{k}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}, \Delta R\right\rangle \\
= & -\frac{1}{4 \lambda_{k}^{3}} \operatorname{Re} \int \Delta^{2} \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left|R_{k}\right|^{2} d x+\frac{1}{2 \lambda_{k}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left|\nabla R_{k}\right|^{2} d x \\
& +O\left(\|R\|_{H^{1}}^{2}+\|\nabla R\|_{L^{2}}\|R\|_{L^{2}}\right) . \tag{3.65}
\end{align*}
$$

We also infer from Lemma 6.2 that

$$
\begin{equation*}
\operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}, f^{\prime}(U) \cdot R\right\rangle=\operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}, f^{\prime}\left(U_{k}\right) \cdot R_{k}\right\rangle+O\left(e^{-\frac{\delta}{T-t}}\right) \tag{3.66}
\end{equation*}
$$

Moreover, since by (6.18),

$$
\begin{equation*}
\left|f^{\prime \prime}(U, R) \cdot R^{2}\right| \leq C\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{d}-1}\right)|R|^{2} \tag{3.67}
\end{equation*}
$$

taking into account $\|U(t)\|_{L^{\infty}} \leq C(T-t)^{-\frac{d}{2}},(2.60)$, (3.51) and (3.63) we get

$$
\begin{align*}
&\left|\frac{1}{2 \lambda_{k}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}, f^{\prime \prime}(U, R) \cdot R^{2}+\eta\right\rangle\right| \\
& \leq C(T-t)^{-1}\left((T-t)^{-2+\frac{d}{2}}\|R\|_{L^{3}}^{3}+\|R\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}}+\|\eta\|_{L^{2}}\|R\|_{L^{2}}\right) \\
& \leq C\left(\frac{D^{2}}{(T-t)^{2}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.68}
\end{align*}
$$

Thus, plugging (3.65), (3.66) and (3.68) into (3.64) we obtain

$$
\begin{align*}
\mathcal{L}_{t, k 2}= & -\frac{1}{4 \lambda_{k}^{3}} \operatorname{Re} \int \Delta^{2} \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left|R_{k}\right|^{2} d x+\frac{1}{2 \lambda_{k}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left|\nabla R_{k}\right|^{2} d x \\
& -\frac{1}{2 \lambda_{k}} \operatorname{Re}\left\langle\Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}, f^{\prime}\left(U_{k}\right) \cdot R_{k}\right\rangle+O\left(\frac{D^{2}}{(T-t)^{2}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.69}
\end{align*}
$$

Regarding $\mathcal{L}_{t, k 3}$ we use equation (3.61) and (6.19) again to get

$$
\begin{equation*}
\mathcal{L}_{t, k 3}=-\frac{1}{2} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left(\nabla R_{k}+\nabla R \Phi_{k}\right), \Delta R+f^{\prime}(U) \cdot R+f^{\prime \prime}(U, R) \cdot R^{2}+\eta\right\rangle . \tag{3.70}
\end{equation*}
$$

Similarly to [53, (5.62)], we have

$$
\begin{align*}
& -\operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot\left(\nabla R_{k}+\nabla R \Phi_{k}\right), \Delta R\right\rangle \\
= & \operatorname{Re} \int \frac{2}{\lambda_{k}} \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left(\nabla R_{k}, \nabla \overline{R_{k}}\right)-\frac{1}{\lambda_{k}} \Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left|\nabla R_{k}\right|^{2} d x+O\left(\frac{D^{2}}{(T-t)^{2}}\right) . \tag{3.71}
\end{align*}
$$

We also have, via Lemma 6.2,

$$
\begin{align*}
& \frac{1}{2} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left(\nabla R_{k}+\nabla R \Phi_{k}\right), f^{\prime}(U) \cdot R\right\rangle \\
= & \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R_{k}, f^{\prime}\left(U_{k}\right) \cdot R_{k}\right\rangle+O\left(e^{-\frac{\delta}{T-t}}\right) . \tag{3.72}
\end{align*}
$$

Moreover, using (2.60), (3.63), (3.67) and integration by parts formula we get

$$
\begin{align*}
& \left|\frac{1}{2} \operatorname{Re}\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left(\nabla R_{k}+\nabla R \Phi_{k}\right), f^{\prime \prime}(U, R) \cdot R^{2}+\eta\right\rangle\right| \\
\leq & C \int(|\nabla R|+|R|)\left(|U|^{\frac{4}{d}-1}+|R|^{\frac{4}{4}-1}\right)|R|^{2} d x+C\left(\lambda_{k}^{-1}\|R\|_{L^{2}}\|\eta\|_{L^{2}}+\|R\|_{L^{2}}\|\nabla \eta\|_{L^{2}}\right) \\
\leq & C\left((T-t)^{-2+\frac{d}{2}}\|R\|_{H^{1}}\|R\|_{L^{4}}^{2}+\|R\|_{H^{1}}\|R\|_{L^{2+\frac{8}{d}}}^{1+\frac{4}{d}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) \\
\leq & C\left(\frac{D^{2}}{(T-t)^{2}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.73}
\end{align*}
$$

Thus, estimates (3.70)-(3.73) together yield that

$$
\begin{align*}
\mathcal{L}_{t, k 3}= & \frac{1}{\lambda_{k}} \operatorname{Re} \int \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left(\nabla R_{k}, \nabla \overline{R_{k}}\right) d x-\frac{1}{2 \lambda_{k}} \operatorname{Re} \int \Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left|\nabla R_{k}\right|^{2} d x \\
& -\left\langle\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R_{k}, f^{\prime}\left(U_{k}\right) \cdot R_{k}\right\rangle+O\left(\frac{D^{2}}{(T-t)^{2}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.74}
\end{align*}
$$

Therefore, combining (3.60), (3.69) and (3.74) altogether we lead to

$$
\begin{align*}
& \frac{d}{d t}\left(\sum_{k=1}^{K} \frac{1}{2} \operatorname{Im} \int \nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x\right) \\
= & \sum_{k=1}^{K} \frac{1}{\lambda_{k}} \operatorname{Re} \int \nabla^{2} \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left(\nabla R_{k}, \nabla \overline{R_{k}}\right) d x-\sum_{k=1}^{K} \frac{1}{4 \lambda_{k}^{3}} \operatorname{Re} \int \Delta^{2} \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right)\left|R_{k}\right|^{2} d x \\
& -\sum_{k=1}^{K} \operatorname{Re}\left(\frac{1}{2 \lambda_{k}} \Delta \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) R_{k}+\nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R_{k}, f^{\prime}\left(U_{k}\right) \cdot R_{k}\right\rangle \\
& +O\left(\frac{D^{2}}{(T-t)^{2}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.75}
\end{align*}
$$

By the integration by parts formula, the third term on the R.H.S. above equals to

$$
\sum_{k=1}^{K} \operatorname{Re} \int \nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla \overline{U_{k}}\left(f^{\prime \prime}\left(U_{k}\right) \cdot R_{k}^{2}\right) d x
$$

where $f^{\prime \prime}\left(U_{k}\right) \cdot R_{k}^{2}$ is as in (2.58) with $U_{k}, R_{k}$ replacing $Q_{k}$ and $\varepsilon_{k}$, respectively. Thus, using the renormalized remainder $\varepsilon_{k}$ given by (2.57) we arrive at

$$
\begin{align*}
& \frac{d}{d t}\left(\sum_{k=1}^{K} \frac{1}{2} \operatorname{Im} \int \nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x\right) \\
= & \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \operatorname{Re} \int \nabla^{2} \chi_{A}(y)\left(\nabla \varepsilon_{k}, \nabla \overline{\varepsilon_{k}}\right) d y-\sum_{k=1}^{K} \frac{1}{4 \lambda_{k}^{3}} \int \Delta^{2} \chi_{A}(y)\left|\varepsilon_{k}\right|^{2} d y \\
& +\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \operatorname{Re} \int\left(\nabla \chi_{A}(y) \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y+O\left(\frac{D^{2}(t)}{(T-t)^{2}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.76}
\end{align*}
$$

This gives the control of Morawetz type functional in (3.56).
Regarding the second term on the R.H.S. of (3.56), using (3.51) we compute

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\gamma_{k}}{\lambda_{k}}\right)=\frac{\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}}{\lambda_{k}^{3}}-\frac{\gamma_{k}}{\lambda_{k}} \frac{\lambda_{k} \dot{\lambda}_{k}+\gamma_{k}}{\lambda_{k}^{2}}=\frac{\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}}{\lambda_{k}^{3}}+O\left(\frac{M o d}{\lambda_{k}^{2}}\right), \tag{3.77}
\end{equation*}
$$

which along with (2.56), (3.53) and (3.54) yields

$$
\begin{align*}
\frac{d}{d t}\left(\sum_{k=1}^{K} \frac{\gamma_{k}}{4 \lambda_{k}}\|x Q\|_{2}^{2}\right)= & \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} M_{k}(t)-\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \int|R(t)|^{2} \Phi_{k} d x \\
& +\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{\mid}}\left|\varepsilon_{k}\right|^{2}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{d}-2} Q_{k}^{2} \bar{\varepsilon}_{k}^{2} d y \\
& +\sum_{k=1}^{K} \frac{1}{\overline{\lambda_{k}^{3}} \operatorname{Re} \int\left(y \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y} \\
& +(T-t)^{-3} O\left((T-t) M o d+(T-t)^{2}\|R\|_{L^{2}}+D^{3}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.78}
\end{align*}
$$

Thus, combining (3.75) and (3.78) altogether and then using the inequality

$$
\begin{equation*}
\int|R|^{2} \Phi_{k} d x \geq \int|R|^{2} \Phi_{k}^{2} d x=\int\left|\varepsilon_{k}\right|^{2} d y \tag{3.79}
\end{equation*}
$$

we arrive at, if $\varepsilon_{k}=\varepsilon_{k, 1}+i \varepsilon_{k, 2}$,

$$
\begin{align*}
\frac{d \mathcal{L}}{d t} \geq & \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}}\left(\int \nabla^{2} \chi\left(\frac{y}{A}\right)\left(\nabla \varepsilon_{k}, \nabla \overline{\varepsilon_{k}}\right) d y+\int\left|\varepsilon_{k}\right|^{2} d y-\int\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}} \varepsilon_{k, 1}^{2}+Q^{\frac{4}{d}} \varepsilon_{k, 2}^{2} d y\right. \\
& \left.-\frac{1}{4 A^{2}} \int \Delta^{2} \chi\left(\frac{y}{A}\right)\left|\varepsilon_{k}\right|^{2} d y\right)+\sum_{k=1}^{K} \frac{2}{d} \frac{1}{\lambda_{k}^{3}} \int\left(A \nabla \chi\left(\frac{y}{A}\right)-y\right) \cdot \nabla Q Q^{\frac{4}{d}-1}\left(\left(1+\frac{4}{d}\right) \varepsilon_{k, 1}^{2}+\varepsilon_{k, 2}^{2}\right) d y \\
& -\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{3}}+O(\widetilde{E r}) \tag{3.80}
\end{align*}
$$

where the error term

$$
\begin{equation*}
\widetilde{E r}:=(T-t)^{-3}\left((T-t) M o d+(T-t)^{2}\|R\|_{L^{2}}+(T-t) D^{2}+D^{3}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.81}
\end{equation*}
$$

Since

$$
\begin{equation*}
\int \nabla^{2} \chi\left(\frac{y}{A}\right)\left(\nabla \varepsilon_{k}, \nabla \overline{\varepsilon_{k}}\right) d y \geq \int \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right)\left|\nabla \varepsilon_{k}\right|^{2} d y \tag{3.82}
\end{equation*}
$$

applying Lemma 6.3 with $\phi(x):=\psi^{\prime \prime}(|x|)$ we lead to

$$
\begin{aligned}
\frac{d \mathcal{L}}{d t} \geq & \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}}\left(C \int \psi^{\prime \prime}\left(\frac{y}{A}\right)\left|\nabla \varepsilon_{k}\right|^{2}+\left|\varepsilon_{k}\right|^{2} d y-\frac{1}{4 A^{2}} \int \Delta^{2} \chi\left(\frac{y}{A}\right)\left|\varepsilon_{k}\right|^{2} d y\right) \\
& +\sum_{k=1}^{K} \frac{2}{d} \frac{1}{\lambda_{k}^{3}} \int\left(A \nabla \chi\left(\frac{y}{A}\right)-y\right) \cdot \nabla Q Q^{\frac{4}{d}-1}\left(\left(1+\frac{4}{d}\right) \varepsilon_{k, 1}^{2}+\varepsilon_{k, 2}^{2}\right) d y \\
& -\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{3}}+O\left(\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \operatorname{Scal}\left(\varepsilon_{k}\right)+\widetilde{E r}\right) .
\end{aligned}
$$

Taking into account for $A$ large enough

$$
\frac{1}{4 A^{2}}\left|\Delta^{2} \chi\left(\frac{y}{A}\right)\right| \leq \frac{1}{4} C \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right), \frac{2}{d}\left(2+\frac{4}{d}\right)\left|A \nabla \chi\left(\frac{y}{A}\right)-y\right|\left|\nabla Q Q^{\frac{4}{d}-1}\right| \leq \frac{1}{4} C \psi^{\prime \prime}\left(\left|\frac{y}{A}\right|\right)
$$

we arrive at

$$
\begin{equation*}
\frac{d \mathcal{L}}{d t} \geq C \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \int\left|\nabla \varepsilon_{k}\right|^{2} e^{-\frac{\mid y}{A}}+\left|\varepsilon_{k}\right|^{2} d y-\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{3}}+O\left(\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \operatorname{Scal}\left(\varepsilon_{k}\right)+\widetilde{E r}\right) \tag{3.83}
\end{equation*}
$$

Note that, by estimates (2.53) and (3.54),

$$
\begin{equation*}
(T-t) M o d \leq C\left((T-t)^{3} D+(T-t) D^{2}+(T-t) \sum_{k=1}^{K}\left|M_{k}\right|+e^{-\frac{\delta}{T-t}}\right) \tag{3.84}
\end{equation*}
$$

Moreover, by Cauchy's inequality, for any $\varepsilon>0$,

$$
\begin{equation*}
(T-t)^{2}\|R\|_{L^{2}} \leq \varepsilon\|R\|_{2}^{2}+\varepsilon^{-1}(T-t)^{4} \tag{3.85}
\end{equation*}
$$

Hence, we infer from (2.6) and (3.81)-(3.85) that

$$
\begin{equation*}
\widetilde{E r} \leq C\left(\varepsilon(T-t)^{-3}\|R\|_{L^{2}}^{2}+E r\right) \leq C\left(\varepsilon \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \int\left|\varepsilon_{k}\right|^{2} d y+E r\right), \tag{3.86}
\end{equation*}
$$

where $E r$ is given by (3.58).
Moreover, similarly to (3.45) and (3.46), we have

$$
\begin{equation*}
\operatorname{Scal}\left(\varepsilon_{k}\right)=O\left(M_{k}^{2}+\|R\|_{L^{2}}^{4}+P^{2}\|R\|_{L^{2}}^{2}+e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}\right) . \tag{3.87}
\end{equation*}
$$

This yields that

$$
\begin{equation*}
\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{3}} \operatorname{Scal}\left(\varepsilon_{k}\right) \leq C(T-t)^{-2}\left(D^{2}+\sum_{k=1}^{K}\left|M_{k}\right|\right)+C e^{-\frac{\delta}{T-t}}=O\left(E_{r}\right) \tag{3.88}
\end{equation*}
$$

Therefore, plugging (3.86) and (3.88) into (3.83) and then taking $\varepsilon$ sufficiently small we obtain (3.57). The proof of Theorem 3.8 is complete.

By virtue of Theorem 3.8, we derive the refined space time estimate of remainder and the precise asymptotic order of parameter $\gamma$, which improve the previous estimates in (3.54) and (2.7) respectively and, more importantly, allow to obtain the energy quantization in the next section.

Theorem 3.9. (Refined estimates of $R$ and $\gamma$ ) There exists $C>0$ such that for $t$ close to $T$,

$$
\begin{equation*}
\sum_{k=1}^{K} \int_{t}^{T} \frac{\left\|R_{k}\right\|_{L^{2}}^{2}}{\lambda_{k}^{3}} d s \leq C \tag{3.89}
\end{equation*}
$$

Moreover, for $1 \leq k \leq K$,

$$
\begin{equation*}
\gamma_{k}(t)-\omega_{k}^{2}(T-t)=o(T-t), \text { as } t \text { close to } T \tag{3.90}
\end{equation*}
$$

Remark 3.10. The space time estimate (3.89) is important to derive the precise asymptotic behavior of $\gamma_{k}$ in (3.90) and to control the remainder near the singularities (see Lemma 3.18 below). Note that, the previous estimate (3.51) is insufficient to get (3.89). Hence, roughly speaking, (3.89) upgrades the convergence rate of $\|R\|_{L^{2}}$ from $T-t$ to $(T-t)^{1+}$.

Remark 3.11. Compared with (2.7), (3.90) gives the precise leading order of the asymptotic behavior of $\gamma_{k}$. This enables to identify the exact value of energy in the next section. It also yields the positivity of $\gamma_{k}$ for $t$ close to $T$, which is important to derive the monotonicity of modified localized virial functional in Theorem 3.13 below.

