ON UNIQUENESS QUESTIONS IN THE THEORY OF VISCOUS FLOW

BY

JOHN G. HEYWOOD

Oregon State University, Cornvallis, Oregon, USA (1)

1. Introduction

In this paper it is shown, by means of counter examples, that for some spatial domains the motion of a viscous incompressible fluid is not uniquely determined by the traditional initial and boundary conditions (i.e., by the applied external forces and by the values of the fluid velocity at an initial instant of time, at the boundary of the spatial domain, and at spatial infinity). In a positive direction, we prove uniqueness for the initial boundary value problem in some classes of spatial domains, and uniqueness for this problem in other classes of domains under appropriate auxiliary conditions. Regarding the uniqueness questions to be considered here, it will be shown that the situation is much the same for the problems of steady flow as for those of nonstationary flow, and much the same for the linear Stokes equations as for the nonlinear Navier-Stokes equations.

In some respects our results are at variance with those given in previous works on the subject, and in some other respects our results may appear at first to be not new. Among the most important papers on theoretical hydrodynamics are some investigations of the existence and uniqueness theory for the boundary value problems of viscous flow within various classes of generalized solutions. Rather remarkably, the uniqueness proofs for these previously studied generalized solution classes do not make use of any properties of the spatial domain, and so the uniqueness theorems for them have in many cases been given for an arbitrary spatial domain. This is the case in the celebrated works [21] of Ladyzhenskaya, [31] of Prodi, and [33] of Serrin, and also in the present author's papers [14, 16, 17]. The uniqueness theorems of these papers are misleading, however, because the classes of generalized solutions to which they apply have been defined in such a way as to exclude from membership, in some domains, some classical and physically important solutions.

---

(1) Research partially supported by the National Research Council of Canada under Operating Grant number A-7179, while the author was at the University of British Columbia.

Until now this has gone without notice. It has been widely believed that for smooth solutions which satisfy appropriate integrability conditions the generalized formulations of the initial boundary value problem studied in these papers are equivalent to the corresponding classical formulation. In fact, this has been stated by Ladyzhenskaya [21, p. 144], Prodi [31, p. 175], and Serrin [33, p. 73]. Our counter examples show that these statements are not correct for some domains, and that the uniqueness theorems for these generalized solutions are not valid for classical solutions. Although a class of generalized solutions which does include all classical solutions that satisfy appropriate integrability conditions has been studied by the present author in [15], the uniqueness proof given in that paper is not complete and is not valid for some domains.

The oversight in all of these papers lies in the identification of two types of function spaces which are not the same for some spatial domains. One of the function spaces consists of the completion in an appropriate norm (different for nonstationary problems than for stationary problems) of the set of all smooth solenoidal vector valued functions with compact support. The other function space consists of the solenoidal functions which belong to the completion, in the same norm, of the set of all smooth vector valued functions with compact support. The only published recognition known to this author that the identity of these function spaces needs to be proved appears in Lions' book [24, p. 67, p. 100], but the proof offered there seems to be incomplete and does not extend, as claimed, to unbounded domains. Until now it appears to have been regarded as merely a technical matter to prove that these function spaces are the same, and it has been generally overlooked that the result might, and does, fail in some domains. It turns out, in fact, that the identity of these function spaces for a given spatial domain is equivalent to uniqueness in that domain for the linear problems of viscous flow. In this respect there is a circularity in the uniqueness proofs contained in the papers mentioned above. The methods of these papers, nevertheless, remain of permanent and undiminished importance, and they are adopted and further developed in our present work.

In this paper we prove the identity of the two types of function spaces for some classes of domains, specifically for bounded domains, exterior domains, half-spaces, and the whole space. We thereby prove uniqueness for both the linear and nonlinear initial boundary value problems (without any condition on the pressure) in these classes of domains; we believe ours is the first valid uniqueness theorem of this type to be given, even in the context of classical solutions and even for a bounded domain. We also prove uniqueness for the linear problem of steady Stokes flow in these same classes of domains by a similar method based upon proving the identity of appropriate function spaces. In the case of a two-dimensional exterior domain our uniqueness theorem is just the Stokes paradox for solutions with finite Dirichlet integrals. Although uniqueness has already been proved for the steady Stokes equations by potential theoretic methods in the case of a bounded domain by Oseen [29] and by Odqvist [28], and in the case of an exterior domain by Finn and Noll [6] and by Chang and Finn [3], our proof has the advantage of being more compatible with the functional analysis approach to existence theorems. We shall consider some specific domains for which the two types of function spaces differ, and show that in these domains the boundary value problems of viscous flow possess multiple solutions; this is true for both the stationary and nonstationary problems, and for both the linear and nonlinear equations. It seems not to have been previously noticed that uniqueness fails in these domains. Under appropriate auxiliary conditions we prove uniqueness in these domains for all of the boundary value problems of viscous flow except the nonlinear stationary problem. Although we do not attempt to prove uniqueness theorems for the nonlinear stationary problem, uniqueness theorems for this problem have been given by Finn for a bounded domain [7] and for an exterior domain [8], and recently Babenko [1] has shown that the class of "physically reasonable" solutions to which Finn's uniqueness theorem in [8] applies includes all solutions with finite Dirichlet integrals.

The simplest context in which to consider an example of the kind of nonuniqueness that interests us is the boundary value problem for the steady Stokes equations. Thus consider the question of whether there may exist, for some domains, nontrivial solutions $\mathbf{u}(x)$, $p(x)$ of

$$\Delta \mathbf{u} = \nabla p \quad \text{in} \quad \Omega \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \tag{3}$$

$$\mathbf{u}(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{4}$$

Here Ω, the domain, represents a region filled with fluid and is required to be an open set of R^n, $n \geqslant 2$; $\partial\Omega$ is the boundary of Ω; $x = (x_1, ..., x_n)$ is the generic point of R^n; $\mathbf{u}(x)$ is a R^n-valued function which represents the fluid velocity; and $p(x)$ is a scalar valued function which represents the pressure. A simple example of a three-dimensional domain for which nontrivial solutions exist is $\Omega = \{x\colon x_1 \neq 0, \text{ or } x_1 = 0 \text{ and } x_2^2 + x_3^2 < 1\}$. We prove that for this domain there exists exactly one solution of problem (1)–(4) which possesses a finite Dirichlet integral and which satisfies the auxiliary condition

$$\int_S \mathbf{u} \cdot \mathbf{n}\, ds = F, \tag{5}$$

where S is the surface $S = \{x\colon x_1 = 0 \text{ and } x_2^2 + x_3^2 < 1\}$, $\mathbf{n} = (1, 0, 0)$ is the unit normal to S, and

F is any prescribed number. The solution $\mathbf{u}$ represents a steady flow with net flux F through an aperture S in a rigid wall occupying the x_2, x_3-plane. It should be remarked that this example can be modified so that the wall has thickness and so that the boundary of Ω is smooth. It will be shown that the pressure tends to a definite limit at infinity in each half-space, $x_1<0$ and $x_1>0$. From a physical point of view it may be more natural to prescribe the pressure drop between the two half-spaces than to prescribe the flux through S. Therefore, we also show that exactly one solution of (1)–(4) having a finite Dirichlet integral is determined by the auxiliary pressure condition

$$\begin{aligned} p(x)&\to p_1 \quad \text{as} \quad |x|\to\infty, \quad x_1<0 \\ p(x)&\to p_2 \quad \text{as} \quad |x|\to\infty, \quad x_1>0. \end{aligned} \tag{6}$$

We will show that the total flux F through the aperture S is proportional to the pressure drop p_1-p_2, and also that for a fixed pressure drop, and for various sized but similarly shaped apertures, the total flux through an aperture is proportional to the cube of its diameter. These results are of particular significance because there is no pressure drop predictable in the theory of potential flow through an aperture; the D'Alembert paradox implies a symmetry of the pressure, upstream and downsteam, for potential flow through an aperture just as for potential flow past an obstacle; see Shinbrot [34, p. 78]. We believe that ours is the first mathematical investigation of flow through an aperture to be based on equations for a viscous fluid; the details are given in section 6.

The same methods used to study flow through an aperture can be applied to somewhat more complicated domains. For instance, if there are two apertures in a wall occupying the x_2, x_3-plane, it will be shown (in section 6) that the fluxes through each cannot be prescribed independently; a solution of (1)–(4) is uniquely determined by the combined net flux through the two apertures from one half-space to the other, or alternatively by the pressure condition (6). It is an easy matter to prove the existence of multiple solutions for quite a large variety of domains; proving uniqueness under appropriate auxiliary conditions is generally more difficult.