Proof of Theorem 3.9. Integrating (3.57) from $t$ to $\tilde{t}$ we have

$$
\begin{equation*}
\sum_{k=1}^{K} \int_{t}^{\tilde{t}} \frac{\left\|R_{k}\right\|_{L^{2}}^{2}}{\lambda_{k}^{3}} d s \leq C\left(|\mathcal{L}(\tilde{t})-\mathcal{L}(t)|+\sum_{k=1}^{K} \int_{t}^{\tilde{t}} \frac{\left|M_{k}\right|}{\lambda_{k}^{3}} d s+\int_{t}^{\tilde{t}} E r d s\right), \tag{3.91}
\end{equation*}
$$

where the error term Er is given by (3.58).
Note that, by estimates (2.9) and (3.51),

$$
\begin{equation*}
|\mathcal{L}(t)| \leq C\left(\|\nabla R\|_{L^{2}}\|R\|_{L^{2}}+\sum_{k=1}^{K}\left|\frac{\gamma_{k}}{\lambda_{k}}\right|\right) \leq C<\infty . \tag{3.92}
\end{equation*}
$$

Moreover, by (3.2),

$$
\begin{equation*}
\sum_{k=1}^{K} \int_{t}^{T} \frac{\left|M_{k}\right|}{\lambda_{k}^{3}} d s \leq C(T-t)^{\zeta} \tag{3.93}
\end{equation*}
$$

and by (3.2) and (3.51),

$$
\begin{equation*}
\int_{t}^{T} E r d s \leq C(T-t) \tag{3.94}
\end{equation*}
$$

Thus, plugging estimates (3.92)-(3.94) into (3.91) and letting $\tilde{t} \rightarrow T$ we obtain (3.89).
Next we prove (3.90). Using (3.77) we get

$$
\begin{equation*}
\int_{t}^{T}\left|\frac{d}{d s}\left(\frac{\gamma_{k}}{\lambda_{k}}\right)\right| d s \leq \int_{t}^{T} \frac{\left|\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right|}{\lambda_{k}^{3}}+C \frac{M o d}{\lambda_{k}^{2}} d s \tag{3.95}
\end{equation*}
$$

Then, using (6.44), (2.6), (3.2), Corollary 3.6 and (3.89) we obtain

$$
\int_{t}^{T}\left|\frac{d}{d s}\left(\frac{\gamma_{k}}{\lambda_{k}}\right)\right| d s \leq C \int_{t}^{T} \frac{\|R\|_{L^{2}}^{2}+\left|M_{k}\right|+(T-s)^{3}}{(T-s)^{3}} d s+C \int_{t}^{T} \frac{M o d}{\lambda_{k}^{2}} d s
$$

$$
\leq C\left(\int_{t}^{T} \frac{\|R\|_{L^{2}}^{2}+(T-s)^{2+\zeta}}{(T-s)^{3}} d s+T-t\right) \leq C<\infty
$$

In particular, this yields that for some $c_{k} \in \mathbb{R}$,

$$
\begin{equation*}
\lim _{t \rightarrow T} \frac{\gamma_{k}(t)}{\lambda_{k}(t)}=c_{k} \tag{3.96}
\end{equation*}
$$

Below, we claim that

$$
\begin{equation*}
c_{k}=\omega_{k}, \quad 1 \leq k \leq K \tag{3.97}
\end{equation*}
$$

Hence, (3.96) and (3.97) together give that

$$
\begin{equation*}
\frac{\gamma_{k}(t)}{\lambda_{k}(t)}-\omega_{k}=o(1), \text { for } t \text { close to } T \tag{3.98}
\end{equation*}
$$

which along with (2.6) yields the desirable estimate (3.90).
It remains to prove (3.97). For this purpose, we see that

$$
\dot{\lambda}_{k}=\frac{\lambda_{k} \dot{\lambda}_{k}+\gamma_{k}}{\lambda_{k}}-\frac{\gamma_{k}}{\lambda_{k}}=O\left(\frac{M o d}{\lambda_{k}}\right)-\frac{\gamma_{k}}{\lambda_{k}} .
$$

Since $\lim _{t \rightarrow T} \lambda_{k}(t)=0$, we get

$$
\begin{equation*}
\lambda_{k}(t)=\int_{t}^{T} \frac{\gamma_{k}}{\lambda_{k}} d s+O\left(\int_{t}^{T} \frac{M o d}{\lambda_{k}} d s\right) \tag{3.99}
\end{equation*}
$$

Note that, by (3.96),

$$
\begin{equation*}
\int_{t}^{T} \frac{\gamma_{k}}{\lambda_{k}} d s=c_{k}(T-t)+o(T-t) \tag{3.100}
\end{equation*}
$$

Moreover, by (3.52),

$$
\begin{equation*}
\int_{t}^{T} \frac{M o d}{\lambda_{k}} d s \leq C \int_{t}^{T}(T-s) d s \leq C(T-t)^{2}=o(T-t) \tag{3.101}
\end{equation*}
$$

Thus, combining (3.99)-(3.101) together we conclude that

$$
\begin{equation*}
\lambda_{k}(t)=c_{k}(T-t)+o(T-t), \tag{3.102}
\end{equation*}
$$

which along with the estimate of $\lambda_{k}$ in (2.6) yields (3.97), thereby finishing the proof.
We close this subsection with the monotonicity of modified localized virial functional below

$$
\begin{equation*}
\mathscr{L}:=\sum_{k=1}^{K} \frac{\gamma_{k}}{2 \lambda_{k}} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x-\sum_{k=1}^{K} \frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|x Q\|_{2}^{2} \tag{3.103}
\end{equation*}
$$

where $\chi_{A}$ is as in (3.56).
Remark 3.12. The modified localized virial functional (3.103) is introduced mainly to derive the refined estimate (4.2) below for the second upgradation purpose. Actually, it enables to cancel the bad $O(1)$ terms in the refined energy estimate (3.109) and thus allows to further upgrade the convergence rate to the second order, see Subsection 4.1 below.

The main monotonicity of modified localized virial functional is formulated in Theorem 3.13.

Theorem 3.13. (Monotonicity of modified localized virial functional) We have that for t close to $T$,

$$
\begin{equation*}
\frac{d \mathscr{L}}{d t} \geq C \sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \int\left|\nabla \varepsilon_{k}\right|^{2} e^{-\frac{W}{A}}+\left|\varepsilon_{k}\right|^{2} d y-\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k}+O(E r) \tag{3.104}
\end{equation*}
$$

where $C>0$ is independent of $t$, and the error term Er is given by (3.58).
Remark 3.14. Note that, the positivity of $\gamma_{k}$, implied by (3.90), is important to derive the monotonicity of modified localized virial functional.

Proof of Theorem 3.13. We first note that, by (2.6) and (3.90),

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\gamma_{k}^{2}}{\lambda_{k}^{2}}\right)=\frac{2 \gamma_{k}}{\lambda_{k}^{4}}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right)-\frac{2 \gamma_{k}^{2}}{\lambda_{k}^{2}} \frac{\lambda_{k} \dot{\lambda}_{k}+\gamma_{k}}{\lambda_{k}^{2}}=\frac{2 \gamma_{k}}{\lambda_{k}^{4}}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right)+O\left(\frac{M o d}{\lambda_{k}^{2}}\right) . \tag{3.105}
\end{equation*}
$$

Using (2.53) and (2.56) we get

$$
\begin{align*}
\frac{d}{d t}\left(\sum_{k=1}^{K} \frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|x Q\|_{2}^{2}\right)= & \sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k}-\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \int|R|^{2} \Phi_{k} d x \\
& +\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{d}}\left|\varepsilon_{k}\right|^{2}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{4}-2} Q_{k}^{2} \bar{\varepsilon}_{k}^{2} d y \\
& +\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \operatorname{Re} \int\left(y \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y+O(\widetilde{E r}) \tag{3.106}
\end{align*}
$$

where the error term $\widetilde{E r}$ is as in (3.81).
Moreover, arguing as in the proof of (3.75) (see also [53, Lemma 5.12]) we get

$$
\begin{align*}
& \frac{d}{d t}\left(\sum_{k=1}^{K} \frac{\gamma_{k}}{2 \lambda_{k}} \operatorname{Im} \int \nabla \chi_{A}\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x\right) \\
&=\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \operatorname{Re} \int \nabla^{2} \chi_{A}(y)\left(\nabla \varepsilon_{k}, \nabla \overline{\varepsilon_{k}}\right) d y-\sum_{k=1}^{K} \frac{\gamma_{k}}{4 \lambda_{k}^{4}} \int \Delta^{2} \chi_{A}(y)\left|\varepsilon_{k}\right|^{2} d y \\
&+\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \operatorname{Re} \int\left(\nabla \chi_{A}(y) \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y \\
&+O\left(\frac{D^{2}(t)}{(T-t)^{2}}+\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\right) . \tag{3.107}
\end{align*}
$$

Therefore, combining (3.106) and (3.107) altogether and using similar arguments as those below (3.78) we obtain (3.104). The proof is complete.
3.3. Energy quantization. This subsection deals with the key energy quantization, that is, the energy of multi-bubble blow-up solutions to (NLS) indeed admits the quantization into the sum of the energies of pseudo-conformal blow-up solutions. The main result is stated in Theorem 3.15.
Theorem 3.15. (Energy quantization) Let v be the multi-bubble blow-up solution to (NLS) satisfying (1.22) and (1.23). Then, we have

$$
\begin{equation*}
E(v)=\sum_{k=1}^{K} \frac{\omega_{k}^{2}}{8}\|y Q\|_{L^{2}}^{2}=\sum_{k=1}^{K} E\left(S_{k}\right), \tag{3.108}
\end{equation*}
$$

where $S_{k}$ is the pseudo-conformal blow-up solution given by (1.10), $1 \leq k \leq K$.
The important consequence of Theorems 3.4 and 3.15 is the following refined energy estimate.
Corollary 3.16. (Refined energy estimate) There exist $C_{1}, C_{2}>0$ such that for any t close to $T$,

$$
\begin{align*}
\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}}\|Q\|_{2}^{2}+C_{1} \frac{D^{2}}{(T-t)^{2}} \leq & \frac{\|y Q\|_{L^{2}}^{2}}{8} \sum_{k=1}^{K}\left(\omega_{k}^{2}-\frac{\gamma_{k}^{2}}{\lambda_{k}^{2}}\right)+\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}+\sum_{k=1}^{K} \frac{M_{k}}{2 \lambda_{k}^{2}} \\
& +C_{2}\left(\sum_{k=1}^{K} \frac{M_{k}^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) \tag{3.109}
\end{align*}
$$

The remaining of this subsection is devoted to the proof of Theorem 3.15. The key ingredient of proof is the $H^{1}$ dispersion of remainder as formulated in Proposition 3.20 below. We shall first prove Lemmas 3.17 and 3.18, which give the average $H^{1}$ dispersion of remainder.

Let $B_{2 \sigma}\left(x_{k}\right):=\left\{x \in \mathbb{R}^{d}:\left|x-x_{k}\right| \leq 2 \sigma\right\}, 1 \leq k \leq K$, where $\sigma$ is as in (2.48). Let $\Omega_{2 \sigma}:=$ $\cup_{k=1}^{K} B_{2 \sigma}\left(x_{k}\right)$ be the interior region near the singularities and $\widetilde{\Omega}_{2 \sigma}:=\mathbb{R}^{d} \backslash \Omega_{2 \sigma}$ be the exterior region. Let $\widetilde{R}:=v-S$ with $S=\sum_{k=1}^{K} S_{k}$.
Lemma 3.17. ( $H^{1}$ dispersion away from the singularities) For $t$ close to $T$, we have

$$
\begin{equation*}
\|\nabla R(t)\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)} \leq\|\nabla \widetilde{R}(t)\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)}+O\left(e^{\left.-\frac{\delta}{T-t}\right)}\right. \tag{3.110}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\lim _{t \rightarrow T} \frac{1}{T-t} \int_{t}^{T} \frac{1}{T-s} \int_{s}^{T}\|\nabla R(r)\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)}^{2} d r d s=0 \tag{3.111}
\end{equation*}
$$

Proof. We note from (2.1) that

$$
R=S-U+\widetilde{R},
$$

which yields that

$$
\begin{equation*}
\|\nabla R\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)} \leq\|\nabla U\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)}+\|\nabla S\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma)}\right.}+\|\nabla \widetilde{R}\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)} . \tag{3.112}
\end{equation*}
$$

Note that, by (2.6), $\left|\alpha_{k}-x_{k}\right|=o(1)$, we may take $t$ sufficiently close to $T$ such that $\left|\alpha_{k}-x_{k}\right|<\sigma$, and so $\left|x-\alpha_{k}\right|>\sigma$ for any $x \in \widetilde{\Omega}_{2 \sigma}$. Then, by the explicit expression of $U_{k}$ in (2.2), (2.9) and (6.8),

$$
\begin{equation*}
\|\nabla U\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)}^{2} \leq \sum_{k=1}^{K} \int_{\left|x-\alpha_{k}\right| \geq \sigma}\left|\nabla U_{k}\right|^{2} d x \leq \sum_{k=1}^{K} \lambda_{k}^{-2} \int_{|y| \geq \frac{\sigma}{\lambda_{k}}}\left(\nabla Q_{k}\right)^{2}(y) d y \leq C e^{-\frac{\delta}{T-1}}, \tag{3.113}
\end{equation*}
$$

where $C, \delta>0$. Similarly, we have

$$
\begin{equation*}
\|\nabla S\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)}^{2} \leq C e^{-\frac{\delta}{T-1}} . \tag{3.114}
\end{equation*}
$$

Thus, putting the above estimates altogether we prove (3.110). Estimate (3.111) then follows from (3.110) and (1.23).

The more delicate interior region near the singularities is treated in Lemma 3.18 below.
Lemma 3.18. ( $H^{1}$ dispersion in average near the singularities) For t close to $T$, we have

$$
\begin{equation*}
\lim _{t \rightarrow T} \frac{1}{T-t} \int_{t}^{T} \frac{1}{T-s} \int_{s}^{T}\|\nabla R(r)\|_{L^{2}\left(\Omega_{2 \sigma}\right)}^{2} d r d s=0 \tag{3.115}
\end{equation*}
$$

Proof. Let $\chi$ be a smooth radial cutoff function such that $\nabla \chi(x)=x$ for $|x| \leq 2 \sigma, \nabla^{2} \chi$ is supported in $|x| \leq 3 \sigma$ and is positive semidefinite on $\mathbb{R}^{d}$. Let $\chi_{k}:=\chi\left(x-x_{k}\right), 1 \leq k \leq K$. We use the local virial functional $\mathbb{L}$ defined by

$$
\begin{equation*}
\mathbb{L}:=\sum_{k=1}^{K} \operatorname{Im} \int \nabla \chi_{k} \cdot \nabla R \bar{R} \Phi_{k} d x, \tag{3.116}
\end{equation*}
$$

where $\Phi_{k}, 1 \leq k \leq K$, are the localization functions given by (2.49).
Note that, by (3.54),

$$
\begin{equation*}
\|\mathbb{L}(t) \mid \leq C\| \nabla R\left\|_{L^{2}}\right\| R(t)\left\|_{L^{2}} \leq C\right\| R(t) \|_{L^{2}} . \tag{3.117}
\end{equation*}
$$

Moreover, using the integration by parts formula we have

$$
\frac{d \mathbb{L}}{d t}=\sum_{k=1}^{K} \operatorname{Re}\left\langle\Delta \chi_{k} R_{k}, i \partial_{t} R\right\rangle+\sum_{k=1}^{K} \operatorname{Re}\left\langle\nabla \chi_{k} \cdot\left(\nabla R_{k}+\nabla R \Phi_{k}\right), i \partial_{t} R\right\rangle,
$$

where $R_{k}=R \Phi_{k}$. Then, in view of equation (3.61), we obtain

$$
\begin{align*}
\frac{d \mathbb{L}}{d t}= & -\sum_{k=1}^{K} \operatorname{Re}\left\langle\Delta \chi_{k} R_{k}, \Delta R\right\rangle-\sum_{k=1}^{K} \operatorname{Re}\left\langle\nabla \chi_{k} \cdot\left(\nabla R_{k}+\nabla R \Phi_{k}\right), \Delta R\right\rangle \\
& -\sum_{k=1}^{K} \operatorname{Re}\left\langle\Delta \chi_{k} R_{k}+\nabla \chi_{k} \cdot\left(\nabla R_{k}+\nabla R \Phi_{k}\right), \eta\right\rangle \\
& \left.-\left.\sum_{k=1}^{K} \operatorname{Re}\left\langle\Delta \chi_{k} R_{k}+\nabla \chi_{k} \cdot\left(\nabla R_{k}+\nabla R \Phi_{k}\right),\right| v\right|^{\frac{4}{d}} v-|U|^{\frac{4}{d}} U\right\rangle \\
= & : \sum_{k=1}^{K}\left(\mathbb{L}_{t, k 1}+\mathbb{L}_{t, k 2}+\mathbb{L}_{t, k 3}+\mathbb{L}_{t, k 4}\right) . \tag{3.118}
\end{align*}
$$

Below we estimate the four terms $\mathbb{L}_{t, k i}, 1 \leq i \leq 4$, separately.
First, since the supports of $\Delta \chi_{k}$ and $R_{j}$ are disjoint for any $j \neq k$, we see that,

$$
\begin{align*}
\mathbb{L}_{t, k 1}=-\operatorname{Re}\left\langle\Delta \chi_{k} R_{k}, \Delta R_{k}\right\rangle & =\int \Delta \chi_{k}\left|\nabla R_{k}\right|^{2} d x-\frac{1}{2} \int \Delta^{2} \chi_{k}\left|R_{k}\right|^{2} d x \\
& =\int \Delta \chi_{k}\left|\nabla R_{k}\right|^{2} d x+O\left(\|R\|_{L^{2}}^{2}\right) . \tag{3.119}
\end{align*}
$$

Next, we claim that

$$
\begin{equation*}
\mathbb{L}_{t, k 2}=2 \operatorname{Re} \int \nabla^{2} \chi_{k}\left(\nabla R_{k}, \nabla \overline{R_{k}}\right) d x-\int \Delta \chi_{k}\left|\nabla R_{k}\right|^{2} d x+2 \operatorname{Re}\left\langle\nabla \chi_{k} \cdot \nabla R, \nabla \Phi_{k} \cdot \nabla R\right\rangle+O\left(\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}\right) \tag{3.120}
\end{equation*}
$$

In order to prove (3.120), we note from $\nabla R \Phi_{k}=\nabla R_{k}-R \nabla \Phi_{k}$ that

$$
\begin{equation*}
\mathbb{L}_{t, k 2}=-2 \operatorname{Re}\left\langle\nabla \chi_{k} \cdot \nabla R_{k}, \Delta R\right\rangle+\operatorname{Re}\left\langle\nabla \chi_{k} \cdot \nabla \Phi_{k} R, \Delta R\right\rangle=: \mathbb{L}_{k, 21}+\mathbb{L}_{k, 22} . \tag{3.121}
\end{equation*}
$$

Using the integration by parts formula we compute

$$
\mathbb{L}_{k, 21}=2 \operatorname{Re} \int \nabla^{2} \chi_{k}\left(\nabla R_{k}, \nabla \bar{R}\right) d x-2 \operatorname{Re} \int \Delta \chi_{k} \nabla R_{k} \cdot \nabla \bar{R} d x-2 \sum_{i, h=1}^{d} \operatorname{Re}\left\langle\partial_{i} \chi_{k} \partial_{h} R_{k}, \partial_{i h} R\right\rangle
$$

Since the supports of $\nabla^{2} \chi_{k}$ and $\Delta \chi_{k}$ are disjoint with $R_{j}, j \neq k$, we infer that

$$
\begin{equation*}
\mathbb{L}_{k, 21}=2 \operatorname{Re} \int \nabla^{2} \chi_{k}\left(\nabla R_{k}, \nabla \overline{R_{k}}\right) d x-2 \int \Delta \chi_{k}\left|\nabla R_{k}\right|^{2} d x-2 \sum_{i, h=1}^{d} \operatorname{Re}\left\langle\partial_{i} \chi_{k} \partial_{h} R_{k}, \partial_{i h} R\right\rangle . \tag{3.122}
\end{equation*}
$$

Moreover, we use the integration by parts formula again to compute

$$
\begin{aligned}
\mathbb{L}_{k, 22} & =-\operatorname{Re} \int \nabla \chi_{k} \cdot \nabla \Phi_{k}|\nabla R|^{2} d x+O\left(\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}\right) \\
& =\operatorname{Re} \int \Delta \chi_{k} \Phi_{k}|\nabla R|^{2} d x+2 \sum_{i, h=1}^{d} \operatorname{Re}\left\langle\partial_{i} \chi_{k} \Phi_{k} \partial_{h} R, \partial_{i h} R\right\rangle+O\left(\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}\right) .
\end{aligned}
$$

Since on the support of $\Delta \chi_{k}, \Phi_{k}=1$ and $\Phi_{j}=0$ for $j \neq k$, it follows that

$$
\begin{equation*}
\mathbb{I}_{k, 22}=\operatorname{Re} \int \Delta \chi_{k}\left|\nabla R_{k}\right|^{2} d x+2 \sum_{i, h=1}^{d} \operatorname{Re}\left\langle\partial_{i} \chi_{k} \Phi_{k} \partial_{h} R, \partial_{i h} R\right\rangle+O\left(\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}\right) \tag{3.123}
\end{equation*}
$$

Note that

$$
\begin{equation*}
-\operatorname{Re}\left\langle\partial_{i} \chi_{k} \partial_{h} R_{k}, \partial_{i h} R\right\rangle+\operatorname{Re}\left\langle\partial_{i} \chi_{k} \Phi_{k} \partial_{h} R, \partial_{i h} R\right\rangle=\operatorname{Re}\left\langle\partial_{i} \chi_{k} \partial_{i} R, \partial_{h} \Phi_{k} \partial_{h} R\right\rangle+O\left(\|R\|_{L^{2}}\|\nabla R\|_{L^{2}}\right) . \tag{3.124}
\end{equation*}
$$

Thus, plugging (3.122), (3.123) and (3.124) into (3.121) we obtain (3.120), as claimed.
Regarding the third term $\mathbb{L}_{t, k 3}$, we move the derivative to $\eta$ and use (3.52) and (3.63) to get

$$
\begin{equation*}
\left|\mathbb{L}_{t, k 3}\right|=O\left(\|R\|_{L^{2}}\|\eta\|_{H^{1}}\right)=O\left(\frac{M o d\|R\|_{L^{2}}}{(T-t)^{3}}+e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}\right)=O\left(\frac{\|R\|_{L^{2}}}{T-t}\right) \tag{3.125}
\end{equation*}
$$

It remains to treat the last term $\mathbb{L}_{t, k 4}$ on the R.H.S. of (3.118). First, since

$$
\begin{equation*}
\left||v|^{\frac{4}{d}} v-|U|^{\frac{4}{d}} U\right| \leq C \sum_{l=1}^{1+\frac{4}{d}}|U|^{1+\frac{4}{d}-l}|R|^{l}, \tag{3.126}
\end{equation*}
$$

using (2.60) and (3.54) we get

$$
\begin{equation*}
\left.\left|\operatorname{Re}\left\langle\Delta \chi_{k} R_{k},\right| v\right|^{\frac{4}{d}} v-|U|^{\frac{4}{d}} U\right\rangle \left\lvert\,=O\left(\frac{\|R\|_{L^{2}}^{2}+D^{3}}{(T-t)^{2}}\right)=O\left(\frac{\|R\|_{L^{2}}}{T-t}+\frac{D^{3}}{(T-t)^{2}}\right)\right. \tag{3.127}
\end{equation*}
$$

Moreover, we claim that

$$
\begin{equation*}
\left.\left|\operatorname{Re}\left\langle\nabla \chi_{k} \cdot\left(\nabla R_{k}+\nabla R \Phi_{k}\right),\right| v\right|^{\frac{4}{d}} v-|U|^{\frac{4}{d}} U\right\rangle \left\lvert\,=O\left(\frac{\|R\|_{L^{2}}}{T-t}+\frac{D^{3}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right)\right. \tag{3.128}
\end{equation*}
$$

This along with (3.127) yields that

$$
\begin{equation*}
\mathbb{L}_{t, k 4}=O\left(\frac{\|R\|_{L^{2}}}{T-t}+\frac{D^{3}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) \tag{3.129}
\end{equation*}
$$

In order to prove (3.128), we use (3.126) and Lemma 6.2 in Appendix to bound

$$
\begin{align*}
\text { L.H.S. of (3.128) } \leq & C\left(\int\left|\nabla \chi_{k}\right|\left|U_{k}\right|^{\frac{4}{d}}\left|R_{k} \nabla R_{k}\right| d x+\int\left|\nabla \chi_{k}\right|\left|U_{k}\right|^{\frac{4}{d}-1}\left|R_{k}^{2} \nabla R_{k}\right| d x\right. \\
& \left.+\sum_{l=3}^{1+\frac{4}{d}} \int\left|\nabla \chi_{k}\right||U|^{1+\frac{4}{d}-l}|R|^{l}|\nabla R| d x+\sum_{l=3}^{1+\frac{4}{d}} \int\left|\nabla \chi_{k}\right||U|^{1+\frac{4}{d}-l}|R|^{l+1} d x+e^{-\frac{\delta}{T-t}}\right) . \tag{3.130}
\end{align*}
$$

Similarly to (3.127), we use (2.60) to bound

$$
\begin{align*}
& \sum_{l=3}^{1+\frac{4}{2}} \int\left|\nabla \chi_{k} \| U\right|^{1+\frac{4}{d}-l}|R|^{l+1} d x \leq C \frac{\|R\|_{L^{2}}^{2}+D^{3}}{(T-t)^{2}} \leq C\left(\frac{\|R\|_{L^{2}}}{T-t}+\frac{D^{3}}{(T-t)^{2}}\right)  \tag{3.131}\\
& \sum_{l=3}^{1+\frac{4}{2}} \int\left|\nabla \chi_{k}\left\|\left.U\right|^{1+\frac{4}{d}-l}|R|^{l}|\nabla R| d x \leq C(T-t)^{-\frac{d}{2}-2+\frac{d}{2} l}\right\| \nabla R\left\|_{L^{2}}\right\| R \|_{L^{2 l}}^{l} \leq C \frac{D^{3}}{(T-t)^{2}}\right. \tag{3.132}
\end{align*}
$$

Next, we deal with the first two integrations on the R.H.S. of (3.130).
Note that for $t$ close to $T,\left|\alpha_{k}(t)-x_{k}\right| \leq \sigma$ and $\nabla \chi(x)=x$ for $|x| \leq 2 \sigma$. Using the renormalized variable $\varepsilon_{k}$ defined by (2.57), (2.9), (3.53), (3.54) and (6.8) below we get

$$
\begin{align*}
& \int\left|\nabla \chi_{k} \| U_{k}\right|^{\frac{4}{\mid}}\left|R_{k} \nabla R_{k}\right| d x \leq \frac{C}{(T-t)^{3}} \int\left|\nabla \chi\left(\lambda_{k} y+\alpha_{k}-x_{k}\right)\right| Q^{\frac{4}{d}}(y)\left|\varepsilon_{k} \nabla \varepsilon_{k}\right| d y \\
& \quad \leq C \frac{\left|\lambda_{k}\right|+\left|\alpha_{k}-x_{k}\right|}{(T-t)^{3}} \int_{|y| \leq \frac{\sigma}{\lambda_{k}}}(1+|y|) Q^{\frac{4}{d}}(y)\left|\varepsilon_{k} \nabla \varepsilon_{k}\right| d y+\frac{C}{(T-t)^{3}} \int_{|y| \geq \frac{\sigma}{\lambda_{k}}} Q^{\frac{4}{\partial}}(y)\left|\varepsilon_{k} \nabla \varepsilon_{k}\right| d y \\
& \quad \leq C\left(\frac{\|R\|_{L^{2}}}{T-t}+e^{-\frac{\delta}{T-t}}\right) \tag{3.133}
\end{align*}
$$

Moreover, by (3.54) and the Gagliardo-Nirenberg inequality (6.1),

$$
\begin{align*}
\int\left|\nabla \chi_{k} \| U_{k}\right|^{\frac{4}{d}-1}\left|R_{k}^{2} \nabla R_{k}\right| d x & \leq C(T-t)^{-2+\frac{d}{2}}\left\|R_{k}\right\|_{L^{4}}^{2}\left\|\nabla R_{k}\right\|_{L^{2}} \\
& \leq C(T-t)^{-2+\frac{d}{2}}\left\|R_{k}\right\|_{L^{2}}^{2-\frac{d}{2}}\left\|\nabla R_{k}\right\|_{L^{2}}^{1+\frac{d}{2}} \leq C \frac{\|R\|_{L^{2}}}{T-t} \tag{3.134}
\end{align*}
$$

Thus, inserting estimates (3.131)-(3.134) into (3.130) we obtain (3.128), as claimed.
Now, combining estimates (3.118), (3.119), (3.120), (3.125) and (3.129) altogether we conclude
$2 \operatorname{Re} \int \nabla^{2} \chi_{k}\left(\nabla R_{k}, \nabla \overline{R_{k}}\right) d x \leq \frac{d \mathbb{L}}{d t}+2\left|\operatorname{Re}\left\langle\nabla \chi_{k} \cdot \nabla R, \nabla \Phi_{k} \cdot \nabla R\right\rangle\right|+C\left(\frac{\|R\|_{L^{2}}}{T-t}+\frac{D^{3}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right)$.

Note that, since the support of $\nabla \Phi_{k}$ is in the exterior region $\widetilde{\Omega}_{2 \sigma}$, Lemma 3.17 yields

$$
\begin{equation*}
\left|\operatorname{Re}\left\langle\nabla \chi_{k} \cdot \nabla R, \nabla \Phi_{k} \cdot \nabla R\right\rangle\right| \leq C\|\nabla R\|_{L^{2}\left(\widetilde{\Omega}_{2 \sigma}\right)}^{2} \leq C\left(\|\nabla \widetilde{R}\|_{L^{2}}^{2}+e^{-\frac{\delta}{T-t}}\right) \tag{3.136}
\end{equation*}
$$

Taking into account $\nabla^{2} \chi_{k}\left(\nabla R_{k}, \nabla \overline{R_{k}}\right) \geq\left|\nabla R_{k}\right|^{2}$ for $x \in B_{2 \sigma}\left(x_{k}\right),\|\nabla R\|_{L^{2}\left(\Omega_{2 \sigma}\right)}^{2}=\sum_{k=1}^{K}\left\|\nabla R_{k}\right\|_{L^{2}\left(B_{2 \sigma}\left(x_{k}\right)\right)}^{2}$, integrating (3.135) from $t$ to $T$ and using (3.51) and the boundary estimate (3.117) we get

$$
\begin{equation*}
\int_{t}^{T}\|\nabla R(s)\|_{L^{2}\left(\Omega_{2 \sigma}\right)}^{2} d s \leq C\left(\|R(t)\|_{L^{2}}+\int_{t}^{T} \frac{\|R(s)\|_{L^{2}}}{T-s} d s+\int_{t}^{T}\|\nabla \widetilde{R}\|_{L^{2}}^{2} d s+o(T-t)\right) \tag{3.137}
\end{equation*}
$$

which along with the Cauchy inequality and the space time estimate (3.89) yields

$$
\begin{align*}
\frac{1}{T-t} \int_{t}^{T}\|\nabla R(s)\|_{L^{2}\left(\Omega_{2 \sigma}\right)}^{2} d s & \leq C\left(\frac{\|R(t)\|_{L^{2}}}{T-t}+\left(\int_{t}^{T} \frac{\|R(s)\|_{L^{2}}^{2}}{(T-s)^{3}} d s\right)^{\frac{1}{2}}+\frac{1}{T-t} \int_{t}^{T}\|\nabla \widetilde{R}\|_{L^{2}}^{2} d s\right)+o(1) \\
& \leq C\left(\frac{\|R\|_{L^{2}}}{T-t}+\frac{1}{T-t} \int_{t}^{T}\|\nabla \widetilde{R}\|_{L^{2}}^{2} d s\right)+o(1) \tag{3.138}
\end{align*}
$$

Furthermore, taking the average of (3.138) again and then using (1.23) and (3.89) we arrive at

$$
\begin{align*}
& \frac{1}{T-t} \int_{t}^{T} \frac{1}{T-s} \int_{s}^{T}\|\nabla R\|_{L^{2}\left(\Omega_{2 \sigma}\right)}^{2} d r d s \\
\leq & C\left(\frac{1}{T-t} \int_{t}^{T} \frac{\|R\|_{L^{2}}}{T-s} d s+\frac{1}{T-t} \int_{t}^{T} \frac{1}{T-s} \int_{s}^{T}\|\nabla \widetilde{R}\|_{L^{2}}^{2} d r d s\right)+o(1) \\
\leq & C\left(\int_{t}^{T} \frac{\|R\|_{L^{2}}^{2}}{(T-s)^{3}} d s\right)^{\frac{1}{2}}+o(1) \rightarrow 0, \text { as } t \rightarrow T . \tag{3.139}
\end{align*}
$$

Therefore, the proof of Lemma 3.18 is complete.
By virtue of Lemmas 3.17 and 3.18, we have the average estimates of $\|R\|_{L^{2}}$ and $\beta$ below.
Proposition 3.19. There exists a universal constant $C>0$ such that for t close to $T$,

$$
\begin{equation*}
\frac{1}{T-t} \int_{t}^{T} \frac{\|R\|_{L^{2}}^{2}+|\beta|^{2}}{(T-s)^{2}} d s \leq C<\infty . \tag{3.140}
\end{equation*}
$$

Proof. By direct calculations,

$$
\begin{align*}
\left(\frac{\left(\alpha_{k}-x_{k}\right) \cdot \beta_{k}}{\lambda_{k}}\right)_{t} & =\frac{\left(\lambda_{k} \dot{\alpha}_{k}-2 \beta_{k}\right) \cdot \beta_{k}+2\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}+\frac{\left(\lambda_{k}^{2} \dot{\beta}_{k}+\beta_{k} \gamma_{k}\right) \cdot\left(\alpha_{k}-x_{k}\right)}{\lambda_{k}^{3}}-\frac{\left(\lambda_{k} \dot{\lambda}_{k}+\gamma_{k}\right)\left(\alpha_{k}-x_{k}\right) \cdot \beta_{k}}{\lambda_{k}^{3}} \\
& =2 \frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}+O\left(\left|\frac{\beta_{k}}{\lambda_{k}^{2}}\right|+\left|\frac{\alpha_{k}-x_{k}}{\lambda_{k}^{3}}\right|+\left|\frac{\left(\alpha_{k}-x_{k}\right) \beta_{k}}{\lambda_{k}^{3}}\right|\right) M o d \tag{3.141}
\end{align*}
$$

Note that, by (2.6), (3.51) and (3.52),

$$
\begin{equation*}
\left|\frac{\left(\alpha_{k}-x_{k}\right) \cdot \beta_{k}}{\lambda_{k}}\right| \leq C\left|\alpha_{k}-x_{k}\right|=o(T-t), \tag{3.142}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\left|\frac{\beta_{k}}{\lambda_{k}^{2}}\right|+\left|\frac{\alpha_{k}-x_{k}}{\lambda_{k}^{3}}\right|+\left|\frac{\left(\alpha_{k}-x_{k}\right) \cdot \beta_{k}}{\lambda_{k}^{3}}\right|\right) \operatorname{Mod}=o(1) \tag{3.143}
\end{equation*}
$$

Hence, integrating (3.141) from $t$ to $T$ and using (2.9), (3.142) and (3.143) we get

$$
\begin{equation*}
\int_{t}^{T} \frac{\left|\beta_{k}\right|^{2}}{(T-s)^{2}} d s=o(T-t), \quad 1 \leq k \leq K \tag{3.144}
\end{equation*}
$$

Moreover, in view of (2.9) and (3.89), we get

$$
\begin{equation*}
\frac{1}{T-t} \int_{t}^{T} \frac{\|R\|_{L^{2}}^{2}}{(T-s)^{2}} d s \leq C \sum_{k=1}^{K} \int_{t}^{T} \frac{\left\|R_{k}\right\|_{2}^{2}}{\lambda_{k}^{3}} d s \leq C<\infty \tag{3.145}
\end{equation*}
$$

Thus, (3.144) and (3.145) yield (3.140) immediately.
As a consequence of Lemmas 3.17, 3.18 and Proposition 3.19, we are now able to upgrade the estimates of remainder and parameter $\beta$ along a sequence, which in particular improve the previous ones in (3.51).

Proposition 3.20. ( $H^{1}$ dispersion along a sequence) There exists a sequence $\left\{t_{n}\right\}$ to $T$ such that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \frac{D\left(t_{n}\right)+\left|\beta\left(t_{n}\right)\right|}{T-t_{n}}=0 \tag{3.146}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\left\|R\left(t_{n}\right)\right\|_{L^{2}}+\left|\beta\left(t_{n}\right)\right|=o\left(T-t_{n}\right), \quad\left\|\nabla R\left(t_{n}\right)\right\|_{L^{2}}=o(1) \tag{3.147}
\end{equation*}
$$

Now, we are ready to prove the key energy quantization in Theorem 3.15.
Proof of Theorem 3.15. We mainly focus on the proof of the first equality in (3.108), as the second one follows from straightforward computations by using the expression of $S_{k}$ in (1.10). It suffices to prove this for a subsequence, due to the energy conservation law of energy.

We infer from (3.33), (3.34) and (3.38) that

$$
\begin{align*}
E(v)= & \sum_{k=1}^{K}\left(\frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}}\|Q\|_{L^{2}}^{2}+\frac{\gamma_{k}^{2}}{8 \lambda_{k}^{2}}\|y Q\|_{L^{2}}^{2}-\frac{\left|\beta_{k}\right|^{2}}{2 \lambda_{k}^{2}} M_{k}-\frac{M_{k}}{2 \lambda_{k}^{2}}\right) \\
& +\frac{1}{2} \operatorname{Re} \int|\nabla R|^{2}+\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}}|R|^{2} \Phi_{k}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}}|R|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}^{2} d x \\
& +O\left(\sum_{k=1}^{K} \frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}\|R\|_{L^{2}}^{2}+\frac{D^{3}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) \\
= & \sum_{i=1}^{3} J_{i} . \tag{3.148}
\end{align*}
$$

Take the sequence $\left\{t_{n}\right\}$ as in Proposition 3.20. Thus, by (2.6), (3.98) and (3.147), we have that along the sequence $\left\{t_{n}\right\}$,

$$
\begin{equation*}
J_{1}=\sum_{k=1}^{K} \frac{\omega_{k}^{2}}{8}\|y Q\|_{L^{2}}^{2}+o(1), \tag{3.149}
\end{equation*}
$$

and

$$
\begin{equation*}
J_{2}=O\left(\left\|\nabla R\left(t_{n}\right)\right\|_{L^{2}}^{2}+\frac{\left\|R\left(t_{n}\right)\right\|_{L^{2}}^{2}}{\left(T-t_{n}\right)^{2}}\right)=o(1) \tag{3.150}
\end{equation*}
$$

Using (2.6), (3.51) and (3.147) again we infer that along the sequence $\left\{t_{n}\right\}$,

$$
\begin{equation*}
J_{3}=o(1) \tag{3.151}
\end{equation*}
$$

Thus, taking into account the conservation law of energy we arrive at

$$
\begin{equation*}
E(v(t))=\lim _{n \rightarrow \infty} E\left(v\left(t_{n}\right)\right)=\sum_{k=1}^{K} \frac{\omega_{k}^{2}}{8}\|y Q\|_{L^{2}}^{2} . \tag{3.152}
\end{equation*}
$$

Therefore, the proof of Theorem 3.15 is complete.