The function spaces which enter into the functional analysis approach to existence and uniqueness questions for problems of steady flow are

$$J_0(\Omega) = \text{Completion of } D(\Omega) \text{ in norm } \|\nabla\boldsymbol{\phi}\|,$$

and

$$J_0^*(\Omega) = \{\boldsymbol{\phi}\colon \boldsymbol{\phi}\in W_0(\Omega) \quad \text{and} \quad \nabla\cdot\boldsymbol{\phi}=0\},$$

where

$$\|\nabla\boldsymbol{\phi}\|^2 = \int_\Omega \sum_{i,j=1}^{n} \left(\frac{\partial\phi_i}{\partial x_j}\right)^2 dx$$

is the Dirichlet integral,

$$D(\Omega) = \{\phi: \phi \in C_0^\infty(\Omega) \quad \text{and} \quad \nabla\cdot\phi = 0\},$$

$$W_0(\Omega) = \text{Completion of } C_0^\infty(\Omega) \text{ in norm } \|\nabla\phi\|,$$

and $C_0^\infty(\Omega)$ is the set of all smooth R^n-valued functions ϕ with compact support in Ω. In section 2 we consider generalized solutions for the various problems of viscous flow, and it is shown there that, for an arbitrary open set Ω, every classical solution of (1)–(4) which possesses a finite Dirichlet integral belongs to $J_0^*(\Omega)$, and also that $\mathbf{u}\equiv 0$ is the only solution of (1) belonging to $J_0^*(\Omega)$ if and only if $J_0(\Omega) = J_0^*(\Omega)$. Clearly $J_0(\Omega) \subset J_0^*(\Omega)$ for every domain Ω. If $J_0(\Omega) \neq J_0^*(\Omega)$, then there is a unique generalized solution of (1)–(4) in each coset of the quotient space $J_0^*(\Omega)/J_0(\Omega)$. The existence-uniqueness problem thus becomes a matter of identifying these cosets in a physically meaningful way through auxiliary conditions; the flux condition (5) for the domain $\Omega = \{x: x_1 \neq 0 \text{ or } x_2^2 + x_3^2 < 1\}$ is an example. In order to prove $J_0(\Omega) = J_0^*(\Omega)$ in the case of a bounded domain, we use a method of "pulling in" from the boundary the support of a given solenoidal vector field $\mathbf{u} \in J_0^*(\Omega)$ so as to obtain approximating solenoidal vector fields with compact support in Ω; these approximating vector fields belong to $J_0(\Omega)$. This method, which is given in section 3, is successful for a large class of bounded domains, however at present there is no method available for treating an arbitrary bounded open set. In order to prove uniqueness for an exterior domain we use a combination of methods. For a region exterior to a sphere we resort to a direct study of problem (1)–(4); we use an "interior type" L^2 estimate for ∇p in a neighborhood of infinity to show that the coefficients of a solution's expansion in spherical harmonics must all vanish. Then, to treat a more general exterior domain we return to a consideration of the function spaces; this enables us to combine the result for a bounded domain with that for the exterior of a sphere. These arguments for an exterior domain are given in section 4. To prove that the two function spaces are the same in the case of a half-space we again study problem (1)–(4) directly; we obtain an L^2 estimate for ∇p in an "interior half-space" and use it to prove that a solution's Fourier transform must vanish. This is done in section 5. The basic Fourier-transform argument was kindly pointed out to the author by Marvin Shinbrot.

A proof that $J_0(\Omega) = J_0^*(\Omega)$ can also be based on potential-theoretic methods in the case of a bounded domain or of an exterior domain. Robert Finn has communicated to the author an argument which shows that a generalized solution of (1) which belongs to $J_0^*(\Omega)$ admits an integral representation in terms of its boundary values and of Green's tensor for the Stokes equations in such domains, which implies the result. To justify this representation, one uses estimates of Odqvist [28] for derivatives of the Green's tensor up to the boundary, and, in the case of an exterior domain, results of Chang and Finn [3] concerning a solution's behavior at infinity. The method of proof given in the present paper, however, has the advantage of making the functional analysis approach to uniqueness questions independent of the potential-theoretic approach, and is also easily modified to investigate the function spaces which arise in the study of nonstationary problems.

The initial boundary value problem for the Navier–Stokes equations is that of finding a solution pair $\mathbf{u}(x, t)$, $p(x, t)$ of

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \Delta\mathbf{u} + \mathbf{f}, \quad (x, t)\in\Omega\times(0, T) \tag{7}$$

$$\nabla\cdot\mathbf{u} = 0, \quad (x, t)\in\Omega\times(0, T) \tag{8}$$

$$\mathbf{u}(x, 0) = \mathbf{a}(x), \quad x\in\Omega \tag{9}$$

$$\mathbf{u}(x, t) = 0, \quad (x, t)\in\partial\Omega\times(0, T) \tag{10}$$

$$\mathbf{u}(x, t)\to 0, \quad \text{as} \quad |x|\to\infty \tag{11}$$

in a space-time cylinder $\Omega\times(0, T)$. Here $\mathbf{f}(x, t)$ is the applied external force density and $\mathbf{a}(x)$ is the prescribed initial velocity; for simplicity we have assumed that the prescribed boundary values and the prescribed limit at infinity are zero, but in [15] we have posed the problem more generally. In section 2 a class of generalized solutions for the problem (7)–(11) is defined which includes all classical solutions for which the energy integral $\int_\Omega \mathbf{u}^2(x, t)\,dx$, the Dirichlet integral $\int_\Omega (\nabla\mathbf{u}(x, t))^2 dx$, and the integral $\int_\Omega \mathbf{u}_t^2(x, t)\,dx$ of the time derivative are square-summable functions of t in $(0, T)$. The principal function spaces which enter into this definition are

$$J_1(\Omega) = \text{Completion of } D(\Omega) \text{ in norm } \|\boldsymbol{\phi}\|_1,$$

and

$$J_1^*(\Omega) = \{\boldsymbol{\phi}\colon \boldsymbol{\phi}\in \mathring{W}_2^1(\Omega) \quad \text{and} \quad \nabla\cdot\boldsymbol{\phi} = 0\},$$

where $\|\boldsymbol{\phi}\|_1^2 = \|\boldsymbol{\phi}\|_{L^2(\Omega)}^2 + \|\nabla\boldsymbol{\phi}\|^2$, and where

$$\mathring{W}_2^1(\Omega) = \text{Completion of } C_0^\infty(\Omega) \text{ in norm } \|\boldsymbol{\phi}\|_1.$$

For a given domain Ω, the question of whether solutions of (7)–(11) are unique is reduced in section 2 to the question of whether $J_1(\Omega) = J_1^*(\Omega)$, and this question is reduced in turn to that of whether there exist nontrivial solutions, with finite norm $\|\mathbf{u}\|_1$, of the time-independent boundary value problem

$$\Delta\mathbf{u} - \mathbf{u} = \nabla p \quad \text{in} \quad \Omega \tag{12}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{13}$$

$$\mathbf{u}=0 \quad \text{on} \quad \partial\Omega \tag{14}$$

$$\mathbf{u}(x)\to 0 \quad \text{as} \quad |x|\to\infty. \tag{15}$$

The methods we use to study problem (12)–(15) are the same as those we use to study problem (1)–(4); so a unified approach to uniqueness questions for both stationary and nonstationary problems is achieved. We prove $J_1(\Omega)=J_1^*(\Omega)$ for bounded domains, exterior domains (including the whole space), and half-spaces in sections 3, 4, and 5 respectively, thereby establishing uniqueness for problem (7)–(11) in these classes of domains. For some other domains, such as the three-dimensional domain $\Omega=\{x\colon x_1\neq 0 \text{ or } x_2^2+x_3^2<1\}$, we show that $J_1(\Omega)\neq J_1^*(\Omega)$; for these domains the problem of properly posing the initial boundary value problem becomes one of characterizing, through physically meaningful auxiliary conditions, the cosets of $J_1^*(\Omega)/J_1(\Omega)$. For the domain $\Omega=\{x\colon x_1\neq 0 \text{ or } x_3^2+x_3^2<1\}$ we prove the uniqueness and (local) existence of a solution of (7)–(11) satisfying the auxiliary condition

$$\int_S \mathbf{u}(x,t)\cdot\mathbf{n}\,ds=F(t), \tag{16}$$

where $F(t)$ is any smooth function of t which satisfies the compatability condition $\int_S \mathbf{a}(x)\cdot\mathbf{n}\,ds=F(0)$.

These results put into new perspective the uniqueness theorems previously given for classical solutions of the initial boundary value problem by Foà [9], Dolidze [5], Graffi [13], and Ito [19]; see also Serrin [32, p. 252]. All of these theorems contain hypotheses concerning the behavior of the pressure; hypotheses that the pressure should possess a certain degree of regularity up to the boundary and (in the case of unbounded domains) should tend in a prescribed manner to a limit at infinity. Our theorems show that an appropriate condition for the pressure, or some alternative auxiliary condition, is indeed necessary for uniqueness in some domains, but that the special geometry of certain classes of domains, particularly of bounded domains and of exterior domains, makes such conditions unnecessary and therefore inappropriate: the behavior of the pressure is already determined by the initial and boundary values, and the limit at infinity, prescribed for the velocity. The theorems previous to ours take no account or advantage of the spatial geometry. It must be remarked that the Graffi uniqueness theorem is exceptional in that it is proved for an exterior domain without assuming that the velocity tends to a limit at infinity, so that the hypothesis made concerning the pressure is necessary for the result. This very interesting theorem indicates the strength of a condition for the pressure. Although it is only stated for an exterior domain, the Graffi theorem is actually valid for an arbitrary (smooth) domain.