## 4. Upgradation of the convergence rate of remainder

This section is mainly devoted to the crucial upgradation of the convergence rate of remainder. First in Subsection 4.1, we upgrade the convergence rate to the second order, by virtue of the refined energy estimate, the monotonicity of modified localized virial functional and the relationship (3.3)(3.6) between localized mass and remainder. Then, in order to further upgrade the convergence rate, we prove the monotonicity of modified generalized energy in Subsection 4.2. Eventually, in Subsection 4.3, we complete the final upgradation to the exponential decay rate.
4.1. Upgradation to the second order. The main result in this step is Theorem 4.1 below, which in particular improves the previous estimates in (3.54) and (3.147) for any $t$ close to $T$.

Theorem 4.1. There exists $C>0$ such that for t close to $T$,

$$
\begin{equation*}
D(t) \leq C(T-t)^{2} . \tag{4.1}
\end{equation*}
$$

Proof. The proof proceeds in two steps below.
Step 1. Upgradation to the order 1+. We first use (3.103) and the precise asymptotic of $\gamma_{k}$ in (3.98) to derive from (3.103) that

$$
\lim _{\bar{t} \rightarrow T} \mathscr{L}(\widetilde{t})-\mathscr{L}(t)=\frac{\|y Q\|_{L^{2}}^{2}}{8} \sum_{k=1}^{K}\left(\frac{\gamma_{k}^{2}(t)}{\lambda_{k}^{2}(t)}-\omega_{k}^{2}\right)+O\left(\|R(t)\|_{L^{2}}\|\nabla R(t)\|_{L^{2}}\right) .
$$

Using (3.98) again we also have that for some $\widetilde{c}>0$,

$$
\tilde{c} \int_{t}^{T} \frac{1}{\lambda_{k}^{3}}\left\|R_{k}\right\|_{L^{2}}^{2} d s \leq \int_{t}^{T} \frac{\gamma_{k}}{\lambda_{k}^{4}}\|R\|_{L^{2}}^{2} d s .
$$

Then, integrating both sides of (3.104) we obtain

$$
\begin{align*}
\tilde{c} \sum_{k=1}^{K} \int_{t}^{T} \frac{\left\|R_{k}\right\|_{L^{2}}^{2}}{\lambda_{k}^{3}} d s \leq & \frac{\|y Q\|_{L^{2}}^{2}}{8} \sum_{k=1}^{K}\left(\frac{\gamma_{k}^{2}(t)}{\lambda_{k}^{2}(t)}-\omega_{k}^{2}\right)+C\|R(t)\|_{L^{2}}\|\nabla R(t)\|_{L^{2}} \\
& +\left|\int_{t}^{T} \sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k} d s\right|+C \int_{t}^{T} E r d s . \tag{4.2}
\end{align*}
$$

where $M_{k}$ and $E r$ are given by (3.1) and (3.58), respectively, and $\widetilde{c}, C>0$.
Thus, combining (4.2) and the refined energy estimate (3.109) altogether and using (3.2), Corollary 3.6 and the inequality

$$
\begin{equation*}
\|R\|_{L^{2}}\|\nabla R\|_{L^{2}} \leq C\|R\|_{L^{2}} \leq \delta \frac{D^{2}}{(T-t)^{2}}+\frac{C^{2}}{\delta}(T-t)^{2}, \tag{4.3}
\end{equation*}
$$

with $\delta$ small enough we obtain that there exists a positive constant $C>0$ such that

$$
\begin{align*}
& \sum_{k=1}^{K} \frac{\left|\beta_{k}(t)\right|^{2}}{\lambda_{k}^{2}}\|Q\|_{2}^{2}+\frac{D^{2}(t)}{(T-t)^{2}}+\sum_{k=1}^{K} \int_{t}^{T} \frac{\left\|R_{k}\right\|_{L^{2}}^{2}}{\lambda_{k}^{3}} d s \\
\leq & C\left((T-t)^{2}+\left|\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{2}}\right|+\left|\sum_{k=1}^{K} \int_{t}^{T} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k} d s\right|+\int_{t}^{T} \frac{D^{2}+\sum_{k=1}^{K}\left|M_{k}\right|}{(T-s)^{2}} d s\right) . \tag{4.4}
\end{align*}
$$

Note that, by (1.23), (3.2) and (3.90),

$$
\begin{align*}
\left|\int_{t}^{T} \sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} M_{k} d s\right| & \leq C \int_{t}^{T} \frac{1}{(T-s)^{3}} \int_{s}^{T} \int_{r}^{T}\|\widetilde{R}(\tau)\|_{L^{2}}^{2} d \tau d r d s+C \int_{t}^{T} \frac{1}{(T-s)^{3}} e^{-\frac{\delta}{T-s}} d s \\
& \leq C \int_{t}^{T} \frac{1}{(T-s)^{2}} \int_{s}^{T} \frac{1}{T-r} \int_{r}^{T}\|\widetilde{R}(\tau)\|_{L^{2}}^{2} d \tau d r d s+C e^{-\frac{\delta}{T-t}} \\
& \leq C(T-t)^{\zeta} . \tag{4.5}
\end{align*}
$$

Using (3.2) again we get that

$$
\begin{equation*}
\left|\sum_{k=1}^{K} \frac{M_{k}}{\lambda_{k}^{2}}\right|+\int_{t}^{T} \frac{D^{2}+\sum_{k=1}^{K}\left|M_{k}\right|}{(T-s)^{2}} d s \leq C(T-t)^{\zeta} \tag{4.6}
\end{equation*}
$$

Thus, plugging (4.5) and (4.6) into (4.4) we obtain that for some $C>0$,

$$
\begin{equation*}
D(t) \leq C(T-t)^{1+\frac{t}{2}} \tag{4.7}
\end{equation*}
$$

Note that, estimate (4.7) already improves the previous one (3.54) and (3.147) for any $t$ close to $T$.
Let us also mention that, even though we have the high order $(T-t)^{2}$ on the R.H.S. of (4.4), the estimate (3.2) of localized mass $M_{k}$ actually restricts the upgradation to the lower order $(T-t)^{1+\frac{\zeta}{2}}$.

In order to further upgrade the convergence rate, the key idea in the next step is to establish a Gronwall type inequality by relating together the localized mass and the quantity $D$.

Step 2. Upgradation to the second order. The goal in this step is to upgrade the convergence rate to the second order $(T-t)^{2}$, i.e., for some $C>0$,

$$
\frac{D^{2}(t)}{(T-t)^{2}} \leq C(T-t)^{2}
$$

For this purpose, we shall establish a Gronwall type estimate from (4.4). Precisely, for $t$ close to $T$ such that $T-t \leq \varepsilon$, using (2.9), (3.3)-(3.6) and (4.4) we obtain the Gronwall type inequality

$$
\begin{equation*}
\frac{D^{2}(t)}{(T-t)^{2}} \leq C_{1}\left((T-t)^{2}+\varepsilon \int_{t}^{T} \frac{D^{2}(s)}{(T-s)^{3}} d s+\varepsilon \int_{t}^{T} \frac{1}{(T-s)^{3}} \int_{s}^{T} \frac{D^{2}(r)}{T-r} d r d s\right), \tag{4.8}
\end{equation*}
$$

where $C_{1}>0$ is independent of $\varepsilon$.
Then, plugging (4.7) into (4.8) we get

$$
\begin{equation*}
\frac{D^{2}(t)}{(T-t)^{2}} \leq C_{1}(T-t)^{2}+\frac{2 C_{1}^{2} \varepsilon}{\zeta}(T-t)^{\zeta} \tag{4.9}
\end{equation*}
$$

We use the induction arguments and suppose that for some $n \geq 1$,

$$
\begin{equation*}
\frac{D^{2}(t)}{(T-t)^{2}} \leq\left(\sum_{k=0}^{n-1} C_{1}^{k+1} \varepsilon^{k}\right)(T-t)^{2}+\frac{2^{n+1} C_{1}^{n+1} \varepsilon^{n}}{\zeta^{n}}(T-t)^{\zeta} \tag{4.10}
\end{equation*}
$$

Note that, (4.9) verifies (4.10) at the preliminary step $n=1$. Then, plugging (4.10) into (4.8) and using straightforward computations we see that estimate (4.10) is still valid with $n+1$ replacing $n$. Thus, the induction arguments yield that, for $\varepsilon$ sufficiently small such that $\frac{C_{1} \varepsilon}{\zeta} \leq \frac{1}{4}$,

$$
\begin{equation*}
\frac{D^{2}(t)}{(T-t)^{2}} \leq \lim _{n \rightarrow \infty}\left(\left(\sum_{k=0}^{n-1} C_{1}^{k+1} \varepsilon^{k}\right)(T-t)^{2}+\frac{2^{n+1} C_{1}^{n+1} \varepsilon^{n}}{\zeta^{n}}(T-t)^{\zeta}\right) \leq 2 C_{1}(T-t)^{2} \tag{4.11}
\end{equation*}
$$

Therefore, the proof of Theorem 4.1 is complete.
As a consequence of Theorems 2.5, 3.1 and 4.1 and estimate (4.4), we obtain the improved estimates of localized mass and geometrical parameters below

Corollary 4.2. There exists $C>0$ such that for every $1 \leq k \leq K$ and any $t$ close to $T$,

$$
\begin{align*}
& \left|M_{k}(t)\right|+\operatorname{Mod}(t) \leq C(T-t)^{4}  \tag{4.12}\\
& \left|\beta_{k}(t)\right| \leq C(T-t)^{2} \tag{4.13}
\end{align*}
$$

4.2. Modified generalized energy. The last key ingredient to fulfill the upgradation to exponential decay rate is the modified generalized energy used in [53], defined by

$$
\mathscr{I}:=\frac{1}{2} \int|\nabla R|^{2} d x+\frac{1}{2} \sum_{k=1}^{K} \int \frac{1}{\lambda_{k}^{2}}|R|^{2} \Phi_{k} d x-\operatorname{Re} \int F(u)-F(U)-f(U) \bar{R} d x
$$

$$
\begin{equation*}
+\sum_{k=1}^{K} \frac{\gamma_{k}}{2 \lambda_{k}} \operatorname{Im} \int\left(\nabla \chi_{A}\right)\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla R \bar{R} \Phi_{k} d x\left(:=\mathscr{I}^{(1)}+\mathscr{I}^{(2)}\right) \tag{4.14}
\end{equation*}
$$

where $F(z):=\frac{d}{2 d+4}|z|^{2+\frac{4}{d}}$ and $f(z):=|z|^{\frac{4}{d}} z, z \in \mathbb{C}$.
The key monotonicity of modified generalized energy is formulated in Theorem 4.3 below.
Theorem 4.3. (Monotonicity of modified generalized energy) For any $\varepsilon>0$ as in Case (I) and Case (II), we have that for close to $T$,

$$
\begin{align*}
\frac{d \mathscr{I}}{d t} \geq & C_{1} \sum_{k=1}^{K} \frac{1}{\lambda_{k}} \int\left(\left|\nabla R_{k}\right|^{2}+\frac{1}{\lambda_{k}^{2}}\left|R_{k}\right|^{2}\right) e^{-\frac{\left|x-\alpha_{k}\right|}{A \lambda_{k}}} d x \\
& -C_{2}\left(\varepsilon \frac{D^{2}(t)}{(T-t)^{3}}+\frac{D^{2}(t)}{(T-t)^{2}}+\sum_{k=1}^{K} \frac{\left|M_{k}(t)\right|}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) \tag{4.15}
\end{align*}
$$

where $C_{1}, C_{2}>0$ are independent of $\varepsilon$ and $t$.
Proof. The proof is similar to that of [53, Theorem 5.9] but requires more delicate estimates of the error terms. Because the Morawetz type functional $\mathscr{I}^{(2)}$ in (4.14) has been already analyzed in (3.107), we will focus on the first functional $\mathscr{I}^{(1)}$ in (4.14).

We use Taylor's expansion and equation (3.61) to compute (see [53, (5.33)])

$$
\begin{align*}
\frac{d \mathscr{I}^{(1)}}{d t}= & -\sum_{k=1}^{K} \dot{\lambda}_{k} \lambda_{k}^{-3} \operatorname{Im} \int|R|^{2} \Phi_{k} d x-\sum_{k=1}^{K} \lambda_{k}^{-2} \operatorname{Im}\left\langle f^{\prime}(U) \cdot R, R_{k}\right\rangle \\
& -\operatorname{Re}\left\langle f^{\prime \prime}(U, R) \cdot R^{2}, \partial_{t} U\right\rangle-\sum_{k=1}^{K} \lambda_{k}^{-2} \operatorname{Im}\left\langle R \nabla \Phi_{k}, \nabla R\right\rangle \\
& -\sum_{k=1}^{K} \lambda_{k}^{-2} \operatorname{Im}\left\langle f^{\prime \prime}(U, R) \cdot R^{2}, R_{k}\right\rangle-\operatorname{Im}\left\langle\Delta R-\sum_{k=1}^{K} \lambda_{k}^{-2} R_{k}+f(u)-f(U), \eta\right\rangle \\
= & \sum_{l=1}^{6} \mathscr{I}_{t, l}^{(1)}, \tag{4.16}
\end{align*}
$$

where $f^{\prime}(U) \cdot R, f^{\prime \prime}(U, R) \cdot R^{2}$ are given by (6.16) and (6.18), respectively.
Note that, since $\left|\frac{\lambda_{k} \hat{\lambda}_{k}+\gamma_{k}}{\lambda_{k}^{4}}\right| \leq \frac{M o d}{\lambda_{k}^{4}} \leq C$ due to (4.12), we compute

$$
\begin{equation*}
\mathscr{I}_{t, 1}^{(1)}=\sum_{k=1}^{K}\left(\frac{\gamma_{k}}{\lambda_{k}^{4}} \int|R|^{2} \Phi_{k} d x-\frac{\lambda_{k} \dot{\lambda}_{k}+\gamma_{k}}{\lambda_{k}^{4}} \int|R|^{2} \Phi_{k} d x\right)=\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \int|R|^{2} \Phi_{k} d x+O\left(D^{2}\right) \tag{4.17}
\end{equation*}
$$

Moreover, we have from [53, (5.35)] that

$$
\begin{align*}
\mathscr{I}_{t, 2}^{(1)}+\mathscr{I}_{t, 3}^{(1)}= & -\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{2}} \operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|U_{k}\right|^{\frac{4}{\mid}}\left|R_{k}\right|^{2}+\frac{2}{d}\left|U_{k}\right|^{\frac{4}{d}-2}{\overline{U_{k}}}^{2} R_{k}^{2} d x \\
& -\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}} \operatorname{Re} \int\left(\frac{x-\alpha_{k}}{\lambda_{k}}\right) \cdot \nabla \overline{U_{k}}\left(f^{\prime \prime}\left(U_{k}\right) \cdot R_{k}^{2}\right) d x+O\left(E_{2}\right), \tag{4.18}
\end{align*}
$$

where $f^{\prime \prime}\left(U_{k}\right) \cdot R_{k}^{2}$ is defined as in (2.58) with $U_{k}$ and $R_{k}$ replacing $Q_{k}$ and $\varepsilon_{k}$, respectively, and

$$
\begin{align*}
E_{2} \leq & \sum_{k=1}^{K} \frac{\left|\beta_{k}\right|}{\left|\lambda_{k}\right|} \int\left|U_{k}\right|^{\frac{4}{d}-1}\left|\nabla U_{k} \| R_{k}\right|^{2} d x+C \sum_{k=1}^{K} \int\left|U_{k}\right|^{\frac{4}{d}}\left|R_{k}\right|^{2} d x \\
& +C \frac{M o d(t)}{(T-t)^{4}}\|R\|_{L^{2}}^{2}+C \sum_{l \geq 3}(T-t)^{\frac{d}{2} l-4-d}\|R\|_{L^{l}}^{l} \tag{4.19}
\end{align*}
$$

Using (2.60), (3.90), Theorem 4.1 and Corollary 4.2 we have

$$
\begin{align*}
E_{2} & \leq C\left((T-t)^{-2} D^{2}+(T-t)^{-\frac{d}{2}\left(\frac{4}{d}-1\right)} D^{2}+\sum_{l=3}^{1++\frac{4}{d}}(T-t)^{\frac{d}{2} l-4-d}(T-t)^{-d\left(\frac{l}{2}-1\right)} D^{l}\right) \\
& \leq C \frac{D^{2}}{(T-t)^{2}} . \tag{4.20}
\end{align*}
$$

Regarding $\mathscr{J}_{t, 4}^{(1)}$, we consider Case (I) and Case (II) separately. In Case (I), since $\sum_{k=1}^{K} \nabla \Phi_{k}=0$, as in (3.25), we bound $\mathscr{I}_{t, 4}^{(1)}$ by

$$
\begin{align*}
\left|\mathscr{I}_{t, 4}^{(1)}\right| & =\left|\left(\frac{1}{\lambda_{k}^{2}}-\frac{1}{\omega^{2}(T-t)^{2}}\right) \operatorname{Im}\left\langle R \nabla \Phi_{k}, \nabla R\right\rangle\right| \\
& \leq \sum_{k=1}^{K} \frac{\left|\omega-\omega_{k} \| \omega+\omega_{k}\right|}{\omega^{2} \omega_{k}^{2}(T-t)^{2}}\|R\|_{L^{2}}\|\nabla R\|_{L^{2}} \leq C \varepsilon \frac{D^{2}}{(T-t)^{3}} \tag{4.21}
\end{align*}
$$

where we have used (2.9) and the inequality $\left|\omega-\omega_{k}\right| \leq \varepsilon$ in Case (I).
In Case (II), since $\|\nabla \Phi\|_{L^{\infty}} \leq C \sigma^{-1} \leq C \varepsilon$, we bound it easily by

$$
\begin{equation*}
\left|\mathscr{I}_{t, 4}^{(1)}\right| \leq \sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}}\left\|\nabla \Phi_{k}\right\|_{L^{\infty}}\|R\|_{L^{2}}\|\nabla R\|_{L^{2}} \leq C \varepsilon \frac{D^{2}}{(T-t)^{3}} \tag{4.22}
\end{equation*}
$$

Moreover, using (2.60), (3.67) and (4.1) we infer that

$$
\begin{align*}
\left|\mathscr{I}_{t, 5}^{(1)}\right| & \leq C(T-t)^{-2}\left(\int|U|^{\frac{4}{d}-1}|R|^{3} d x+\|R\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}}\right) \\
& \leq C(T-t)^{-2}\left((T-t)^{-2} D^{3}+(T-t)^{-2} D^{2+\frac{4}{d}}\right) \leq C \frac{D^{2}}{(T-t)^{2}} . \tag{4.23}
\end{align*}
$$

Finally, regarding $\mathscr{I}_{t, 6}^{(1)}$, since

$$
|f(u)-f(U)| \leq C\left(|U|^{\frac{4}{d}}+|R|^{\frac{4}{d}}\right)|R| \leq C\left((T-t)^{-2}+|R|^{\frac{4}{d}}\right)|R|,
$$

using the integration by parts formula and (3.63) we have

$$
\begin{aligned}
\left|\mathscr{I}_{t, 6}^{(1)}\right| & \leq C\left(\|\nabla \eta\|_{L^{2}}\|\nabla R\|_{L^{2}}+(T-t)^{-2}\|R\|_{L^{2}}\|\eta\|_{L^{2}}+\|R\|_{H^{1}}^{1+\frac{4}{d}}\|\eta\|_{L^{2}}\right) \\
& \leq C\left(\frac{M o d}{(T-t)^{4}} D+\left(\frac{D}{T-t}\right)^{1+\frac{4}{d}} \frac{M o d}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) .
\end{aligned}
$$

Note that, by (2.53) and (4.1),

$$
M o d \leq C\left(\sum_{k=1}^{K}\left|M_{k}\right|+(T-t)^{2} D+e^{-\frac{\delta}{T-t}}\right)
$$

Taking into account Theorem 4.1 we obtain

$$
\begin{equation*}
\left|\mathscr{I}_{t, 6}^{(1)}\right| \leq C\left(\frac{D^{2}}{(T-t)^{2}}+\sum_{k=1}^{K} \frac{\left|M_{k}\right|}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) \tag{4.24}
\end{equation*}
$$

Now, combining estimates (4.16)-(4.24) altogether and using $\varepsilon_{k}$ in (2.57) we conclude that

$$
\begin{align*}
\frac{d \mathscr{I}^{(1)}}{d t}= & \sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}}\left\|\varepsilon_{k}\right\|_{L^{2}}^{2}-\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{\mid}}\left|\varepsilon_{k}\right|^{2}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{d}-2}{\overline{Q_{k}}}^{2} \varepsilon_{k}^{2} d y \\
& -\sum_{k=1}^{K} \frac{\gamma_{k}}{\lambda_{k}^{4}} \operatorname{Re} \int y \cdot \nabla \overline{Q_{k}}\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y \\
& +O\left(\frac{D^{2}}{(T-t)^{2}}+\varepsilon \frac{D^{2}}{(T-t)^{3}}+\sum_{k=1}^{K} \frac{\left|M_{k}\right|}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) . \tag{4.25}
\end{align*}
$$

Therefore, combining (3.107) and (4.25) and arguing as those below (3.80) we prove (4.15).
4.3. Upgradation to the exponential order. We are now ready to fulfill the final upgradation to the exponential decay rate. The main result is formulated in Theorem 4.4 below.

Theorem 4.4. (Exponential decay of remainder) There exist $C, \delta>0$ such that for t close to $T$,

$$
\begin{equation*}
D(t) \leq C e^{-\frac{\delta}{T-t}} \tag{4.26}
\end{equation*}
$$

Proof. On one hand, expanding the modified generalized energy $\mathscr{I}$ up to the second order yields

$$
\begin{equation*}
\mathscr{I}=\frac{1}{2} \operatorname{Re} \int|\nabla R|^{2}+\sum_{k=1}^{K} \frac{1}{\lambda_{k}^{2}}|R|^{2} \Phi_{k}-\left(1+\frac{2}{d}\right)|U|^{\frac{4}{t}}|R|^{2}-\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}^{2} d x+O(\widehat{E} r), \tag{4.27}
\end{equation*}
$$

where the error term

$$
\begin{equation*}
\widehat{E} r=\sum_{l=3}^{2+\frac{4}{7}}|U|^{2+\frac{4}{d}-l}|R|^{l} d x+\|\nabla R\|_{L^{2}}\|R\|_{L^{2}} \tag{4.28}
\end{equation*}
$$

Using (2.60) and taking $t$ close to $T$ such that $T-t \leq \varepsilon$ we get

$$
\begin{equation*}
\widehat{E} r \leq \sum_{l=3}^{2+\frac{4}{d}}(T-t)^{-\frac{d}{2}\left(2+\frac{4}{d}-l\right)}(T-t)^{-d\left(\frac{1}{2}-1\right)} D^{l}+\frac{D^{2}}{T-t} \leq C \frac{D^{2}}{T-t} \leq C \varepsilon \frac{D^{2}}{(T-t)^{2}} \tag{4.29}
\end{equation*}
$$

Moreover, arguing as in the proof of (3.39) we deduce that the main order on the R.H.S. of (4.27) is bounded from below by

$$
\begin{equation*}
\tilde{c} \frac{D^{2}}{(T-t)^{2}}+O\left(\sum_{k=1}^{K} \frac{M_{k}^{2}}{(T-t)^{2}}+e^{-\frac{\delta}{T-t}}\right) . \tag{4.30}
\end{equation*}
$$

Note that, by (3.3) and (4.1), for $t$ close to $T$ such that $T-t \leq \varepsilon$,

$$
\begin{aligned}
\frac{M_{k}^{2}}{(T-t)^{2}} & \leq \frac{C}{(T-t)^{2}}\left(\int_{t}^{T} \frac{D^{2}}{T-s} d s\right)^{2}+C e^{-\frac{\delta}{T-t}} \\
& \leq C(T-t)^{4} \int_{t}^{T} \frac{D^{2}}{(T-s)^{3}} d s+C e^{-\frac{\delta}{T-t}}
\end{aligned}
$$

$$
\begin{equation*}
\leq C \varepsilon \int_{t}^{T} \frac{D^{2}}{(T-s)^{3}} d s+C e^{-\frac{\delta}{T-t}} \tag{4.31}
\end{equation*}
$$

Thus, taking $\varepsilon$ small enough we conclude from (4.27), (4.29) and (4.30) that

$$
\begin{equation*}
\mathscr{I}(t) \geq \tilde{c} \frac{D^{2}(t)}{(T-t)^{2}}-C \varepsilon \int_{t}^{T} \frac{D^{2}}{(T-s)^{3}} d s-C e^{-\frac{\delta}{T-t}} \tag{4.32}
\end{equation*}
$$

On the other hand, for $t$ close to $T$ such that $T-t<\varepsilon$, we have

$$
\frac{D^{2}}{(T-t)^{2}} \leq \varepsilon \frac{D^{2}}{(T-t)^{3}}, \frac{1}{(T-t)^{2}} \int_{t}^{T} \frac{D^{2}}{T-s} d s \leq \int_{t}^{T} \frac{D^{2}}{(T-s)^{3}} d s
$$

Then, Theorem 4.3 and (3.3) yield that for $t$ close to $T$,

$$
\begin{equation*}
\frac{d \mathscr{I}}{d t} \geq-C\left(\varepsilon \frac{D^{2}(t)}{(T-t)^{3}}+\int_{t}^{T} \frac{D^{2}}{(T-s)^{3}} d s+e^{-\frac{\delta}{T-t}}\right) \tag{4.33}
\end{equation*}
$$

Thus, combining (4.32) and (4.33) and using the fundamental theorem of calculus we obtain

$$
\begin{equation*}
\tilde{c} \frac{D^{2}(t)}{(T-t)^{2}} \leq \mathscr{I}(\tilde{t})+C \int_{t}^{T}\left(\varepsilon \frac{D^{2}(s)}{(T-s)^{3}}+\int_{s}^{T} \frac{D^{2}}{(T-r)^{3}} d r\right) d s+C e^{-\frac{\delta}{T-t}} . \tag{4.34}
\end{equation*}
$$

Since by (4.1), (4.27) and (4.29),

$$
\begin{equation*}
\mathscr{I}(\tilde{t}) \leq C \frac{D^{2}(\tilde{t})}{(T-\tilde{t})^{2}} \leq C(T-\tilde{t})^{2} \rightarrow 0, \text { as } \tilde{t} \rightarrow T \tag{4.35}
\end{equation*}
$$

Letting $\tilde{t} \rightarrow T$ in (4.34) and using estimate (4.35) we obtain a new Gronwall type inequality

$$
\begin{equation*}
\frac{D^{2}(t)}{(T-t)^{2}} \leq C_{2} \int_{t}^{T}\left(\varepsilon \frac{D^{2}(s)}{(T-s)^{3}}+\int_{s}^{T} \frac{D^{2}}{(T-r)^{3}} d r\right) d s+C_{2} e^{-\frac{\delta}{T-t}} \tag{4.36}
\end{equation*}
$$

The keypoint here is that, the second order $(T-t)^{2}$ in (4.8) now has been replaced by the much faster exponential rate $e^{-\frac{\delta}{T-t}}$.

Now, we claim that there exist $C, \delta>0$ such that for $t$ close to $T$,

$$
\begin{equation*}
D(t) \leq C e^{-\frac{\delta}{T-t}} \tag{4.37}
\end{equation*}
$$

In order to prove (4.37), we take $t$ close to $T$ such that $T-t \leq \varepsilon$ and $e^{-\frac{\delta}{T-t}} \leq(T-t) e^{-\frac{\delta}{2(T-t)}}$. Then, inserting (4.1) into (4.36) we get

$$
\begin{equation*}
\frac{D^{2}}{(T-t)^{2}} \leq C_{2}^{2} \varepsilon(T-t)^{2}+C_{2}^{2}(T-t) e^{-\frac{\delta}{2(T-t)}} . \tag{4.38}
\end{equation*}
$$

Again, we use the induction arguments and assume that for some $n \geq 1$,

$$
\begin{equation*}
\frac{D^{2}}{(T-t)^{2}} \leq \sum_{k=0}^{n-1} 2^{k} C_{2}^{k+2} \varepsilon^{k}(T-t) e^{-\frac{\delta}{2(T-t)}}+C_{2}^{n+1} \varepsilon^{n}(T-t)^{2} \tag{4.39}
\end{equation*}
$$

Note that, (4.39) holds at the initial step $n=1$ due to (4.38). Moreover, plugging (4.39) into (4.36) and using straightforward computations we see that (4.39) is also valid with $n+1$ replacing $n$.

Thus, the induction arguments yield that (4.39) holds for all $n \geq 1$. In particular, taking $\varepsilon$ small enough such that $C_{2} \varepsilon \leq \frac{1}{2}$ we obtain

$$
\frac{D^{2}(t)}{(T-t)^{2}} \leq \lim _{n \rightarrow \infty}\left(\left(\sum_{k=0}^{n-1} 2^{k} C_{2}^{k+1} \varepsilon^{k}\right)(T-t) e^{-\frac{\delta}{2(T-t)}}+C_{2}^{n+1} \varepsilon^{n}(T-t)^{2}\right) \leq 2 C_{2}(T-t) e^{-\frac{\delta}{2(T-t)}},
$$

which yields (4.37), thereby completing the proof of Theorem 4.4.

As a consequence of Theorems 2.5, 3.1 and 4.4, we obtain the exponential decay estimates of localized mass and modulation equations, which in particular improve the previous estimate (4.12).

Corollary 4.5. (Exponential decay of localized mass and modulation equations) There exist $C, \delta>$ 0 such that for t close to $T$,

$$
\begin{equation*}
\sum_{k=1}^{K}\left|M_{k}(t)\right|+\operatorname{Mod}(t) \leq C e^{-\frac{\delta}{T-t}} . \tag{4.40}
\end{equation*}
$$

We conclude this section with the upgradation to the exponential decay rate of geometrical parameters. Let us set

$$
\begin{equation*}
\mathcal{P}_{0, k}=\left(\lambda_{0, k}, \alpha_{0, k}, \beta_{0, k}, \gamma_{0, k}, \theta_{0, k}\right)=\left(\omega_{k}(T-t), x_{k}, 0, \omega_{k}^{2}(T-t), \omega_{k}^{-2}(T-t)^{-1}+\vartheta_{k}\right), \tag{4.41}
\end{equation*}
$$

which are the geometrical parameters corresponding to the pseudo-conformal blow-up solutions $S_{k}, 1 \leq k \leq K$. Let $\mathcal{P}_{k}=\left(\lambda_{k}, \alpha_{k}, \beta_{k}, \gamma_{k}, \theta_{k}\right)$ be the modulation parameters as in Theorem 2.1 and set $\left|\mathcal{P}_{k}-\mathcal{P}_{0, k}:=\left|\lambda_{k}-\omega_{k}(T-t)\right|+\left|\alpha_{k}(t)-x_{k}\right|+\left|\beta_{k}(t)\right|+\left|\gamma_{k}(t)-\omega_{k}^{2}(T-t)\right|+\left|\theta_{k}(t)-\omega_{k}^{-2}(T-t)^{-1}-\vartheta_{k}\right|\right.$.

Corollary 4.6. (Exponential decay of geometrical parameters) There exist $C, \delta>0$ such that for $t$ close to $T$,

$$
\begin{equation*}
\sum_{k=1}^{K}\left|\mathcal{P}_{k}(t)-\mathcal{P}_{0, k}(t)\right| \leq C e^{-\frac{\delta}{T-t}} . \tag{4.42}
\end{equation*}
$$

Proof. (i) Estimates of $\lambda_{k}$ and $\gamma_{k}$. We compute that, by (2.9), (3.51) and (4.40),

$$
\begin{equation*}
\left|\frac{d}{d t}\left(\frac{\gamma_{k}}{\lambda_{k}}\right)\right|=\frac{\left|\lambda_{k}^{2} \dot{\gamma}_{k}-\lambda_{k} \dot{\lambda}_{k} \gamma_{k}\right|}{\lambda_{k}^{3}} \leq \frac{\left|\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right|}{\lambda_{k}^{3}}+\left|\frac{\gamma_{k}}{\lambda_{k}}\right| \frac{\left|\gamma_{k}+\lambda_{k} \dot{\lambda}_{k}\right|}{\lambda_{k}^{2}} \leq C \frac{\operatorname{Mod}(t)}{\lambda_{k}^{3}} \leq C e^{-\frac{\delta}{3(T-t)}} . \tag{4.43}
\end{equation*}
$$

Then, taking into account (3.98) we infer that

$$
\begin{equation*}
\left|\left(\frac{\gamma_{k}}{\lambda_{k}}\right)(t)-\omega_{k}\right| \leq \int_{t}^{T}\left|\frac{d}{d r}\left(\frac{\gamma_{k}}{\lambda_{k}}\right)\right| d r \leq C e^{-\frac{\delta}{3(T-t)}} . \tag{4.44}
\end{equation*}
$$

Moreover, by (4.44),

$$
\left|\frac{d}{d t}\left(\lambda_{k}-\omega_{k}(T-t)\right)\right|=\left|\dot{\lambda}_{k}+\frac{\gamma_{k}}{\lambda_{k}}+\omega_{k}-\frac{\gamma_{k}}{\lambda_{k}}\right| \leq \frac{M o d}{\lambda_{k}}+C e^{-\frac{\delta}{3(T-t)}} \leq C e^{-\frac{\delta}{3(T-t)}},
$$

which along with (2.6) yields that

$$
\begin{equation*}
\left|\lambda_{k}(t)-\omega_{k}(T-t)\right| \leq \int_{t}^{T}\left|\frac{d}{d r}\left(\lambda_{k}-\omega_{k}(T-r)\right)\right| d r \leq C e^{-\frac{\delta}{3(T-t)}}, \tag{4.45}
\end{equation*}
$$

thereby yielding the estimate of $\lambda_{k}$ in (4.42).
Similarly, by (4.37), (4.40) and (4.44),

$$
\left|\frac{d}{d t}\left(\gamma_{k}-\omega_{k}^{2}(T-t)\right)\right|=\left|\dot{\gamma}_{k}+\frac{\gamma_{k}^{2}}{\lambda_{k}^{2}}+\omega_{k}^{2}-\frac{\gamma_{k}^{2}}{\lambda_{k}^{2}}\right| \leq \frac{M o d}{\lambda_{k}^{2}}+C\left|\omega_{k}-\frac{\gamma_{k}}{\lambda_{k}}\right| \leq C e^{-\frac{\delta}{3(T-t)}} .
$$

Taking into account (3.90) we obtain

$$
\begin{equation*}
\left|\gamma_{k}(t)-\omega_{k}^{2}(T-t)\right| \leq \int_{t}^{T}\left|\frac{d}{d r}\left(\gamma_{k}(r)-\omega_{k}^{2}(T-r)\right)\right| d r \leq C e^{-\frac{\delta}{3(T-t)}} . \tag{4.46}
\end{equation*}
$$

which yields the estimate of $\gamma_{k}$ in (4.42).
(ii) Estimates of $\beta_{k}$ and $\alpha_{k}$. We use the refined estimate of $\beta_{k}$ in Corollary 3.16 and (4.40) to get

$$
\begin{equation*}
\left|\beta_{k}\right|^{2} \leq C(T-t)^{2} \sum_{k=1}^{K}\left|\frac{\gamma_{k}}{\lambda_{k}}-\omega_{k}\right|+C e^{-\frac{\delta}{t-t}} . \tag{4.47}
\end{equation*}
$$

which along with (4.44) yields that

$$
\begin{equation*}
\left|\beta_{k}(t)\right| \leq C e^{-\frac{\delta}{6(T-t)}}, \tag{4.48}
\end{equation*}
$$

thereby implying the estimate of $\beta_{k}$ in (4.42).
Moreover, by (4.40) and (4.48),

$$
\begin{equation*}
\left|\dot{\alpha}_{k}\right|=\left|\frac{\lambda_{k} \dot{\alpha}_{k}-2 \beta_{k}}{\lambda_{k}}+\frac{2 \beta_{k}}{\lambda_{k}}\right| \leq \frac{M o d}{\lambda_{k}}+\frac{2\left|\beta_{k}\right|}{\lambda_{k}} \leq C e^{-\frac{\delta}{\lambda(T-1)}} . \tag{4.49}
\end{equation*}
$$

Hence, taking into account $\lim _{t \rightarrow T} \alpha_{k}(t)=x_{k}$ we infer that

$$
\begin{equation*}
\left|\alpha_{k}(t)-x_{k}\right| \leq \int_{t}^{T}\left|\dot{\alpha}_{k}(r)\right| d r \leq C e^{-\frac{\delta}{\lambda(T-t)}} \tag{4.50}
\end{equation*}
$$

which yields the estimate of $\alpha_{k}$ in (4.42).
(iii) Estimate of $\theta_{k}$. By (4.40), (4.45) and (4.48),

$$
\begin{align*}
\left|\frac{d}{d t}\left(\theta_{k}-\left(\omega_{k}^{-2}(T-t)^{-1}+\vartheta_{k}\right)\right)\right| & =\left|\frac{\lambda_{k}^{2} \dot{\theta}_{k}-1-\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}+\frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}+\frac{1}{\lambda_{k}^{2}}-\frac{1}{\omega_{k}^{2}(T-t)^{2}}\right| \\
& \leq \frac{M o d}{\lambda_{k}^{2}}+\frac{\left|\beta_{k}\right|^{2}}{\lambda_{k}^{2}}+\frac{\left|\lambda_{k}-\omega_{k}(T-t)\right|\left|\lambda_{k}+\omega_{k}(T-t)\right|}{\omega_{k}^{2} \lambda_{k}^{2}(T-t)^{2}} \leq C e^{-\frac{\delta}{4(T-t)}} . \tag{4.51}
\end{align*}
$$

In view of (2.7), $\lim _{t \rightarrow T}\left|\theta_{k}-\left(\omega_{k}^{-2}(T-t)^{-1}+\vartheta_{k}\right)\right|=0$, we obtain that for $t$ close to $T$,

$$
\left|\theta_{k}-\left(\omega_{k}^{-2}(T-t)^{-1}+\vartheta_{k}\right)\right| \leq \int_{t}^{T}\left|\frac{d}{d r}\left(\theta_{k}-\left(\omega_{k}^{-2}(T-r)^{-1}+\vartheta_{k}\right)\right)\right| d r \leq C e^{-\frac{\delta}{4(T-t)}},
$$

thereby proving the estimate of $\theta_{k}$ in (4.42).
Therefore, the proof of Corollary 4.6 is complete.