2. Generalized solutions

It turns out to be most convenient to give our uniqueness theorems, even for classical solutions, within the framework of a class of generalized solutions. We will consider first the nonlinear initial boundary value problem. In order that the principal point of difficulty to which this paper is devoted will not be obscured by technicalities related to the introduction of generalized solutions, let us examine briefly the formal uniqueness argument which we seek to justify. Suppose that $\mathbf{u}$ and $\bar{\mathbf{u}}$ are two solutions of problem (7)–(11) with corresponding pressures p and $\bar{p}$. Letting $\mathbf{w}=\mathbf{u}-\bar{\mathbf{u}}$ and $q=\bar{p}-p$, and operating formally with equations (7)–(11), one obtains

$$\tfrac{1}{2}\|\mathbf{w}(t)\|^2+\int_0^t\|\nabla\mathbf{w}\|^2d\tau=\int_0^t\int_\Omega \mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx\,d\tau+\int_0^t\int_\Omega \mathbf{w}\cdot\nabla q\,dx\,d\tau \tag{17}$$

where $\|\mathbf{w}(t)\|^2=\int_\Omega \mathbf{w}^2(x,t)\,dx$ and $\mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}=\sum_{i,j=1}^n w_i(\partial w_j/\partial x_i)u_j$. Among the operations leading to (17) are several integrations by parts which may be considered to be formally justified by the boundary conditions (9)–(11). Our concern is with the second integral on the right, the term which involves the pressure. It will be shown that this term, in effect, vanishes for some classes of spatial domains without making any hypotheses beyond (7)–(11), except that the fluid velocity and its first derivatives should be square-summable, while for some other classes of spatial domains further hypotheses are necessary and natural. Assuming that the pressure term in (17) does vanish and that $\mathbf{u}$ is a classical solution, the uniqueness argument can be completed as follows. Without any real loss of generality one may assume $|\mathbf{u}(x,t)|$ is bounded by a constant C for all $(x,t)\in\Omega\times(0,T)$, so that for all $t\in(0,T)$ there holds

$$\left|\int_\Omega \mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx\right|\leqslant C\|\nabla\mathbf{w}\|\cdot\|\mathbf{w}\|\leqslant\|\nabla\mathbf{w}\|^2+\tfrac{1}{4}C^2\|\mathbf{w}\|^2. \tag{18}$$

By combining (18) with (17) one obtains

$$\|\mathbf{w}(t)\|^2\leqslant\tfrac{1}{2}C^2\int_0^t\|\mathbf{w}\|^2\,d\tau, \tag{19}$$

which implies that $\mathbf{w}$ vanishes, as may be seen by setting $F(t)=\int_0^t\|\mathbf{w}\|^2d\tau$ and observing that (19) becomes $F'(t)\leqslant\frac{1}{2}C^2F(t)$.

It is evident that this argument can only be applied to solutions $\mathbf{u}$ such that $\mathbf{u}$ and $\nabla\mathbf{u}$ are square-summable over $\Omega\times(0,T)$; we will confine our attention to such solutions and assume in addition that $\mathbf{u}_t$ is square-summable over $\Omega\times(0,T)$. The question of whether every classical solution of (7)–(11) must satisfy these integrability conditions seems worthy of consideration, but is beyond the scope of this paper; see, however, Ma [25]. We study the following class of generalized solutions which includes all classical solutions that meet these integrability conditions and also generalized solutions obtained by the method of Kiselev and Ladyzhenskaya [20].

Definition. We call a function $\mathbf{u}(x, t)$ a generalized solution of (7)–(11) in $(0, T]$ if and only if:

$$\mathbf{u} \in L^2(0, T; J_1^*(\Omega)) \quad \text{and} \quad \mathbf{u}_t \in L^2(0, T; L^2(\Omega)), \tag{20}$$

either $|\mathbf{u}(x, t)|$ is uniformly bounded in $\Omega \times (0, T)$, or $\Omega \subset R^n$ and for some $q > n$ the integral $\int_\Omega |\mathbf{u}(x, t)|^q dx$ is uniformly bounded for t in $(0, T)$, (21)

$$\mathbf{u}(x, t) \to \mathbf{a}(x) \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad t \to 0, \tag{22}$$

$$\int_0^T \int_\Omega \{\mathbf{u}_t \cdot \boldsymbol{\phi} + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \phi + \nabla \mathbf{u} : \nabla \boldsymbol{\phi} - \mathbf{f} \cdot \boldsymbol{\phi}\} \, dx \, dt = 0 \quad \text{for all} \quad \boldsymbol{\phi} \in L^2(0, T; J_1(\Omega)). \tag{23}$$

Further, if $\mathbf{u}$ is a generalized solution in $(0, T']$ for all $0 < T' < T$, we call $\mathbf{u}$ a generalized solution in $(0, T)$; here $T = \infty$ is allowed.

Here we have denoted by $L^2(0, T; V)$, with V taken to be either $L^2(\Omega)$ or $J_1^*(\Omega)$ or $J_1(\Omega)$, the set of all V-valued measurable functions $\mathbf{u}(\cdot, t)$ such that $\int_0^t \|\mathbf{u}(t)\|_V^2 dt$ is finite. The proof that every classical solution is a generalized solution is based primarily on the following two lemmas. Lemma 1, well known for smoothly bounded domains, can be proved for arbitrary open sets by potential theoretic methods, Deny and Lions [4, p. 359]; we give a direct and elementary proof, valid for an arbitrary open set, at the end of this section. Lemma 2 is well known; see [21, p. 27] and [16].

LEMMA 1. *Let Ω be an arbitrary open set of R^n. Suppose that $u \in C(\overline{\Omega})$, that $u = 0$ on $\partial\Omega$, that u has generalized first derivatives, and that the integrals $\int_\Omega u^2 dx$ and $\int_\Omega (\nabla u)^2 dx$ are finite. Then $u \in \overset{\circ}{W}{}_2^1(\Omega)$.*

LEMMA 2. *Let Ω be an arbitrary open set of R^n. If $\mathbf{u} \in L^2_{\mathrm{loc}}(\Omega)$, then $\int_\Omega \mathbf{u} \cdot \boldsymbol{\phi} \, dx = 0$ for all $\boldsymbol{\phi} \in D(\Omega)$ if and only if $\mathbf{u} = \nabla p$ for some $p \in L^2_{\mathrm{loc}}(\Omega)$ with $\nabla p \in L^2_{\mathrm{loc}}(\Omega)$.*

A function $\mathbf{u}(x, t)$ is called a classical solution of (7)–(11) if $\mathbf{u}$ is continuous in $\overline{\Omega} \times [0, T)$, if its derivatives $\mathbf{u}_{x_i}$, $\mathbf{u}_{x_i x_j}$, and $\mathbf{u}_t$ are continuous in $\Omega \times (0, T)$, and if the conditions (7)–(11) are satisfied for some $p(x, t) \in C^1(\Omega \times (0, T))$ in the senses appropriate to continuous functions. Now if in addition $\mathbf{u}$, $\mathbf{u}_{x_i}$, and $\mathbf{u}_t$ are square-summable over $\Omega \times (0, T)$, it is a routine matter to check that condition (20) follows from Lemma 1 and the definition of $J_1^*(\Omega)$. Condition (21) holds at least on every subinterval $(0, T']$ of $(0, T)$ in virtue of conditions (10) and (11). Since $\mathbf{u}, \mathbf{u}_t \in L^2(0, T; L^2(\Omega))$, certainly $\mathbf{u}(x, t)$ converges in $L^2(\Omega)$ as $t \to 0$; by (9) the limit must be $\mathbf{a}(x)$, and thus (22) holds. It is a routine matter to show that the set of all functions $\boldsymbol{\phi} \in C_0^\infty(\Omega \times [0; T])$, such that $\nabla \cdot \boldsymbol{\phi} = 0$, forms a dense subset of $L^2(0, T; J_1(\Omega))$; for such functions, (23) follows from (7) and Lemma 2. Since $\mathbf{u}$ is bounded, $\mathbf{u} \cdot \nabla \mathbf{u} \in L^2(0, T; L^2(\Omega))$; if we also assume $\mathbf{f} \in L^2(0, T; L^2(\Omega))$, we can obtain (23) for all $\boldsymbol{\phi} \in L^2(0, T; J_1(\Omega))$ by passing to a limit from solenoidal functions $\boldsymbol{\phi} \in C_0^\infty(\Omega \times [0, T])$. We have proved:

THEOREM 1. *A classical solution* $\mathbf{u}$ *of* (7)–(11) *is a generalized solution if* $\mathbf{u}$, $\nabla \mathbf{u}$, $\mathbf{u}_t \in L^2(\Omega \times (0, T))$. *Here it is assumed that* $\mathbf{f} = \mathbf{f}^* + \nabla q$ *where* q, $\nabla q \in L^2_{\text{loc}}(\Omega \times (0, T))$ *and* $\mathbf{f}^* \in L^2(\Omega \times (0, T))$.