## 5. Proof of main results

In this section, we prove the main results in Theorems 1.3, 1.5 and 1.7.
5.1. Proof of Theorem 1.3. By virtue of Theorem 4.4 and Corollary 4.6, we have that

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} U_{k}(t)\right\|_{H^{1}}+\sum_{k=1}^{K}\left|\mathcal{P}_{k}(t)-\mathcal{P}_{0, k}(t)\right| \leq C e^{-\frac{\delta}{T-t}}, \tag{5.1}
\end{equation*}
$$

where $\mathcal{P}_{k}$ and $\mathcal{P}_{0, k}$ are as in Corollary 4.6 , and $C, \delta>0$.
Moreover, using similar computations as in (2.26) and (2.46) and using (5.1) we obtain

$$
\begin{equation*}
\left\|\sum_{k=1}^{K} U_{k}(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}} \leq C e^{-\frac{\delta}{T-t}} . \tag{5.2}
\end{equation*}
$$

Thus, combining (5.1) and (5.2) altogether we obtain that for $t$ close to $T$

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}} \leq\left\|v(t)-\sum_{k=1}^{K} U_{k}(t)\right\|_{H^{1}}+\left\|\sum_{k=1}^{K} U_{k}(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}} \leq C e^{-\frac{\delta}{T-t}} . \tag{5.3}
\end{equation*}
$$

In particular, this yields that for any $\zeta \in(0,1)$,

$$
\begin{equation*}
\left\|v(t)-\sum_{k=1}^{K} S_{k}(t)\right\|_{H^{1}} \leq C(T-t)^{3+\zeta} \tag{5.4}
\end{equation*}
$$

Thus, by virtue of Theorem 1.1, we obtain the uniqueness of multi-bubble blow-up solutions satisfying (1.22) and (1.23). In particular, the unique multi-bubble blow-up solution coincides with the one constructed in [53]. Thus, in view of (1.19), we obtain the exponential convergence (1.24).

Therefore, the proof of Theorem 1.3 is complete.
5.2. Proof of Theorems $\mathbf{1 . 5}$ and 1.7. In order to prove Theorem 1.5, let us first derive the convergence rate of $u-\sum_{k=1}^{K} W_{k}$ in the pseudo-coformal space $\Sigma$ from that in the energy space $H^{1}$.
Lemma 5.1. Assume that $u$ is a multi-soliton to equation (NLS) satisfying that for some $\mu>2$,

$$
\begin{equation*}
\left\|u(s)-\sum_{k=1}^{K} W_{k}(s)\right\|_{H^{1}}=O\left(\frac{1}{s^{\mu}}\right), \text { as } s \rightarrow+\infty \tag{5.5}
\end{equation*}
$$

where $\left\{W_{k}\right\}$ are the solitary waves given by (1.7) with distinct speeds $\left\{v_{k}\right\}$. Then, we have

$$
\begin{equation*}
\left\|x\left(u(s)-\sum_{k=1}^{K} W_{k}(s)\right)\right\|_{L^{2}}=O\left(\frac{1}{s^{\mu-2}}\right), \text { as } s \rightarrow+\infty . \tag{5.6}
\end{equation*}
$$

Moreover, in the single soliton case where $K=1$, if (5.5) holds with $\mu>1$ and the propagation speed $v_{1}=0$, then we have

$$
\begin{equation*}
\left\|x\left(u(s)-W_{1}(s)\right)\right\|_{L^{2}}=O\left(\frac{1}{s^{\mu-1}}\right), \text { as } s \rightarrow+\infty \tag{5.7}
\end{equation*}
$$

Proof. Set $z:=u-W$ with $W:=\sum_{k=1}^{K} W_{k}, f(z)=|z|^{\frac{4}{z}} z$. Let $\varphi$ be a smooth radial cut-off function such that $\varphi(x)=|x|^{2}$ for $|x| \leq 1, \varphi(x)=0$ for $|x| \geq 2,0 \leq \varphi \leq 1$, and for a universal constant $C>0$

$$
\begin{equation*}
|\nabla \varphi(x)|^{2} \leq C \varphi(x) \tag{5.8}
\end{equation*}
$$

$\operatorname{Set} \varphi_{A}(x):=A^{2} \varphi\left(\frac{x}{A}\right)$ for $A>0$ and

$$
\begin{equation*}
M_{A}(s):=\int \varphi_{A}|z(s)|^{2} d x \tag{5.9}
\end{equation*}
$$

By equations (NLS) and (1.7), $z$ satisfies the equation

$$
\begin{equation*}
i z_{s}+\Delta z+f(z)+f(W, z)+H=0 \tag{5.10}
\end{equation*}
$$

where $f(W, z)$ denotes the difference

$$
\begin{equation*}
f(W, z):=f(z+W)-f(W)-f(z), \tag{5.11}
\end{equation*}
$$

and the remainder $H$ contains the remaining coupling terms

$$
\begin{equation*}
H=f(W)-\sum_{k=1}^{K} f\left(W_{k}\right) \tag{5.12}
\end{equation*}
$$

Then, using the integration by parts formula we get

$$
\begin{equation*}
\frac{d}{d s} M_{A}=-2 \operatorname{Im}\left\langle\nabla \varphi_{A} z, \nabla z\right\rangle+2 \operatorname{Im}\left\langle\varphi_{A} z, f(W, z)+H\right\rangle \tag{5.13}
\end{equation*}
$$

Note that, by Hölder's inequality, (5.5) and (5.8),

$$
\begin{equation*}
\left|\operatorname{Im}\left\langle\nabla \varphi_{A} z, \nabla z\right\rangle\right| \leq C\|z\|_{H^{1}} M_{A}^{\frac{1}{2}} \leq C \frac{M_{A}^{\frac{1}{2}}}{s^{\mu}}, \tag{5.14}
\end{equation*}
$$

where $C$ is a universal positive constant independent of $s$ and $A$.
Moreover, since

$$
\begin{equation*}
|f(W, z)| \leq C\left(|W|^{\frac{4}{d}}|z|+|W||z|^{\frac{4}{d}}\right), \tag{5.15}
\end{equation*}
$$

using the exponential decoupling between $W_{j}$ and $W_{k}, j \neq k$, we infer that

$$
\begin{aligned}
\left|\operatorname{Im}\left\langle\varphi_{A} z, f(W, z)+H\right\rangle\right| & \leq C \sum_{k=1}^{K} \int \varphi_{A}|z|\left(\left|W_{k}\right|^{\frac{4}{d}}|z|+\left|W_{k} \| z\right|^{\frac{4}{d}}\right) d x+C M_{A}^{\frac{1}{2}} e^{-\delta s} \\
& \leq C \sum_{k=1}^{K} M_{A}^{\frac{1}{2}}\left(\left\|\varphi_{A}^{\frac{1}{2}} W_{k}^{\frac{4}{d}}\right\|_{L^{4}}\|z\|_{H^{1}}+\left\|\varphi_{A}^{\frac{1}{2}} W_{k}\right\|_{L^{4}}\|z\|_{H^{1}}^{\frac{4}{d}}+e^{-\delta s}\right)
\end{aligned}
$$

where $C$ is a universal constant. Note that, by (1.7) and the change of variables

$$
\left\|\varphi_{A}^{\frac{1}{2}} W_{k}\right\|_{L^{4}}+\left\|\varphi_{A}^{\frac{1}{2}} W_{k}^{\frac{4}{2}}\right\|_{L^{4}} \leq C\left(\left\|x W_{k}\right\|_{L^{4}}+\left\|x W_{k}^{\frac{4}{a}}\right\|_{L^{4}}\right) \leq C s .
$$

Taking into account (5.5) we get

$$
\begin{equation*}
\left|\operatorname{Im}\left\langle\varphi_{A} z, f(W, z)+H\right\rangle\right| \leq C \frac{M_{A}^{\frac{1}{2}}}{s^{\mu-1}} \tag{5.16}
\end{equation*}
$$

Note that, one decay rate is lost here.
Then, plugging (5.14) and (5.16) into (5.13) we get

$$
\begin{equation*}
\left|\frac{d}{d s} M_{A}\right| \leq C \frac{M_{A}^{\frac{1}{2}}}{s^{\mu-1}}, \tag{5.17}
\end{equation*}
$$

which yields that, for any $s, \tilde{s} \in \mathbb{R}^{+}, s<\tilde{s}$,

$$
\begin{equation*}
\left|M_{A}^{\frac{1}{2}}(\tilde{s})-M_{A}^{\frac{1}{2}}(s)\right| \leq C\left(\frac{1}{s^{\mu-2}}-\frac{1}{\tilde{s}^{\mu-2}}\right), \tag{5.18}
\end{equation*}
$$

where $C>0$ is independent of $s$ and $A$. Since $\mu>2$, and by (5.5), $M_{A}(\tilde{s}) \rightarrow 0$ as $\tilde{s} \rightarrow \infty$, we let $\tilde{s} \rightarrow \infty$ and get

$$
\begin{equation*}
\left|M_{A}^{\frac{1}{2}}(s)\right| \leq \frac{C}{s^{\mu-2}} . \tag{5.19}
\end{equation*}
$$

Since $C$ is independent of $s$ and $A$, we may let $A \rightarrow \infty$ and use Fatou's lemma to obtain

$$
\begin{equation*}
\|x z(s)\|_{L^{2}} \leq \frac{C}{s^{\mu-2}} \tag{5.20}
\end{equation*}
$$

which yields (5.6).
In the single soliton case $K=1$, if the speed $v_{1}=0$, then we have the improved estimate

$$
\left\|\varphi_{A}^{\frac{1}{2}} W_{1}\right\|_{L^{4}}+\left\|\varphi_{A}^{\frac{1}{2}} W_{1}^{\frac{4}{d}}\right\|_{L^{4}} \leq\left\|x W_{1}\right\|_{L^{4}}+\left\|x W_{1}^{\frac{4}{d}}\right\|_{L^{4}} \leq C,
$$

and thus the above arguments yield

$$
\begin{equation*}
\left|\operatorname{Im}\left\langle\varphi_{A} z, f(W, z)+H\right\rangle\right| \leq C \frac{M_{A}^{\frac{1}{2}}}{s^{\mu}}, \tag{5.21}
\end{equation*}
$$

which improves the previous estimate (5.16). Thus, arguing as above we get (5.7).

Proof of Theorem 1.5. First note that, for any $v, u \in \Sigma, v=P_{T}(u)$ given by (1.6), one has

$$
\begin{align*}
& \|v(t)\|_{L^{2}}=\left\|u\left(\frac{1}{T-t}\right)\right\|_{L^{2}},  \tag{5.22}\\
& \|x v(t)\|_{L^{2}}=(T-t)\left\|y u\left(\frac{1}{T-t}\right)\right\|_{L^{2}},  \tag{5.23}\\
& \|\nabla v(t)\|_{L^{2}} \leq C\left(\frac{1}{T-t}\left\|\nabla u\left(\frac{1}{T-t}\right)\right\|_{L^{2}}+\left\|y u\left(\frac{1}{T-t}\right)\right\|_{L^{2}}\right) . \tag{5.24}
\end{align*}
$$

Now, let $u_{i}$ denote the multi-soliton to (NLS) satisfying (1.26) and $W=\sum_{k=1}^{K} W_{k}$ be the sum of $K$ solitary waves. Let $v_{i}=P_{T}\left(u_{i}\right), S=P_{T}(W)$ be the blow-up solutions corresponding to $u_{i}$ and $W$, respectively, $i=1$, 2. We infer from Lemma 5.1, (1.26), (5.22) and (5.24) that

$$
\left\|v_{i}(t)-S(t)\right\|_{H^{1}} \leq \frac{1}{(T-t)}\left\|\left(u_{i}-W\right)\left(\frac{1}{T-t}\right)\right\|_{H^{1}}+\left\|y\left(u_{i}-W\right)\left(\frac{1}{T-t}\right)\right\|_{L^{2}} \leq C(T-t)^{\zeta},
$$

which verifies the condition (1.25).
Thus, by virtue of Theorem 1.3, we obtain that $v_{1} \equiv v_{2}$, which, via the pseudo-conformal transformation (1.6), in turn yields that $u_{1} \equiv u_{2}$, and thus the uniqueness of multi-solitions follows. Moreover, estimate (1.27) follows from (1.24) and estimates (5.22)-(5.24).

Therefore, the proof of Theorem 1.5 is complete.
Next, we prove Theorem 1.7. The key ingredient is the monotonicity of virial functional in Lemma 5.2, which enables to obtain an improved space time estimate of gradient in Corollary 5.3.

Lemma 5.2. (Monotonicity of virial functional) Assume that u is a multi-soliton to (NLS) such that

$$
\begin{equation*}
\left\|u(s)-\sum_{k=1}^{K} W_{k}(s)\right\|_{\Sigma}=O\left(\frac{1}{s^{\mu}}\right), \text { for s large enough }, \tag{5.25}
\end{equation*}
$$

where $\mu>0,\left\{W_{k}\right\}$ are the solitary waves given by (1.7) with distinct speeds $\left\{v_{k}\right\}$. Assume additionally that $v_{k} \neq 0,1 \leq k \leq K$. Let I denote the virial functional

$$
\begin{equation*}
I:=\operatorname{Im} \int x \cdot \nabla z \bar{z} d x, \text { with } z:=u-\sum_{k=1}^{K} W_{k} . \tag{5.26}
\end{equation*}
$$

Then, there exists $C>0$ such that for slarge enough,

$$
\begin{equation*}
\frac{d I}{d s} \geq\|\nabla z(s)\|_{L^{2}}^{2}-C\left(\frac{1}{s^{2 \mu+1}}+e^{-\delta s}\right) \tag{5.27}
\end{equation*}
$$

Proof. Using equation (5.10) we compute

$$
\begin{equation*}
\frac{d I}{d s}=2 \operatorname{Im} \int \Lambda z \bar{z}_{s} d x=-2 \operatorname{Re}\langle\Lambda z, \Delta z+f(z)\rangle-2 \operatorname{Re}\langle\Lambda z, f(W, z)\rangle-2 \operatorname{Re}\langle\Lambda z, H\rangle, \tag{5.28}
\end{equation*}
$$

where $\Lambda:=\frac{d}{2} \mathrm{I}_{\mathrm{d}}+x \cdot \nabla$ and $f(W, z)$ is as in (5.11).
For the first term on the R.H.S. of (5.28), using the integration by parts formula we compute

$$
\begin{equation*}
-2 \operatorname{Re}\langle\Lambda z, \Delta z+f(z)\rangle=2\|\nabla z\|_{L^{2}}^{2}-\frac{2 d}{d+2}\|z\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}}=4 E(z) . \tag{5.29}
\end{equation*}
$$

Moreover, by the Gagliardo-Nirenberg inequality (6.1) in Appendix,

$$
\begin{equation*}
E(z)=\frac{1}{2} \int|\nabla z|^{2} d x-\frac{d}{2 d+4} \int|z|^{2+\frac{4}{d}} d x \geq\left(\frac{1}{2}-C\|z\|_{L^{2}}^{\frac{4}{d}}\right)\|\nabla z\|_{L^{2}}^{2}, \tag{5.30}
\end{equation*}
$$

where $C>0$. Then, in view of (5.25), we may take $s$ large enough such that $C\|z(s)\|_{L^{2}}^{\frac{4}{d}} \leq \frac{1}{4}$ and get

$$
\begin{equation*}
E(z(s)) \geq \frac{1}{4}\|\nabla z(s)\|_{L^{2}}^{2} \tag{5.31}
\end{equation*}
$$

Plugging this into (5.29) we get

$$
\begin{equation*}
-2 \operatorname{Re}\langle\Lambda z, \Delta z+f(z)\rangle \geq\|\nabla z(s)\|_{L^{2}}^{2} \tag{5.32}
\end{equation*}
$$

Regarding the second inner product on the R.H.S. of (5.28), using the integration by parts formula we move the derivative onto $W$ and compute

$$
\begin{align*}
\operatorname{Re}\langle\Lambda z, f(z+W)-f(W)-f(z)\rangle= & \operatorname{Re}\left\langle x \cdot \nabla W, f(z+W)-f(W)-f^{\prime}(W) \cdot z\right\rangle \\
& +O\left(\sum_{l=3}^{1++\frac{4}{d}} \int|W|^{2+\frac{4}{d}-l}|z|^{l} d x\right), \tag{5.33}
\end{align*}
$$

which along with the exponential decoupling between different traveling waves yields that

$$
\begin{align*}
|\operatorname{Re}\langle\Lambda z, f(z+W)-f(W)-f(z)\rangle| \leq & C \sum_{k=1}^{K} \sum_{l=2}^{\frac{4}{d}} \int|x|\left|\nabla W_{k}\right|\left|W_{k}\right|^{1+\frac{4}{d}-l}|z|^{l} d x \\
& +C \sum_{k=1}^{K} \sum_{l=3}^{1+\frac{4}{d}} \int\left|W_{k}\right|^{2+\frac{4}{d}-l}|z|^{l} d x+C e^{-\delta s} \tag{5.34}
\end{align*}
$$