The solution of the initial boundary value problem constructed by Hopf [18] was proved by him to satisfy a list of properties which included the condition $\mathbf{u} \in L^2(0, T; J_1(\Omega))$. The properties listed by Hopf have been subsequently taken by some authors as defining properties for a class of "weak solutions". Thus, solutions belonging to solution classes studied by Prodi [30, 31], Lions [23, 24], and Serrin [33] are required to satisfy the condition $\mathbf{u} \in L^2(0, T; J_1(\Omega))$. Hidden properties of this condition have made possible some uniqueness theorems, without auxiliary conditions, which are not valid for classical solutions. Kiselev and Ladyzhenskaya [20] and Ladyzhenskaya [21], while requiring only that $\mathbf{u}$ belong to $L^2(0, T; J_1^*(\Omega))$ and not to $L^2(0, T; J_1(\Omega))$, have required that (23) be satisfied by all $\boldsymbol{\phi} \in L^2(0, T; J_1^*(\Omega))$. Thus, for domains such that $J_1^*(\Omega) \not\subset J_1(\Omega)$, their condition (23) implies something more than equation (7) and results in excluding some classical solutions from membership in their solution class (the paper [20], however, treats only bounded domains). Consequently they were able to prove uniqueness within their solution class, without auxiliary conditions, even for an arbitrary spatial domain [21]. For the class of generalized solutions we defined above, the uniqueness argument of Kiselev and Ladyzhenskaya reduces the uniqueness problem for (7)–(11) to a matter of determining the cosets of $J_1^*(\Omega)/J_1(\Omega)$.

PROPOSITION 1. *Suppose that* $\mathbf{u}$ *and* $\bar{\mathbf{u}}$ *are two generalized solutions of* (7)–(11) *which belong to the same coset of* $L^2(0, T; J_1^*(\Omega))/L^2(0, T; J_1(\Omega))$. *Then* $\mathbf{u} = \bar{\mathbf{u}}$.

Proof. Consider $\mathbf{w} = \mathbf{u} - \bar{\mathbf{u}}$. Let $\tau \in (0, T)$ be arbitrary, and let $\boldsymbol{\phi}(x, t)$ be equal to $\mathbf{w}(x, t)$ for $t \leq \tau$, and vanish for $t > \tau$. The assumption that $\mathbf{u}$ and $\bar{\mathbf{u}}$ belong to the same coset means that $\mathbf{u} - \bar{\mathbf{u}} \in L^2(0, T; J_1(\Omega))$; therefore $\boldsymbol{\phi} \in L^2(0, T; J_1(\Omega))$. Thus we can subtract (23) for $\bar{\mathbf{u}}$ from (23) for $\mathbf{u}$ to obtain

$$\int_0^\tau \int_\Omega \{\mathbf{w}_t \cdot \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{w} - \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} \cdot \mathbf{w} + \nabla \mathbf{w} : \nabla \mathbf{w}\}\, dx\, dt = 0.$$

We write $\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{w}-\bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}}\cdot\mathbf{w}=\mathbf{w}\cdot\nabla\mathbf{u}\cdot\mathbf{w}+\bar{\mathbf{u}}\cdot\nabla\mathbf{w}\cdot\mathbf{w}$, and observe that $\int_\Omega \mathbf{w}\cdot\nabla\mathbf{u}\cdot\mathbf{w}\,dx= -\int_\Omega \mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx$ and $\int_\Omega \bar{\mathbf{u}}\cdot\nabla\mathbf{w}\cdot\mathbf{w}\,dx=0$. This follows from the integration by parts identity $\int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\,dx=-\int_\Omega(\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}\,dx-\int_\Omega \mathbf{u}\cdot\nabla\mathbf{w}\cdot\mathbf{v}\,dx$ which is easily verified for any three pointwise bounded vector fields $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}\in \mathring{W}_2^1(\Omega)$. Instead of assuming that the vector fields are pointwise bounded it is enough to know, if $\Omega\subset R^n$, that they belong to $L^q(\Omega)$ with $q\geqslant n$; in passing to the limit from functions in $C_0^\infty(\Omega)$ one then uses Hölder's inequality and the Sobolev inequality $\|\boldsymbol{\phi}\|_{L^p(\Omega)}\leqslant C\|\boldsymbol{\phi}\|_1$, which is valid for p satisfying $p^{-1}+n^{-1}+\frac{1}{2}=1$ if $n>2$, and satisfying $2\leqslant p<\infty$ if $n=2$. Since $\mathbf{w}_t\cdot\mathbf{w}=\frac{1}{2}(d/dt)\mathbf{w}^2$, and since $\mathbf{w}(x,0)=0$, one obtains

$$\tfrac{1}{2}\|\mathbf{w}(t)\|^2+\int_0^t\|\nabla\mathbf{w}\|^2d\tau=\int_0^t\int_\Omega \mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx\,d\tau. \tag{24}$$

If $|\mathbf{u}(x,t)|$ is uniformly bounded in $\Omega\times(0,T)$, the nonlinear term in (24) can be estimated as in (18), so that (19) is obtained and the proof is completed. If $|\mathbf{u}(x,t)|$ is perhaps not uniformly bounded, but $\int_\Omega|\mathbf{u}(x,t)|^q dx<C$ for all $t\in(0,T)$ for some $q>n$, then one can combine Hölder's inequality with the Sobolev inequality $\|\mathbf{w}\|_{L^p(\Omega)}\leqslant C\|\nabla\mathbf{w}\|^{n/q}\|\mathbf{w}\|^{1-n/q}$ which is valid for $p^{-1}+q^{-1}+\frac{1}{2}=1$, to obtain

$$\left|\int_\Omega \mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx\right|\leqslant\|\mathbf{u}\|_{L^q(\Omega)}\cdot\|\nabla\mathbf{w}\|\cdot\|\mathbf{w}\|_{L^p(\Omega)}\leqslant C\|\nabla\mathbf{w}\|^{1+n/q}\cdot\|\mathbf{w}\|^{1-n/q}\leqslant\|\nabla\mathbf{w}\|^2+C\|\mathbf{w}\|^2. \tag{25}$$

Here Young's inequality $ab\leqslant s^{-1}a^s+t^{-1}b^t$, valid for $s^{-1}+t^{-1}=1$, has been used in the last step; and C simply denotes a constant which may change values from step to step. The proof of Proposition 1 is now completed by using (25) in place of (18).

In order to determine the cosets of $J_1^*(\Omega)/J_1(\Omega)$, we shall frequently exploit the equivalence of showing that $J_1^*(\Omega)=J_1(\Omega)$ and of proving uniqueness for problem (12)–(15) in Ω.

PROPOSITION 2. *Let Ω be an arbitrary open set of R^n. Then $J_1^*(\Omega)=J_1(\Omega)$ if and only if the only function $\mathbf{w}\in J_1^*(\Omega)$, such that $\int_\Omega(\nabla\mathbf{w}:\nabla\boldsymbol{\phi}+\mathbf{w}\cdot\boldsymbol{\phi})dx=0$ for all $\boldsymbol{\phi}\in D(\Omega)$, is $\mathbf{w}\equiv 0$.*

Proof. In order to show that $J_1^*(\Omega)\subset J_1(\Omega)$, let $\mathbf{u}$ be an arbitrary element of $J_1^*(\Omega)$. Clearly $\int_\Omega(\nabla\mathbf{u}:\nabla\boldsymbol{\phi}+\mathbf{u}\cdot\boldsymbol{\phi})dx$ defines a bounded linear functional on $\boldsymbol{\phi}\in J_1(\Omega)$. Thus, since $J_1(\Omega)$ is a Hilbert space, there is an element $\mathbf{v}\in J_1(\Omega)$ such that $\int_\Omega(\nabla\mathbf{u}:\nabla\boldsymbol{\phi}+\mathbf{u}\cdot\boldsymbol{\phi})dx= \int_\Omega(\nabla\mathbf{v}:\nabla\boldsymbol{\phi}+\mathbf{v}\cdot\boldsymbol{\phi})dx$ for all $\boldsymbol{\phi}\in J_1(\Omega)$. Let $\mathbf{w}=\mathbf{u}-\mathbf{v}$. Then $\mathbf{w}\in J_1^*(\Omega)$ and $\int_\Omega(\nabla\mathbf{w}:\nabla\boldsymbol{\phi}+\mathbf{w}\cdot\boldsymbol{\phi})dx= 0$ for all $\boldsymbol{\phi}\in D(\Omega)$. This, by assumption, implies that $\mathbf{w}=0$ and hence that $\mathbf{u}=\mathbf{v}\in J_1(\Omega)$.

If $J_1^*(\Omega)=J_1(\Omega)$, and if $\mathbf{w}\in J_1^*(\Omega)$ satisfies $\int_\Omega(\nabla\mathbf{w}:\nabla\boldsymbol{\phi}+\mathbf{w}\cdot\boldsymbol{\phi})dx=0$ for all $\boldsymbol{\phi}\in D(\Omega)$, then, since $D(\Omega)$ is dense in $J_1(\Omega)=J_1^*(\Omega)$, one obtains $\int_\Omega(\nabla\mathbf{w}:\nabla\mathbf{w}+\mathbf{w}\cdot\mathbf{w})dx=0$. Thus $\mathbf{w}\equiv 0$.

The next lemma, which is essentially due to Ladyzhenskaya [21], supplies a means of estimating the L^2-norm of ∇p in a neighborhood of infinity or in an "interior" half-space. We write $\Omega' \subset\subset \Omega$ to mean that the closure $\bar{\Omega}'$ of Ω' is a compact subset of Ω, and we denote $\|\mathbf{u}\|_{L^2(\Omega')}$ simply by $\|\mathbf{u}\|_{\Omega'}$. The domain Ω may be an arbitrary open subset of R^n.

LEMMA 3. *Let* $\mathbf{u}$ *be a vector field such that* $\mathbf{u}$, $\nabla\mathbf{u} \in L^2_{\text{loc}}(\Omega)$ *and* $\nabla\cdot\mathbf{u}=0$. *Suppose* $\int_\Omega(\nabla\mathbf{u}:\nabla\boldsymbol{\phi}-\mathbf{f}\cdot\boldsymbol{\phi})\,dx=0$ *holds for some* $\mathbf{f}\in L^2_{\text{loc}}(\Omega)$ *and all* $\boldsymbol{\phi}\in D(\Omega)$. *Let* $\Omega'' \subset\subset \Omega' \subset\subset \Omega$. *Let* ζ *be a continuously differentiable, and piecewise twice continuously differentiable, real valued function* $\zeta\colon \Omega\to[0,1]$, *such that* $\zeta\equiv 1$ *in* Ω'', *and* $\zeta\equiv 0$ *in* $\Omega-\Omega'$. *Then* $u_{x_ix_j}\in L^2_{\text{loc}}(\Omega)$ *and*

$$\|\mathbf{u}_{x_ix_j}\|_{\Omega''} \leqslant C_\zeta\|\nabla\mathbf{u}\|_{\Omega'-\Omega''} + C_\zeta\|\mathbf{f}\|_{\Omega'} + \|(\Delta\zeta)\mathbf{u}\|_{\Omega'}, \tag{26}$$

where the constant C_ζ *depends only on the maximum value of the first derivatives of* ζ.

Proof. Let $\boldsymbol{\phi}$ be the vector field with ith component

$$\phi_i = \left\{-\zeta^2\Delta u_{\varrho i} + \sum_{j,l,m=1}^{n}(\delta_i^l\delta_j^m - \delta_i^m\delta_j^l)\left(\frac{\partial}{\partial x_j}\zeta^2\right)\left(\frac{\partial}{\partial x_l}u_{\varrho m}\right)\right\}_\varrho,$$

where δ_i^j is one if $i=j$ and zero if $i\neq j$, and the subscript ϱ denotes an averaging convolution $g_\varrho(x)=\int g(x-\varrho y)\omega(y)\,dy$ with kernel $\omega\in C_0^\infty(|x|<1)$ satisfying $\int\omega(x)\,dx=1$. For small enough ϱ the support of $\boldsymbol{\phi}$ is in Ω', and one may check by direct computation that $\boldsymbol{\phi}\in D(\Omega)$. By using the well known identities $\int\phi_\varrho\psi\,dx=\int\phi\psi_\varrho\,dx$ and $(\phi_\varrho)_{x_i}=(\phi_{x_i})_\varrho$ for the averaging convolution, we may write the identity $\int_\Omega\nabla\mathbf{u}\colon\nabla\boldsymbol{\phi}\,dx=\int_\Omega\mathbf{f}\cdot\boldsymbol{\phi}\,dx$ in the form

$$\begin{aligned}\int_\Omega(\zeta\Delta\mathbf{u}_\varrho)^2\,dx &+ \int_\Omega\sum_{i=1}^n(\zeta\Delta u_{\varrho i})\sum_{j,l,m=1}^n(\delta_i^l\delta_j^m-\delta_i^m\delta_j^l)\left(2\frac{\partial}{\partial x_j}\zeta\right)\left(\frac{\partial}{\partial x_l}u_{\varrho m}\right)dx\\ &= \int_\Omega(\zeta\Delta\mathbf{u}_\varrho)\cdot(-\zeta\mathbf{f}_\varrho)\,dx + \int_\Omega\sum_{i=1}^n(\zeta\mathbf{f}_{\varrho i})\sum_{j,l,m=1}^n(\delta_i^l\delta_j^m-\delta_i^m\delta_j^l)\left(2\frac{\partial}{\partial x_j}\zeta\right)\left(\frac{\partial}{\partial x_l}u_{\varrho m}\right)dx.\end{aligned}$$

By using the Schwarz inequality, and also the inequality $ab\leqslant\frac{1}{4}a^2+b^2$, we obtain

$$\|\zeta\Delta\mathbf{u}_\varrho\|^2\leqslant C_\zeta\|\nabla\mathbf{u}\|^2_{\Omega'-\Omega''}+\|\mathbf{f}\|^2_{\Omega'}.$$

Since $\Delta(\zeta\mathbf{u}_\varrho)=\zeta\Delta\mathbf{u}_\varrho+2\Delta\zeta\cdot\nabla\mathbf{u}_\varrho+(\Delta\zeta)\mathbf{u}_\varrho$, we get

$$\|\Delta(\zeta\mathbf{u}_\varrho)\|\leqslant C_\zeta\|\nabla\mathbf{u}\|_{\Omega'-\Omega''}+\|\mathbf{f}\|_{\Omega'}+\|(\Delta\zeta)\mathbf{u}_\varrho\|.$$

Finally, noting that $\|\Delta\mathbf{v}\|^2_{\Omega'}=\Sigma^n_{i,j=1}\|\mathbf{v}_{x_ix_j}\|^2_{\Omega'}$ holds for functions $v\in C_0^\infty(\Omega')$, as may be shown through integration by parts, and taking the limit as $\varrho\to 0$, we obtain (26).

PROPOSITION 3. *Let* $\mathbf{u}$ *be a vector field such that* $\mathbf{u}$, $\nabla\mathbf{u}\in L^2_{\text{loc}}(\Omega)$ *and* $\nabla\cdot\mathbf{u}=0$. *Then* $\int_\Omega(\nabla\mathbf{u}\colon\nabla\boldsymbol{\phi}+\mathbf{u}\cdot\boldsymbol{\phi})\,dx=0$ *holds for all* $\boldsymbol{\phi}\in D(\Omega)$ *if and only if* $\mathbf{u}\in C^\infty(\Omega)$ *and* $\Delta\mathbf{u}-\mathbf{u}=\nabla p$ *for some harmonic function* p.

Proof. Suppose that $\int_\Omega (\nabla \mathbf{u} : \nabla \boldsymbol{\phi} + \mathbf{u}\cdot\boldsymbol{\phi})dx = 0$ for all $\boldsymbol{\phi} \in D(\Omega)$. Then, by the lemma, $\mathbf{u}$ has second derivatives $\mathbf{u}_{x_i x_j} \in L^2_{\text{loc}}(\Omega)$. Clearly $\nabla\cdot\mathbf{u}_{x_i} = 0$. Since the derivatives of functions $\boldsymbol{\phi} \in D(\Omega)$ also belong to $D(\Omega)$, one has $\int_\Omega (\nabla \mathbf{u}_{x_i} : \nabla\boldsymbol{\phi} + \mathbf{u}_{x_i}\cdot\boldsymbol{\phi})\,dx = -\int_\Omega (\nabla\mathbf{u} : \nabla\boldsymbol{\phi}_{x_i} + \mathbf{u}\cdot\boldsymbol{\phi}_{x_i})dx = 0$ for all $\boldsymbol{\phi} \in D(\Omega)$. Thus, by the lemma again, $\mathbf{u}$ has locally square-summable third derivatives. An induction argument shows that $\mathbf{u}$ has locally square-summable derivatives of all orders, and therefore, by a well known theorem of Sobolev, $\mathbf{u} \in C^\infty(\Omega)$. Finally, we observe that Lemma 2 implies the existence of p, and that $\Delta p = \nabla\cdot\nabla p = \nabla\cdot\Delta\mathbf{w} - \nabla\cdot\mathbf{w} = 0$.

We turn now to a consideration of the boundary value problem (1)–(4) for the steady Stokes equations.

Definition. We call a function $\mathbf{u}(x)$ a generalized solution of (1)–(4) if and only if $\mathbf{u} \in J_0^*(\Omega)$ and $\int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\phi}\,dx = 0$ for all $\boldsymbol{\phi} \in J_0(\Omega)$.

Since the Stokes equations are linear, uniqueness questions for the more general inhomogeneous boundary value problems reduce to uniqueness questions for problem (1)–(4); thus in this paper we consider only problem (1)–(4). To prove that every classical solution of (1)–(4) which possesses a finite Dirichlet integral satisfies our definition of generalized solution we need the following lemma, in addition to Lemma 2.