For the quadratic terms above, since $\left|x-v_{k} s\right| \geq \frac{v_{k} \mid s}{2}$ for $|x| \leq \frac{v_{k} \mid s}{2}$, and $\nabla W_{k}, W_{k} \in L^{\infty}$, we infer from the exponential decay property of $Q$ that

$$
\begin{align*}
\int|x|\left|\nabla W_{k} \| W_{k}\right|^{\frac{4}{d}-1}|z|^{2} d x & \leq C\left(\int_{|x| \geq \frac{v_{k} \mid s}{2}}\left|x Q\left(\frac{x-v_{k} s}{\omega_{k}}\right) z^{2}\right| d x+\int_{|x| \leq \leq \frac{v_{k} \mid s}{2}}\left|x Q\left(\frac{x-v_{k} s}{\omega_{k}}\right) z^{2}\right| d x\right) \\
& \leq C\left(\|x z\|_{L^{2}}\|z\|_{L^{2}\left(|x| \geq \frac{v_{k} \mid s}{2}\right)}+e^{-\delta s}\right) \\
& \leq C\left(\frac{\|x z\|_{L^{2}}^{2}}{\left|v_{k}\right| s}+e^{-\delta s}\right) \leq C\left(\frac{1}{s^{2 \mu+1}}+e^{-\delta s}\right), \tag{5.35}
\end{align*}
$$

where we also used (5.25) and $v_{k} \neq 0$ in the last step, $1 \leq k \leq K$.
Moreover, for the higher order terms with $l \geq 3$, similarly we have

$$
\begin{equation*}
\int|x|\left|\nabla W_{k} \| W_{k}\right|^{1+\frac{4}{d}-1}|z|^{l} d x+\int\left|W_{k}\right|^{2+\frac{4}{d}-l}|z|^{l} d x \leq C\left(\|z\|\left\|_{\Sigma}\right\| z \|_{L^{2(l-1)}\left(|x| \geq \frac{v_{k} \mid s}{2}\right)}^{l-1}+e^{-\delta s}\right) . \tag{5.36}
\end{equation*}
$$

Note that, by the Gagliardo-Nirenberg inequality (6.1),

$$
\begin{equation*}
\|z\|_{L^{2(l-1)}\left(|x| \geq \frac{\left.v_{k} \mid s\right)}{2}\right)}^{l-1} \leq C\left(\|z\|_{L^{2}\left(|x| \geq \frac{v_{k} \mid s}{3}\right)}^{l-1+d \frac{l d}{2}}\|\nabla z\|_{L^{2}\left(|x| \geq \frac{\left.v_{k} \mid s\right)}{3}\right)}^{\frac{l d}{2}-d}+\left(\frac{1}{s}\right)^{l-1}\|z\|_{L^{2}\left(|x| \geq \frac{\left.\left|v_{k}\right| s\right)}{3}\right)}^{l-1}\right) \leq \frac{C}{s^{\mu+1}} . \tag{5.37}
\end{equation*}
$$

This along with (5.25) yields that, for $3 \leq l \leq 1+\frac{4}{d}$,

$$
\begin{equation*}
\int|x|\left|\nabla W_{k}\right|\left|W_{k}\right|^{1+\frac{4}{d}-1}|z|^{l} d x+\int\left|W_{k}\right|^{2+\frac{4}{d}-l}|z|^{l} d x \leq C\left(\frac{1}{s^{2 \mu+1}}+e^{-\delta s}\right) . \tag{5.38}
\end{equation*}
$$

Thus, plugging (5.35) and (5.38) into (5.34) we obtain

$$
\begin{equation*}
|\operatorname{Re}\langle\Lambda z, f(z+W)-f(W)-f(z)\rangle| \leq C\left(\frac{1}{s^{2 \mu+1}}+e^{-\delta s}\right) . \tag{5.39}
\end{equation*}
$$

For the last term on the R.H.S. of (5.28), again using the exponential decoupling between different traveling waves we get

$$
\begin{equation*}
|\operatorname{Re}\langle\Lambda z, H\rangle| \leq\|z\|_{L^{2}}\|\Lambda H\|_{L^{2}} \leq C e^{-\delta s} . \tag{5.40}
\end{equation*}
$$

Therefore, plugging (5.31), (5.39) and (5.40) into (5.28) we prove (5.27) and finish the proof.
Corollary 5.3. Assume that $u$ is a multi-soliton to equation (NLS) such that

$$
\begin{equation*}
\left\|u(s)-\sum_{k=1}^{K} W_{k}(s)\right\|_{\Sigma}=O\left(\frac{1}{s^{\mu}}\right), \text { for s large enough } \tag{5.41}
\end{equation*}
$$

where $\mu>0$ and $\left\{W_{k}\right\}$ are as in (5.25). Then, we have

$$
\begin{equation*}
\int_{s}^{+\infty}\left\|\nabla u(r)-\sum_{k=1}^{K} \nabla W_{k}(r)\right\|_{L^{2}}^{2} d r=O\left(\frac{1}{s^{2 \mu}}\right), \text { for s large enough } \tag{5.42}
\end{equation*}
$$

In particular, if $\mu=\frac{1}{2}+\zeta, \zeta \in(0,1)$, then we have

$$
\begin{equation*}
\int_{s}^{+\infty}\left\|\nabla u(r)-\sum_{k=1}^{K} \nabla W_{k}(r)\right\|_{L^{2}}^{2} d r=O\left(\frac{1}{s^{1+2 \zeta}}\right), \text { for s large enough, } \tag{5.43}
\end{equation*}
$$

and for $v:=P_{T} u$ and $S_{k}:=P_{T}\left(W_{k}\right), 0<T<\infty$,

$$
\begin{equation*}
\frac{1}{T-t} \int_{t}^{T}\left\|\nabla v(r)-\sum_{k=1}^{K} \nabla S_{k}(r)\right\|_{L^{2}}^{2} d r=O\left((T-t)^{2 \zeta}\right), \quad \text { for t close to } T . \tag{5.44}
\end{equation*}
$$

Remark 5.4. Compared with (5.41), (5.42) allows to gain half more convergence rate along some sequence $\left\{s_{n}\right\}$ to infinity, namely,

$$
\begin{equation*}
\left\|\nabla u\left(s_{n}\right)-\sum_{k=1}^{K} \nabla W_{k}\left(s_{n}\right)\right\|_{L^{2}}=O\left(\frac{1}{s_{n}^{\mu+\frac{1}{2}}}\right) . \tag{5.45}
\end{equation*}
$$

Proof of Corollary 5.3. Let $z:=u-\sum_{k=1}^{K} W_{k}, \widetilde{R}:=v-\sum_{k=1}^{K} S_{k}$ and $I$ be the virial functional as in (5.26). Note that, by (5.41),

$$
\begin{equation*}
|I(s)| \leq\|z\|_{\Sigma}^{2} \leq \frac{1}{s^{2 \mu}} \tag{5.46}
\end{equation*}
$$

Moreover, integrating (5.27) from $s$ to $\tilde{s}$ and using the boundary estimate (5.46) we get

$$
\begin{equation*}
\int_{s}^{\tilde{s}}\|\nabla z\|_{L^{2}}^{2} d r \leq C|I(\tilde{s})-I(s)|+C \int_{s}^{\tilde{s}} \frac{1}{r^{1+2 \mu}}+e^{-\delta r} d r \leq C\left(\frac{1}{s^{2 \mu}}+\frac{1}{\tilde{s}^{2 \mu}}\right) . \tag{5.47}
\end{equation*}
$$

Letting $\tilde{s}$ tend to infinity we obtain (5.42).
In particular, if $\mu=\frac{1}{2}+\zeta$, we have that for some $C>0$,

$$
\begin{equation*}
\int_{s}^{+\infty}\|\nabla z(r)\|_{L^{2}}^{2} d r \leq \frac{C}{s^{1+2 \zeta}} \tag{5.48}
\end{equation*}
$$

which yields that, for the rescaled time $t$ defined by $s=\frac{1}{T-t}$,

$$
\begin{equation*}
\int_{t}^{T} \frac{\left\|\nabla z\left(\frac{1}{T-r}\right)\right\|_{L^{2}}^{2}}{(T-r)^{2}} d r \leq C(T-t)^{1+2 \zeta} \tag{5.49}
\end{equation*}
$$

Since $\widetilde{R}=P_{T}(z)$, by (5.24),

$$
\begin{equation*}
\|\nabla \widetilde{R}(r)\|_{L^{2}}^{2} \leq C\left(\frac{\left\|\nabla z\left(\frac{1}{T-r}\right)\right\|_{L^{2}}^{2}}{(T-r)^{2}}+\left\|y z\left(\frac{1}{T-r}\right)\right\|_{L^{2}}^{2}\right) . \tag{5.50}
\end{equation*}
$$

Thus, by virtue of (5.41) with $\mu=\frac{1}{2}+\zeta$ and (5.49) we arrive at

$$
\begin{equation*}
\int_{t}^{T}\|\nabla \widetilde{R}\|_{L^{2}}^{2} d r \leq C \int_{t}^{T} \frac{\left\|\nabla z\left(\frac{1}{T-r}\right)\right\|_{L^{2}}^{2}}{(T-r)^{2}} d r+C \int_{t}^{T}\left\|z\left(\frac{1}{T-r}\right)\right\|_{\Sigma}^{2} d r \leq C(T-t)^{1+2 \zeta} \tag{5.51}
\end{equation*}
$$

which yields (5.44), thereby finishing the proof of Corollary 5.3.
We are now ready to prove Theorem 1.7.
Proof of Theorem 1.7. Let $z$ and $\widetilde{R}$ be as in Corollary 5.3. Using (5.22), (5.24) and (1.29) we have that for $t$ close to $T$,

$$
\|\widetilde{R}(t)\|_{L^{2}}+(T-t)\|\nabla \widetilde{R}(t)\|_{L^{2}} \leq\left\|z\left(\frac{1}{T-t}\right)\right\|_{H^{1}}+(T-t)\left\|y z\left(\frac{1}{T-t}\right)\right\|_{L^{2}} \leq C(T-t)^{\frac{1}{2}+\zeta},
$$

which verifies the condition (1.22). Moreover, the condition (1.23) is now verified by (5.44).
Thus, by virtue of Theorem 1.3, we obtain the uniqueness of multi-bubble blow-up solutions, which in turn yields the uniqueness of multi-solitons via the pseudo-conformal transformations. Therefore, the proof of Theorem 1.7 is complete.

At last, we end this section with the proof of Corollary 1.9.
Proof of Corollary 1.9. The uniqueness in (1.32) is an application of Theorem 1.5 to the case $K=1$. Moreover, (1.33) can be proved by the improved convergence rate in (5.7) and similar arguments as in the proof of Theorem 1.5. Therefore, the proof of Corollary 1.9 is complete.

## 6. Appendix

In this appendix we collect the tools used in the previous sections and the proof of modulation equations in Theorem 2.5.
Lemma 6.1. ([8, Theorem 1.3.7]) Let $d \geq 1$ and $2 \leq p<\infty$. Then, there exists $C>0$ such that

$$
\begin{equation*}
\|f\|_{L^{p}} \leq C\|f\|_{L^{2}}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\nabla f\|_{L^{2}}^{d\left(\frac{1}{2}-\frac{1}{p}\right)}, \quad \forall f \in H^{1} \tag{6.1}
\end{equation*}
$$

In particular, for any $1<p<\infty$,

$$
\begin{equation*}
\|f\|_{L^{p}} \leq C\|f\|_{H^{1}}, \quad \forall f \in L^{p} . \tag{6.2}
\end{equation*}
$$

Lemma 6.2. (Decoupling estimates [53, Lemma 3.1]) For every $1 \leq k \leq K$, set

$$
\begin{equation*}
G_{k}(t, x):=\lambda_{k}^{-\frac{d}{2}} g_{k}\left(t, \frac{x-\alpha_{k}}{\lambda_{k}}\right) e^{i \theta_{k}}, \text { with } g_{k}(t, y):=g(y) e^{i\left(\beta_{k}(t) \cdot y-\frac{1}{4} \gamma_{k}(t)|y|^{2}\right)}, \tag{6.3}
\end{equation*}
$$

where $g \in C_{b}^{2}\left(\mathbb{R}^{d}\right)$ decays exponentially fast at infinity

$$
\left|\partial^{\nu} g(y)\right| \leq C e^{-\delta|y|}, \quad|v| \leq 2
$$

with $C, \delta>0, \mathcal{P}_{k}:=\left(\lambda_{k}, \alpha_{k}, \beta_{k}, \gamma_{k}, \theta_{k}\right) \in C\left(\left[T^{*}, T\right) ; \mathbb{X}\right)$ satisfies that for $T^{*}$ close to $T$

$$
\begin{equation*}
\left|\left(\alpha_{k}(t)-x_{k}\right) \cdot \mathbf{v}_{1}\right| \leq \sigma,\left|x_{k}-\alpha_{k}(t)\right| \leq 1, \frac{1}{2} \leq \frac{\lambda_{k}(t)}{\left|\omega_{k}(T-t)\right|} \leq 2, \quad t \in\left[T^{*}, T\right) \tag{6.4}
\end{equation*}
$$

and $\left|\beta_{k}\right|+\left|\gamma_{k}\right| \leq 1$,

$$
\begin{equation*}
C\left(T-T^{*}\right)\left(1+\max _{1 \leq k \leq K}\left|x_{k}\right|\right) \leq 1, \tag{6.5}
\end{equation*}
$$

where $C$ is sufficiently large but independent of $T$. Then, there exist $C, \delta>0$ such that for any $1 \leq k \neq l \leq K, m \in \mathbb{N}$, for any multi-index $v$ with $|v| \leq 2$ and for any $T^{*}$ close to $T$,

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left|x-\alpha_{l}\right|^{n}\left|\partial^{\nu} G_{l}(t)\right|\left|x-\alpha_{k}\right|^{m}\left|G_{k}(t)\right| d x \leq C e^{-\frac{\delta}{T-t}}, \quad t \in\left[T^{*}, T\right) \tag{6.6}
\end{equation*}
$$

Moreover, for any $h \in L^{1}$ or $L^{2}, 1 \leq k \neq l \leq K, m, n \in \mathbb{N}$, multi-index $v$ with $|v| \leq 2$ and $T^{*}$ close to $T$,

$$
\begin{equation*}
\int_{\mathbb{R}^{d}}\left|x-\alpha_{l}\right|^{n}\left|\partial^{\nu} G_{l}(t) \| x-\alpha_{k}\right|^{m}|h| \Phi_{k} d x \leq C e^{-\frac{\delta}{T-l}} \min \left\{\|h\|_{L^{1}},\|h\|_{L^{2}}\right\}, \quad t \in\left[T^{*}, T\right) \tag{6.7}
\end{equation*}
$$

Coercivity of linearized operators. We recall the linearized operators from [50, 52, 53, 56]. Let $Q$ denote the ground state that solves the elliptic equation (1.1). It is known that $Q$ is smooth and decays at infinity exponentially fast, i.e., there exist $C, \delta>0$ such that for any multi-index $|v| \leq 3$,

$$
\begin{equation*}
\left|\partial_{x}^{v} Q(x)\right| \leq C e^{-\delta|x|}, \quad x \in \mathbb{R}^{d} . \tag{6.8}
\end{equation*}
$$

Define the linearized operator $L=\left(L_{+}, L_{-}\right)$around the ground state by

$$
\begin{equation*}
L_{+}:=-\Delta+I-\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}}, \quad L_{-}:=-\Delta+I-Q^{\frac{4}{d}} \tag{6.9}
\end{equation*}
$$

which has the generalized null space spanned by $\left\{Q, x Q,|x|^{2} Q, \nabla Q, \Lambda Q, \rho\right\}$. Here, the operator $\Lambda:=\frac{d}{2} I_{d}+x \cdot \nabla$, and $\rho$ is the unique radial solution to the equation

$$
\begin{equation*}
L_{+} \rho=-|x|^{2} Q \tag{6.10}
\end{equation*}
$$

One has the exponential decay of $\rho$ (see, e.g., [28, 42])

$$
\begin{equation*}
|\rho(x)|+|\nabla \rho(x)| \leq C e^{-\delta|x|} \tag{6.11}
\end{equation*}
$$

where $C, \delta>0$, and the algebraic identities (see, e.g., [56, (B.1), (B.10), (B.15)])

$$
\begin{align*}
& L_{+} \nabla Q=0, \quad L_{+} \Lambda Q=-2 Q, \quad L_{+} \rho=-|x|^{2} Q \\
& L_{-} Q=0, \quad L_{-} x Q=-2 \nabla Q, \quad L_{-}|x|^{2} Q=-4 \Lambda Q \tag{6.12}
\end{align*}
$$

For any complex valued $H^{1}$ function $f=f_{1}+i f_{2}$ in terms of the real and imaginary parts, set

$$
\begin{equation*}
(L f, f):=\int f_{1} L_{+} f_{1} d x+\int f_{2} L_{-} f_{2} d x \tag{6.13}
\end{equation*}
$$

The scalar product along the unstable directions in the null space is defined by

$$
\begin{equation*}
\left.\operatorname{Scal}(f)=\left\langle f_{1}, Q\right\rangle^{2}+\left\langle f_{1}, x Q\right\rangle^{2}+\left.\left\langle f_{1},\right| x\right|^{2} Q\right\rangle^{2}+\left\langle f_{2}, \nabla Q\right\rangle^{2}+\left\langle f_{2}, \Lambda Q\right\rangle^{2}+\left\langle f_{2}, \rho\right\rangle^{2} \tag{6.14}
\end{equation*}
$$

The localized coercivity of linearized operators is stated in Lemma 6.3 below.
Lemma 6.3. (Localized coercivity [53, Corollary 3.4]) Let $\phi$ be a positive smooth radial function on $\mathbb{R}^{d}$, such that $\phi(x)=1$ for $|x| \leq 1, \phi(x)=e^{-|x|}$ for $|x| \geq 2,0<\phi \leq 1$, and $\left|\frac{\nabla \phi}{\phi}\right| \leq C$ for some $C>0 . \operatorname{Set} \phi_{A}(x):=\phi\left(\frac{x}{A}\right), A>0$. Then, for A large enough we have

$$
\begin{equation*}
\int\left(|f|^{2}+|\nabla f|^{2}\right) \phi_{A}-\left(1+\frac{4}{d}\right) Q^{\frac{4}{d}} f_{1}^{2}-Q^{\frac{4}{d}} f_{2}^{2} d x \geq C_{1} \int\left(|f|^{2}+|\nabla f|^{2}\right) \phi_{A} d x-C_{2} \operatorname{Scal}(f) \tag{6.15}
\end{equation*}
$$

where $C_{1}, C_{2}>0$, and $f_{1}, f_{2}$ are the real and imaginary parts of $f$, respectively.