LEMMA 4. *Let Ω be an arbitrary open set of R^n. Suppose that $u \in C(\overline{\Omega})$, that $u = 0$ on $\partial\Omega$, that $u(x) \to 0$ (continuously) as $|x| \to \infty$, that u has generalized first derivatives, and that $\int_\Omega (\nabla u)^2 dx$ is finite. Then $u \in W_0(\Omega)$. The assumption that $u(x) \to 0$ as $|x| \to \infty$ may be dropped altogether if $n = 2$, and may be replaced by the weaker assumption, that $\int_\Omega [(u(x) - c)^2/|x|^2]dx < \infty$ implies $c = 0$, if $n \geq 3$.*

Proof. If Ω is bounded, the inequality $\|u\| \leq C_\Omega \|\nabla u\|$ holds, and Lemma 4 follows immediately from Lemma 1. If Ω is unbounded, we consider a sequence of truncations of u of the form $\zeta_k u$, where $\zeta_k(x)$ is a continuous and piecewise continuously differentiable real-valued function of $r = |x|$, such that for some numbers $0 < a_k < b_k$, $\zeta_k(r) \equiv 1$ for $r \leq a_k$, $0 \leq \zeta_k(r) \leq 1$ for $a_k \leq r \leq b_k$, and $\zeta_k(r) \equiv 0$ for $r \geq b_k$. Clearly $\zeta_k u \in \mathring{W}_2^1(\Omega) \subset W_0(\Omega)$. We will give a particular sequence of functions $\{\zeta_k\}$ such that the truncations $\zeta_k u$ converge to u in Dirichlet norm. Observe that

$$\int_\Omega \{\nabla(\zeta_k u - u)\}^2 dx \leq 4\int_{\Omega,\, a_k \leq r \leq b_k} (u\nabla\zeta_k)^2 dx + 6\int_{\Omega,\, r \geq a_k} (\nabla u)^2\, dx. \tag{27}$$

If $a_k \to \infty$ as $k \to \infty$, the second integral on the right will tend to zero. In order to show that the first integral on the right tends to zero, for an appropriate choice of functions ζ_k, we need the inequalities

$$\int_\Omega \frac{u^2}{r^2}dx < \infty \quad \text{for} \quad n \geq 3, \quad \text{and} \quad \int_{\Omega,\, r\geq 2} \frac{u^2}{r^2 \log^2 r} < \infty \quad \text{for} \quad n=2, \tag{28}$$

which are valid for every function $u \in C(\bar{\Omega})$ such that $u=0$ on $\partial\Omega$, such that $\int_\Omega (\nabla u)^2 dx < \infty$, and, if $n \geq 3$, such that $\int_\Omega [(u(x)-c)^2/|x|^2]dx < \infty$ implies $c=0$. The inequality (28) for $n \geq 3$ is due to Finn [8], who based its proof on an inequality of Payne and Weinberger; we will give another proof here which is based upon properties of harmonic functions, and which yields the inequality for $n=2$ as well. Before proving (28), let us show how (27) and (28) can be combined to complete the proof of Lemma 4. For the case of $n \geq 3$ we follow Finn [8, p. 368]. Let $a_k=k$, $b_k=2k$, and $\zeta_k(r)=(2k-r)/r$ for $k \leq r \leq 2k$. Then $|\nabla\zeta_k|=2k/r^2$ for $k<r<2k$, and therefore

$$\int_{\Omega,\, k\leq r\leq 2k} (u\nabla\zeta_k)^2 dx \leq 4\int_{\Omega,\, k\leq r\leq 2k} \frac{k^2}{r^2}\frac{u^2}{r^2}dx \leq 4\int_{\Omega,\, k\leq r} \frac{u^2}{r^2}dx \to 0 \quad \text{as} \quad k\to\infty.$$

For the case $n=2$, let $a_k=e^k$, $b_k=e^{2k}$, and $\zeta_k(r)=2-(1/\mathrm{k})\log r$ for $e^k \leq r \leq e^{2k}$. Then $|\nabla\zeta_k| = 1/(kr)$ for $e^k<r<e^{2k}$, and therefore

$$\int_{\Omega,\, e^k\leq r\leq e^{2k}} (u\nabla\zeta_k)^2 dx \leq \int_{\Omega,\, e^k\leq r\leq e^{2k}} \frac{u^2}{k^2r^2}dx \leq 4\int_{\Omega,\, e^k\leq r\leq e^{2k}} \frac{u^2}{r^2\log^2 r}dx \to 0 \quad \text{as} \quad k\to\infty.$$

It remains to prove (28). First consider the case $n \geq 3$. Extend the domain of definition of u to all of R^n by setting $u \equiv 0$ in the complement Ω^c of Ω. Clearly $\int \nabla u: \nabla\phi\, dx$ defines a bounded linear functional on $\phi \in W_0(R^n)$. Thus there exists an element v of $W_0(R^n)$ such that $\int \nabla(u-v): \nabla\phi\, dx = 0$ for all $\phi \in C_0^\infty(R^n)$. Clearly $u-v$ is harmonic in R^n and $\int \{\nabla(u-v\}^2 dx$ is finite. Thus $u=v+c$ for some constant c. Now $\int_{R^n} v^2/r^2 dx \leq 4(n-2)^{-2}\int_{R^n}(\nabla v)^2 dx$, as may be checked through an easily justified integration by parts for functions $v \in C_0^\infty(R^n)$:

$$\int \frac{v^2}{|x|^2}dx = \frac{-1}{n-2}\int \sum_{i=1}^n \frac{\partial v^2}{\partial x_i}\frac{x_i}{|x|^2}dx = \frac{-2}{n-2}\int \sum_{i=1}^n v\frac{\partial v}{\partial x_i}\frac{x_i}{|x|^2}dx$$
$$\leq \frac{2}{n-2}\left(\int \sum_{i=1}^n \frac{v^2}{|x|^2}\frac{x_i^2}{|x|^2}dx\right)^{1/2}\left(\int \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{1/2}.$$

Thus $\int_{R^n}[(u-c)^2/r^2]dx \leq 4(n-2)^{-2}\int_{R^n}(\nabla u)^2 dx$, and this implies (28) because of the assumption that $u\to 0$ as $|x|\to\infty$, even in the weakened form.

We argue a little differently if $n=2$ in order to avoid dealing with the Hilbert space $W_0(R^2)$ whose elements are equivalence classes of functions which differ by constants. As before, we define u throughout R^2 by setting $u\equiv 0$ in Ω^c. Then let $\bar{u}$ be a truncation of u which equals 0 for $|x| \leq 3/2$, and which equals u for $|x| \geq 2$. We may assume $\int(\nabla\bar{u})^2 dx < \infty$, so that $\int_{|x|>1}\nabla\bar{u}\colon\nabla\phi\,dx$ defines a bounded linear functional on $\phi\in W_0(|x|>1)$. Let v be the element of $W_0(|x|>1)$ which satisfies $\int_{|x|>1}\nabla(\bar{u}-v)\colon\nabla\phi\,dx=0$ for all $\phi\in C_0^\infty(|x|>1)$. Clearly $\bar{u}-v$ is harmonic in $|x|>1$, and $\int_{|x|>1}\{\nabla(\bar{u}-v)\}^2dx<\infty$. Thus $\bar{u}-v$ tends to a constant as $|x|\to\infty$, and hence $\bar{u}-v$ is a bounded function. Now $\int_{|x|>1}v^2/(r^2\log^2 r)\,dx\leqslant 4\int_{|x|>1}(\nabla v)^2dx$, as may be easily checked through an integration by parts for functions $v\in C_0^\infty(|x|>1)$:

$$\int_{|x|>1}\frac{v^2\,dx}{|x|^2\log^2|x|}=\int_{|x|>1}\sum_{i=1}^{2}\frac{\partial v^2}{\partial x_i}\frac{x_i}{|x|^2\log|x|}\,dx=2\int_{|x|>1}\sum_{i=1}^{2}v\frac{\partial v}{\partial x_i}\,\frac{x_i}{|x|^2\log|x|}\,dx$$

$$\leqslant 2\left(\int_{|x|>1}\sum_{i=1}^{2}\frac{v^2}{|x|^2\log^2|x|}\,\frac{x_i^2}{|x|^2}\,dx\right)^{1/2}\left(\int_{|x|>1}\sum_{i=1}^{n}\left(\frac{\partial v}{\partial x_i}\right)^2\right)^{1/2}.$$

Since

$$\int_{|x|>2}\frac{dx}{|x|^2\log^2|x|}<\infty,$$

and since $|u|\leqslant|v|+c$, (28) follows.

A vector field $\mathbf{u}(x)$ is called a classical solution of (1)–(4) if $\mathbf{u}\in C(\bar{\Omega})\cap C^2(\Omega)$ and if conditions (1)–(4) are satisfied, for some $p(x)\in C^1(\Omega)$, in the senses appropriate to continuous functions. The following theorem is an immediate consequence of Lemmas 2 and 4.

THEOREM 2. *A classical solution of* (1)–(4) *is a generalized solution if* $\int_\Omega(\nabla\mathbf{u})^2dx$ *is finite. The assumption* (4), *that* $\mathbf{u}(x)\to 0$ *as* $|x|\to\infty$, *is unnecessary if* $\Omega\subset R^2$, *and may be replaced by the weaker assumption, that* $\int_\Omega[(\mathbf{u}(x)-\mathbf{c})^2/|x|^2]dx<\infty$ *implies* $\mathbf{c}=0$, *if* $\Omega\subset R^n$ *with* $n\geqslant 3$.

PROPOSITION 4. *Let* Ω *be an arbitrary open set of* R^n. *Then* $J_0^*(\Omega)=J_0(\Omega)$ *if and only if the only generalized solution of* (1)–(4) *is* $\mathbf{u}=0$. *If* $\mathbf{u}$ *and* $\bar{\mathbf{u}}$ *are two generalized solutions of* (1)–(4) *which belong to the same coset of* $J_0^*(\Omega)/J_0(\Omega)$, *then* $\mathbf{u}=\bar{\mathbf{u}}$.

Proof. To say that $\mathbf{u}$ and $\bar{\mathbf{u}}$ belong to the same coset of $J_0^*(\Omega)/J_0(\Omega)$ means just that $\mathbf{u}-\bar{\mathbf{u}}\in J_0(\Omega)$; thus if $\mathbf{u}$ and $\bar{\mathbf{u}}$ are also generalized solutions of (1)–(4) it readily follows that $\int_\Omega\{\nabla(\mathbf{u}-\bar{\mathbf{u}})\}^2dx=0$. Clearly this implies $\mathbf{u}=0$ is the unique generalized solution of (1)–(4) if $J_0^*(\Omega)=J_0(\Omega)$. On the other hand, suppose the only generalized solution of (1)–(4) is $\mathbf{u}=0$, and suppose that $\mathbf{v}\in J_0^*(\Omega)$. Since $\int_\Omega\nabla\mathbf{v}\colon\nabla\phi\,dx$ defines a bounded linear functional on $\phi\in J_0(\Omega)$, there exists an element $\mathbf{w}$ of $J_0(\Omega)$ such that $\int_\Omega\nabla\mathbf{w}\colon\nabla\boldsymbol{\phi}\,dx=\int_\Omega\nabla\mathbf{v}\colon\nabla\boldsymbol{\phi}\,dx$ for all $\boldsymbol{\phi}\in J_0(\Omega)$. Clearly $\mathbf{w}-\mathbf{v}$ is a generalized solution of (1)–(4), and hence $\mathbf{w}-\mathbf{v}=0$. Thus we have $\mathbf{v}=\mathbf{w}\in J_0(\Omega)$.

The proof of the next proposition is similar to that of Proposition 3.

PROPOSITION 5. *Let* $\mathbf{u}$ *be a vector field such that* $\mathbf{u}$, $\nabla\mathbf{u}\in L^2_{\mathrm{loc}}(\Omega)$ *and* $\nabla\cdot\mathbf{u}=0$. *Then* $\int_\Omega \nabla\mathbf{u}\colon \nabla\phi\, dx=0$ *holds for all* $\phi\in D(\Omega)$ *if and only if* $\mathbf{u}\in C^\infty(\Omega)$ *and* $\Delta\mathbf{u}=\nabla p$ *for some harmonic function* p.

We will now give a proof of Lemma 1. The author wishes to thank Professor C. A. Swanson for suggestions which have led to improvements in the proof. We begin by noting that the case of an unbounded open set Ω can be reduced to the case of a bounded set by considering truncations of u; the argument is much simpler than in Lemma 4 because $u\in L^2(\Omega)$. Thus we consider only the case of bounded Ω. We need some preliminary facts which are true for bounded sets.

(i) The set $C_0(\Omega)\cap \mathrm{Lip}\,(\Omega)$ of all Lipschitz continuous functions with compact support in Ω is contained in $\mathring{W}^1_2(\Omega)$.

To prove this, suppose that $u\in C_0(\Omega)\cap \mathrm{Lip}\,(\Omega)$, and let $u_\varrho(x)=\int u(y)\omega[(x-y)/\varrho]\varrho^{-n}dy$ where $\omega(x)$ is an averaging kernel, with $\omega\in C_0^\infty(|x|<1)$ and $\int\omega\, dx=1$. Of course $u_\varrho\to u$ in $L^2(\Omega)$ as $\varrho\to 0$; thus we need only show that $\|(\partial/\partial x_i)u_\varrho\|$ remains bounded uniformly in ϱ as $\varrho\to 0$. It suffices to show that $|(\partial/\partial x_i)u_\varrho|$ is bounded uniformly in x and ϱ. Let K be the Lipschitz constant for u, so that $|u(x-\varrho y)-u(x)|\leqslant K\varrho|y|$. Then

$$\left|\frac{\partial}{\partial x_i}u_\varrho(x)\right|=\left|\varrho^{-1}\int_{|y|<1}u(x-\varrho y)\,\omega_{y_i}(y)\,dy\right|$$
$$\leqslant\left|\varrho^{-1}\int_{|y|<1}u(x)\,\omega_{y_i}(y)\,dy\right|+\varrho^{-1}\int_{|y|<1}K\varrho|y|\cdot|\omega_{y_i}(y)|\,dy\leqslant CK, \tag{*}$$

because the first integral on the right vanishes through integration by parts.

(ii) If $v, w\in W^1_2(\Omega)$, then $u(x)=\max\,(v(x), w(x))\in W^1_2(\Omega)$. As usual, $W^1_2(\Omega)$ denotes those functions in $L^2(\Omega)$ which have weak derivatives in $L^2(\Omega)$.

For every $\varepsilon>0$, choose v_ε, $w_\varepsilon\in C^\infty(\Omega)$ such that $\|v-v_\varepsilon\|_1<\varepsilon$ and $\|w-w_\varepsilon\|_1<\varepsilon$; this is possible by a theorem of Meyers and Serrin [26]. Let $u_\varepsilon(x)=\max\,(v_\varepsilon(x), w_\varepsilon(x))$. It may be checked that $|\max\,(\alpha,\beta)-\max\,(\bar\alpha,\bar\beta)|\leqslant|\alpha-\bar\alpha|+|\beta-\bar\beta|$ holds for any numbers α, $\bar\alpha$, β, and $\bar\beta$. Thus $\|u-u_\varepsilon\|\leqslant\|v-v_\varepsilon\|+\|w-w_\varepsilon\|$; so $u_\varepsilon\to u$ in $L^2(\Omega)$ as $\varepsilon\to 0$. That u_ε has weak derivatives $\partial u_\varepsilon/\partial x_i$ can be seen by applying (i) to truncations of u_ε; if $\zeta\in C_0^\infty(\Omega)$ and $\zeta\equiv 1$ in a subdomain Ω' of Ω, then $\zeta u_\varepsilon\in C_0(\Omega)\cap\mathrm{Lip}\,(\Omega)\subset\mathring{W}^1_2(\Omega)$ and $u_\varepsilon=\zeta u_\varepsilon$ in Ω'. To show that $\|\partial u_\varepsilon/\partial x_i\|$ is finite and remains uniformly bounded as $\varepsilon\to 0$, one uses the fact that, for almost all x, $|\partial u_\varepsilon/\partial x_i|\leqslant C\max\,(|\nabla v_\varepsilon|, |\nabla w_\varepsilon|)$. This last inequality can be rigorously proved by noting that u_ε is locally Lipschitz continuous and that, in a sufficiently small neighborhood of any given point x, the Lipschitz constant is approximately equal to $\max\,(|\nabla v_\varepsilon(x)|, |\nabla w_\varepsilon(x)|)$; one then applies inequality (*).

(iii) If $N_\delta = \{x: x \in \Omega \text{ and } |x-y| < \delta \text{ for some } y \in \partial\Omega\}$, then the measure of N_δ tends to zero as $\delta \to 0$.

To prove this, remember that Ω can be formed as a countable union of open, balls, $\Omega = \bigcup_{i=1}^{\infty} O_i$, and also that meas $(\Omega) = \lim_{n\to\infty}$ meas $(\bigcup_{i=1}^{n} O_i)$. Now given $\varepsilon > 0$, choose N such that meas $(\Omega - \bigcup_{i=1}^{N} O_i) < \varepsilon/2$. Then meas $(N_\delta) =$ meas $(N_\delta \cap \bigcup_{i=1}^{N} O_i)$ $+$ meas $(N_\delta \cap (\Omega - \bigcup_{i=1}^{N} O_i)) < \Sigma_{i=1}^{N}$ meas $(N_\delta \cap O_i) + \varepsilon/2$. Since for every i, meas $(N_\delta \cap O_i) \to 0$ as $\delta \to 0$, we can choose a value for δ such that meas $(N_\delta \cap O_i) < \varepsilon/(2N)$ for all $i = 1, 2, ..., N$. For this value of δ, meas $N_\delta < N \cdot \varepsilon/(2N) + \varepsilon/2 = \varepsilon$.

The proof of Lemma 1 is completed as follows. For every number $\delta > 0$, let $\zeta_\delta(r)$ be a continuously differentiable non-increasing function of $r \geq 0$, such that $\zeta_\delta(r) \equiv 1$ for $0 \leq r \leq \delta/2$, $|(d/dr)\zeta_\delta(r)| < 3/\delta$ for $\delta/2 \leq r \leq \delta$, and $\zeta_\delta(r) \equiv 0$ for $r \geq \delta$. Also for $\delta > 0$, choose a finite number of points $y_i \in \partial\Omega$ such that $\partial\Omega \subset \bigcup_i \{x: |x-y_i| < \delta/2\}$; this is possible because $\partial\Omega$ is compact. Finally, let $\eta_\delta(x) = \max_i \zeta_\delta(|x-y_i|)$. Clearly $\eta_\delta \equiv 1$ in some open set containing $\partial\Omega$, and clearly $\eta_\delta \equiv 0$ outside N_δ. It follows from (ii) that $\eta_\delta \in W_2^1(\Omega)$. Moreover $\int_{N\delta} (\nabla\eta_\delta)^2 dx \leq (3/\delta)^2$ meas (N_δ) and $\int_{N\delta} (\eta_\delta)^2 dx \leq$ meas (N_δ). Therefore $\delta\eta_\delta \to 0$ in $W_2^1(\Omega)$ as $\delta \to 0$, by (iii).