Below we prove Theorem 2.5. For this purpose, we set $f(z):=|z|^{\frac{4}{d}} z, d=1,2$, and

$$
\begin{align*}
& f^{\prime}(U) \cdot R:=\partial_{z} f(U) R+\partial_{\bar{z}} f(U) \bar{R}=\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}} R+\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R},  \tag{6.16}\\
& f^{\prime \prime}(U) \cdot R^{2}:=\frac{1}{d}\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}-2} \bar{U} R^{2}+\frac{2}{d}\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}-2} U|R|^{2}+\frac{1}{d}\left(\frac{2}{d}-1\right)|U|^{\frac{4}{d}-4} U^{3} \bar{R}^{2},  \tag{6.17}\\
& f^{\prime \prime}(U, R) \cdot R^{2}:=R^{2} \int_{0}^{1} t \int_{0}^{1} \partial_{z z} f(U+s t R) d s d t+2|R|^{2} \int_{0}^{1} t \int_{0}^{1} \partial_{z \bar{z}} f(U+s t R) d s d t \\
& \quad+\bar{R}^{2} \int_{0}^{1} t \int_{0}^{1} \partial_{\bar{z} z} f(U+s t R) d s d t . \tag{6.18}
\end{align*}
$$

Then, we have the expansion

$$
\begin{equation*}
f(U+R)=f(U)+f^{\prime}(U) \cdot R+f^{\prime \prime}(U, R) \cdot R^{2} . \tag{6.19}
\end{equation*}
$$

We also get from equation (NLS) and (2.1) that (see [53, (4.11)])

$$
\begin{align*}
& i \partial_{t} R+\sum_{k=1}^{K}\left(\Delta R_{k}+\left(1+\frac{2}{d}\right)\left|U_{k}\right|^{\frac{4}{d}} R_{k}+\frac{2}{d}\left|U_{k}\right|^{\frac{4}{d}-2} U_{k}^{2} \overline{R_{k}}+i \partial_{t} U_{k}+\Delta U_{k}+\left|U_{k}\right|^{\frac{4}{d}} U_{k}\right)+\sum_{k=1}^{K} f^{\prime \prime}\left(U_{k}\right) \cdot R^{2} \\
= & -H_{1}-H_{2}-H_{3} . \tag{6.20}
\end{align*}
$$

Here, $H_{1}, H_{2}$ and $H_{3}$ contain the interactions between different blow-up profiles

$$
\begin{align*}
& H_{1}:=\left(1+\frac{2}{d}\right)|U|^{\frac{4}{d}} R+\frac{2}{d}|U|^{\frac{4}{d}-2} U^{2} \bar{R}-\sum_{k=1}^{K}\left(\left(1+\frac{2}{d}\right)\left|U_{k}\right|^{\frac{4}{d}} R_{k}+\frac{2}{d}\left|U_{k}\right|^{\frac{4}{d}-2} U_{k}^{2} \overline{R_{k}}\right),  \tag{6.21}\\
& H_{2}:=|U|^{\frac{4}{d}} U-\sum_{k=1}^{K}\left|U_{k}\right|^{\frac{4}{d}} U_{k},  \tag{6.22}\\
& H_{3}:=f^{\prime \prime}(U, R) \cdot R^{2}-\sum_{k=1}^{K} f^{\prime \prime}\left(U_{k}\right) \cdot R^{2} . \tag{6.23}
\end{align*}
$$

Moreover, we get from equation (NLS) and (2.2) that (see [53, (4.14)])

$$
\begin{align*}
i \partial_{t} U_{k}+\Delta U_{k}+\left|U_{k}\right|^{\frac{4}{d}} U_{k}=\frac{e^{i \theta_{k}}}{\lambda_{k}^{2+\frac{d}{2}}}\{ & -\left(\lambda_{k}^{2} \dot{\theta}_{k}-1-\left|\beta_{k}\right|^{2}\right) Q_{k}-\left(\lambda_{k}^{2} \dot{\beta}_{k}+\gamma_{k} \beta_{k}\right) \cdot y Q_{k}+\frac{1}{4}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right)|y|^{2} Q_{k} \\
& \left.-i\left(\lambda_{k} \dot{\alpha}_{k}-2 \beta_{k}\right) \cdot \nabla Q_{k}-i\left(\lambda_{k} \dot{\lambda}_{k}+\gamma_{k}\right) \Lambda Q_{k}\right\}\left(t, \frac{x-\alpha_{k}}{\lambda_{k}}\right), \tag{6.24}
\end{align*}
$$

where $Q_{k}$ is given by (2.15), $1 \leq k \leq K$. The following identity also holds (see [53, (4.28)])

$$
\begin{equation*}
\Delta Q_{k}-Q_{k}+\left|Q_{k}\right|^{\frac{4}{2}} Q_{k}=\left|\beta_{k}-\frac{\gamma_{k}}{2}\right|^{2} Q_{k}-i \gamma_{k} \Lambda Q_{k}+2 i \beta_{k} \cdot \nabla Q_{k} . \tag{6.25}
\end{equation*}
$$

Proof of Theorem 2.5. The proof is similar to that of [53, Proposition 4.3], it mainly relies on the almost orthogonality in Lemma 2.4 and the decoupling Lemma 6.2.

Taking the inner product of (6.20) with $\Lambda_{k} U_{k}$ and then taking the real part we get

$$
\begin{aligned}
& \left.-\operatorname{Im}\left\langle\partial_{t} R, \Lambda_{k} U_{k}\right\rangle+\left.\operatorname{Re}\left\langle\Delta R_{k}+\left(1+\frac{2}{d}\right)\right| U_{k}\right|^{\frac{4}{d}} R_{k}+\frac{2}{d}\left|U_{k}\right|^{\frac{4}{d}-2} U_{k}^{2} \overline{R_{k}}, \Lambda_{k} U_{k}\right\rangle \\
& \left.+\left.\operatorname{Re}\left\langle i \partial_{t} U_{k}+\Delta U_{k}+\right| U_{k}\right|^{\frac{4}{d}} U_{k}, \Lambda_{k} U_{k}\right\rangle+\operatorname{Re}\left\langle f^{\prime \prime}\left(U_{k}\right) \cdot R^{2}, \Lambda_{k} U_{k}\right\rangle
\end{aligned}
$$

$$
\begin{align*}
= & -\operatorname{Re}\left\langle\sum_{j \neq k}\left(\Delta R_{j}+\left(1+\frac{2}{d}\right)\left|U_{j}\right|^{\frac{4}{d}} R_{j}+\frac{2}{d}\left|U_{j}\right|^{\frac{4}{4}-2} U_{j}^{2} \overline{R_{j}}\right)+H_{1}, \Lambda_{k} U_{k}\right\rangle \\
& -\operatorname{Re}\left\langle\sum_{j \neq k}\left(i \partial_{t} U_{j}+\Delta U_{j}+\left|U_{j}\right|^{\frac{4}{d}} U_{j}\right)+H_{2}, \Lambda_{k} U_{k}\right\rangle \\
& -\operatorname{Re}\left\langle\sum_{j \neq k}\left(f^{\prime \prime}\left(U_{j}\right) \cdot R^{2}+H_{3}, \Lambda_{k} U_{k}\right\rangle .\right. \tag{6.26}
\end{align*}
$$

For the R.H.S. of equation (6.26), we claim that

$$
\begin{equation*}
\lambda_{k}^{2} \times(\text { R.H.S. of }(6.26))=O\left(e^{-\frac{\delta}{T-t}}\left(1+\operatorname{Mod}+\|R\|_{L^{2}}+\|R\|_{L^{2}}^{2}\right)+D^{3}(t)\right) \tag{6.27}
\end{equation*}
$$

To this end, we have from [53, (4.19), (4.20)] that the first two terms on the R.H.S. of (6.26) are bounded by

$$
\begin{equation*}
C(T-t)^{-2} e^{-\frac{\delta}{T-t}}\left(1+\operatorname{Mod}+\|R\|_{L^{2}}\right) \tag{6.28}
\end{equation*}
$$

where $C, \delta>0$. Moreover, By Lemma 6.2,

$$
\begin{equation*}
\left\lvert\, \operatorname{Re}\left\langle\sum_{j \neq k}\left(f^{\prime \prime}\left(U_{j}\right) \cdot R^{2}, \Lambda_{k} U_{k}\right\rangle\right| \leq C(T-t)^{-2} e^{-\frac{\delta}{T-t}\|R\|_{L^{2}}^{2} .}\right. \tag{6.29}
\end{equation*}
$$

Using (6.18) we also have

$$
\begin{align*}
&\left|\operatorname{Re}\left\langle H_{3}, \Lambda U_{k}\right\rangle\right| \leq C \sum_{\tilde{z}, z^{*} \in\{z, \bar{z}\}}\left(\left.\langle | R\right|^{2} \int_{0}^{1} t \int_{0}^{1}\left|\partial_{\tilde{z} \tilde{z}^{*}} f(U)-\sum_{k=1}^{K} \partial_{\tilde{z} z^{*}} f\left(U_{k}\right)\right| d s d t,\left|\Lambda_{k} U_{k}\right|\right\rangle \\
&\left.\left.+\left.\langle | R\right|^{2} \int_{0}^{1} t \int_{0}^{1}\left|\partial_{\tilde{z} z^{*}} f(U+s t R)-\partial_{\tilde{z^{*}}} f(U)\right| d s d t,\left|\Lambda_{k} U_{k}\right|\right\rangle\right) . \tag{6.30}
\end{align*}
$$

Using Lemma 6.2 again we see that the first inner product above only contributes $e^{-\frac{\delta}{T-t}}\|R\|_{L^{2}}^{2}$, while the second one can be bounded by

$$
\begin{equation*}
\int|R|^{3}\left(|U|^{\frac{4}{d}-2}+|R|^{\frac{4}{d}-2}\right)\left|\Lambda_{k} U_{k}\right| d x \leq C(T-t)^{-2}\left(\left\|\varepsilon_{k}\right\|_{L^{3}}^{3}+\left\|\varepsilon_{k}\right\|_{L^{1+\frac{4}{d}}}^{1+\frac{4}{d}}\right) \leq C(T-t)^{-2} D^{3}, \tag{6.31}
\end{equation*}
$$

where the renormalized remainder $\varepsilon_{k}$ is given by (2.57) and we also used (6.2) in the last step.
Thus, combining estimates (6.28)-(6.31) and using (2.9) we obtain (6.27), as claimed.
Regarding the L.H.S. of (6.26), using the almost orthogonality (2.51), equation (6.24) and (6.25) we have that (see [53, (4.26), (4.27), (4.31)])

$$
\begin{align*}
\operatorname{Im}\left\langle\partial_{t} R, \Lambda_{k} U_{k}\right\rangle= & \operatorname{Im}\left\langle\Lambda_{k} R_{k}, \partial_{t} U_{k}\right\rangle+O\left(e^{-\frac{\delta}{T-t}}(1+M o d)\|R\|_{L^{2}}\right),  \tag{6.32}\\
\lambda_{k}^{2} \operatorname{Im}\left\langle\Lambda_{k} R_{k}, \partial_{t} U_{k}\right\rangle= & \operatorname{Re}\left\langle\varepsilon_{k}, \Lambda Q_{k}\right\rangle+\gamma_{k} \operatorname{Im}\left\langle\Lambda \varepsilon_{k}, \Lambda Q_{k}\right\rangle-2 \beta_{k} \operatorname{Im}\left\langle\Lambda \varepsilon_{k}, \nabla Q_{k}\right\rangle \\
& +O\left(\left(M o d+P^{2}\right)\|R\|_{L^{2}}\right), \tag{6.33}
\end{align*}
$$

and

$$
\begin{equation*}
\left.\left.\lambda_{k}^{2} \operatorname{Re}\left\langle i \partial_{t} U_{k}+\Delta U_{k}+\right| U_{k}\right|^{\frac{4}{d}} U_{k}, \Lambda_{k} U_{k}\right\rangle=-\frac{1}{4}\|y Q\|_{2}^{2}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right)+O\left(\left|\beta_{k}\right| M o d\right) \tag{6.34}
\end{equation*}
$$

Moreover, by (6.17) and the change of variables,

$$
\lambda_{k}^{2} \operatorname{Re}\left\langle f^{\prime \prime}\left(U_{k}\right) \cdot R^{2}, \Lambda_{k} U_{k}\right\rangle=\operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{d}}\left|\varepsilon_{k}\right|^{2}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{d}-2} Q_{k}^{2}{\overline{\varepsilon_{k}}}^{2} d y
$$

$$
\begin{equation*}
+\operatorname{Re} \int\left(y \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y+O\left(e^{\left.-\frac{\delta}{T-1}\|R\|_{L^{2}}^{2}\right) .}\right. \tag{6.35}
\end{equation*}
$$

Thus, combining (6.32)-(6.35) altogether and rearranging the terms according to the orders of renormalized variable $\varepsilon_{k}$ we get

$$
\begin{align*}
& \lambda_{k}^{2} \times(\text { L.H.S. of }(6.26)) \\
= & -\frac{1}{4}\|y Q\|_{2}^{2}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right) \\
& \left.+\left.\operatorname{Re}\left\langle\Delta \varepsilon_{k}-\varepsilon_{k}+\left(1+\frac{2}{d}\right)\right| Q_{k}\right|^{\frac{4}{d}} \varepsilon_{k}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{d}-2} Q_{k}^{2} \overline{\varepsilon_{k}}, \Lambda Q_{k}\right\rangle  \tag{6.36}\\
& -\gamma_{k} \operatorname{Im}\left\langle\Lambda \varepsilon_{k}, \Lambda Q_{k}\right\rangle+2 \beta_{k} \operatorname{Im}\left\langle\Lambda \varepsilon_{k}, \nabla Q_{k}\right\rangle  \tag{6.37}\\
& +\operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{d}}\left|\varepsilon_{k}\right|^{2}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{d}-2} Q_{k}^{2} \bar{\varepsilon}_{k}^{2} d y+\operatorname{Re} \int\left(y \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y  \tag{6.38}\\
& +O\left(\left(M o d+P^{2}\right)\|R\|_{L^{2}}+P M o d+e^{-\frac{\delta}{T-t}}(1+M o d)\right) . \tag{6.39}
\end{align*}
$$

For the linear part on the R.H.S. above, using the linearized operators $L_{ \pm}$in (6.9) we have (see [53, (4.37)])

$$
\begin{equation*}
(6.36)+(6.37)=M_{k}-\int|R|^{2} \Phi_{k} d x+O\left(\left(P^{2}+e^{-\frac{\delta}{T-t}}\right)\|R\|_{L^{2}}\right) \tag{6.40}
\end{equation*}
$$

Hence, we obtain that

$$
\begin{align*}
\lambda_{k}^{2} \times(\text { L.H.S. of }(6.26))= & -\frac{1}{4}\|y Q\|_{L^{2}}^{2}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right)+M_{k}-\int|R|^{2} \Phi_{k} d x \\
& +\operatorname{Re} \int\left(1+\frac{2}{d}\right)\left|Q_{k}\right|^{\frac{4}{\mid}}\left|\varepsilon_{k}\right|^{2}+\frac{2}{d}\left|Q_{k}\right|^{\frac{4}{d}-2} Q_{k}^{2} \bar{\varepsilon}_{k}^{2} d y \\
& +\operatorname{Re} \int\left(y \cdot \nabla \overline{Q_{k}}\right)\left(f^{\prime \prime}\left(Q_{k}\right) \cdot \varepsilon_{k}^{2}\right) d y \\
& +O\left(\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right) M o d+P^{2}\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right) \tag{6.41}
\end{align*}
$$

which along with (2.8) and (6.27) yields (2.56).
Moreover, we can also bound the quadratic terms of $\varepsilon_{k}$ in (6.38) by

$$
\begin{equation*}
(6.38) \leq C\left\|\varepsilon_{k}\right\|_{L^{2}}^{2} \leq C\|R\|_{L^{2}}^{2} \tag{6.42}
\end{equation*}
$$

This along with (6.41) yields that

$$
\begin{align*}
\lambda_{k}^{2} \times(\text { L.H.S. of }(6.26))= & -\frac{1}{4}\|y Q\|_{2}^{2}\left(\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right)+M_{k} \\
& +O\left(\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right) M o d+P^{2}\|R\|_{L^{2}}+\|R\|_{L^{2}}^{2}+e^{-\frac{\delta}{T-t}}\right) \tag{6.43}
\end{align*}
$$

Thus, combining (6.27) and (6.43) together we obtain that for every $1 \leq k \leq K$,

$$
\begin{equation*}
\left|\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right| \leq C\left(\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right) M o d+\left|M_{k}\right|+P^{2}\|R\|_{L^{2}}+\|R\|_{L^{2}}^{2}+D^{3}+e^{-\frac{\delta}{T-t}}\right) \tag{6.44}
\end{equation*}
$$

which along with (2.54) and (2.61) yields that

$$
\begin{equation*}
\sum_{k=1}^{K}\left|\lambda_{k}^{2} \dot{\gamma}_{k}+\gamma_{k}^{2}\right| \leq C\left(\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right) M o d+\sum_{k=1}^{K}\left|M_{k}\right|+P^{2} D+D^{2}+e^{-\frac{\delta}{T-t}}\right) \tag{6.45}
\end{equation*}
$$

Similar arguments apply also to the remaining four modulation equations. Actually, taking the inner products of equation (6.20) with $i\left(x-\alpha_{k}\right) U_{k}, i\left|x-\alpha_{k}\right|^{2} U_{k}, \nabla U_{k}, \varrho_{k}$, respectively, then taking the real parts and using analogous arguments as above, we obtain

$$
\begin{equation*}
M o d \leq C\left(\left(P+\|R\|_{L^{2}}+e^{-\frac{\delta}{T-t}}\right) M o d+\sum_{k=1}^{K}\left|M_{k}\right|+P^{2} D+D^{2}+e^{-\frac{\delta}{T-t}}\right) . \tag{6.46}
\end{equation*}
$$

Therefore, using (2.6) and (2.7) we may take $t$ close to $T$ such that $P(t)+\|R(t)\|_{L^{2}}+e^{-\frac{\delta}{T-t}}$ is sufficiently small to obtain (2.53). The proof of Theorem 2.5 is complete.

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