Now let $u^+(x) = \max(u(x), 0)$ and $u^-(x) = \min(u(x), 0)$. It follows from (ii) that u^+ and u^- belong to $W_2^1(\Omega)$. We will show that u^+ and u^- each belong to $\mathring{W}_2^1(\Omega)$ as well, which implies that $u \in \mathring{W}_2^1(\Omega)$. To this end, let $u_\delta^+(x) = \max(u^+(x), \delta\eta_\delta(x))$; certainly $u_\delta^+ \in W_2^1(\Omega)$, by (ii). Since $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, we have $u_\delta^+ = \delta$ in a neighborhood of $\partial\Omega$. It follows that $u_\delta^+ - \delta$, which vanishes in a neighborhood of $\partial\Omega$, belongs to $\mathring{W}_2^1(\Omega)$. Since u_δ^+ equals u^+ everywhere in Ω except in part of N_δ where it equals $\delta\eta_\delta$, it follows that $\|u_\delta^+ - u^+\|_1 \to 0$ as $\delta \to 0$, because meas $N_\delta \to 0$ and $\|\delta\eta_\delta\|_1 \to 0$. Clearly $u_\delta^+ - \delta$, which belongs to $\mathring{W}_2^1(\Omega)$, also converges to u^+ in $W_2^1(\Omega)$ as $\delta \to 0$; thus u^+ and similarly u^- belong to $\mathring{W}_2^1(\Omega)$.

Remark. Consider, for an arbitrary open (bounded) set Ω, the question of uniqueness for the Dirichlet problem: $\Delta u = f$ in Ω; $u = 0$ on $\partial\Omega$. One has the existence and uniqueness of a "generalized solution" belonging to $\mathring{W}_2^1(\Omega)$ by the Riesz representation theorem. One also has uniqueness for classical solutions by the maximum principle. To show that the classical solution (when it exists) is the same as the "generalized solution", one can use Lemma 1 (for unbounded Ω, Lemma 4), which implies that every classical solution with a finite Dirichlet integral belongs to the generalized solution class, and is therefore subject to the uniqueness theorem of that solution class.

3. Uniqueness in bounded domains

We will give a method here, by which it can be proved, for some bounded domains Ω, that $J_0^*(\Omega) = J_0(\Omega)$ and that $J_1^*(\Omega) = J_1(\Omega)$. In virtue of Propositions 1 and 4, this implies uniqueness for generalized solutions of problems (1)–(4) and (7)–(11) in these domains. Since for functions belonging to $\mathring{W}_2^1(\Omega)$, the norms $\|\nabla\cdot\|$ and $\|\cdot\|_1$ are equivalent in the case of a bounded domain, $J_0^*(\Omega)=J_1^*(\Omega)$ and $J_0(\Omega)=J_1(\Omega)$, and it is not necessary to treat separately the function spaces for stationary and non-stationary problems. The basic idea of our method can be very simply described by considering a star-like domain Ω. To be precise, suppose the closure $\bar{\Omega}_\varrho$ of $\Omega_\varrho=\{x\colon x=\varrho y \text{ for some } y\in\Omega\}$ is contained in Ω for every positive $\varrho<1$. Then if $\mathbf{u}\in J_0^*(\Omega)$, it is easy to see that $\mathbf{u}_\varrho$ defined by $\mathbf{u}_\varrho(x)=\mathbf{u}(x/\varrho)$ for $x\in\Omega_\varrho$, and by $\mathbf{u}_\varrho(x)=0$ for $x\in\Omega-\Omega_\varrho$,will belong to $J_0(\Omega)$. One merely observes that $\mathbf{u}_\varrho\in\mathring{W}_2^1(\Omega)$, that $\nabla\cdot\mathbf{u}_\varrho=0$, and that the support of $\mathbf{u}_\varrho$ lies in a compact subset of Ω; it follows that the averages of $\mathbf{u}_\varrho$, obtained by averaging convolutions with small radii, belong to $D(\Omega)$ and converge to $\mathbf{u}_\varrho$ in norm $\|\cdot\|_1$ as the radii tend to zero. It is obvious that $\|\mathbf{u}_\varrho-\mathbf{u}\|_1\to 0$ as $\varrho\to 1$, and this proves that $J_0^*(\Omega)=J_0(\Omega)$. To make this argument work for a more general class of domains, we need a more general class of transformations with which to "pull in" the support of $\mathbf{u}$ from $\Omega\varrho$, than the contractions of star-like domains.

THEOREM 3. *If Ω is a bounded open set of R^n, a condition which is sufficient to ensure that $J_0^*(\Omega)=J_0(\Omega)$ and that $J_1^*(\Omega)=J_1(\Omega)$, and hence sufficient to ensure uniqueness for generalized solutions of problems* (1)–(4) *and* (7)–(11) *in Ω, is that there should exist a one-parameter family $\{T_\varrho\}$ of maps $T_\varrho\colon\Omega\to\Omega$, say for $\varrho\in(0,1]$, with the properties:*

(a) *T_1 is the identity map,*
(b) *T_ϱ is one-to-one for every $\varrho\in(0,1]$, and the closure of $T_\varrho(\Omega)$ is contained in Ω for every $\varrho\in(0,1)$, and*
(c) *$T_\varrho(x)$ and its first and second derivatives $(\partial/\partial x_i)T_\varrho(x)$, $(\partial^2/\partial x_i\partial x_j)T_\varrho(x)$ are uniformly continuous functions of $(\varrho,x)\in(0,1]\times\Omega$.*

For an arbitrarily given function $\mathbf{u}\in J_0^*(\Omega)$, we need to define another function $\mathbf{u}_\varrho$, which we call the image of $\mathbf{u}$ under T_ϱ, which is solenoidal, has support in $T_\varrho(\Omega)$, and belongs to $\mathring{W}_2^1(\Omega)$. These properties ensure, through an averaging argument, that $\mathbf{u}_\varrho\in J_0(\Omega)$. The following formula is due to Ford [10]; see also Ford and Heywood [11]. We define $\mathbf{u}_\varrho$ implicitely in $T_\varrho(\Omega)$ by the condition that

$$\mathbf{u}_\varrho(T_\varrho(x))=\frac{\nabla T_\varrho(x)\cdot\mathbf{u}(x)}{\mathcal{J}_\varrho(x)} \tag{29}$$

should hold for all $x\in\Omega$, and we set $\mathbf{u}_\varrho(x)\equiv 0$ for all $x\in\Omega-T_\varrho(\Omega)$. Here $(\nabla T_\varrho)_{ij}=(\partial/\partial x_j)T_{\varrho i}$, $\mathcal{J}_\varrho$ is the Jacobian det (∇T_ϱ) of T_ϱ, and $(\nabla T_\varrho\cdot\mathbf{u})_i=\Sigma_{j=1}^n(\partial T_{\varrho i}/\partial x_j)u_j$. The proof of Theorem 3 is contained in the following two lemmas.

LEMMA 5. *There exists a number $\delta < 1$ such that the Jacobian $\mathcal{J}_\varrho(x)$ of T_ϱ is positive and bounded away from zero for $(\varrho, x) \in [\delta, 1] \times \Omega$. The inverse S_ϱ of T_ϱ, and its derivatives $(\partial/\partial x_i)S_\varrho(x)$, are uniformly continuous functions of $(\varrho, x) \in [\delta, 1] \times T_\varrho(\Omega)$. For $\varrho \in [\delta, 1]$, $\mathbf{u}_\varrho$ belongs to $C_0^1(\Omega)$ if $\mathbf{u}$ belongs to $C_0^1(\Omega)$, and the map $\mathring{W}_2^1(\Omega) \to \mathring{W}_2^1(\Omega)$ defined by $\mathbf{u} \to \mathbf{u}_\varrho$ is continuous in norm $\|\cdot\|_1$, uniformly in ϱ. Finally, if $\mathbf{u} \in \mathring{W}_2^1(\Omega)$, then $\|\mathbf{u}_\varrho - \mathbf{u}\|_1 \to 0$ as $\varrho \to 1$.*

Proof. The existence of δ is ensured by the uniform continuity of $\mathcal{J}_\varrho(x)$ in $(0, 1] \times \Omega$, and by the fact that $\mathcal{J}_1(x) \equiv 1$. To check that the inverse and its derivatives are uniformly continuous see, for instance, Buck [2, p. 216]. To see that $\mathbf{u} \in C_0^1(\Omega)$ implies that $\mathbf{u}_\varrho \in C_0^1(\Omega)$, one may inspect the explicit formula

$$u_{\varrho i}(x) = \frac{\sum_{j=1}^{n} \dfrac{\partial T_{\varrho i}}{\partial x_j}(S_\varrho(x)) \cdot u_j(S_\varrho(x))}{\mathcal{J}_\varrho(S_\varrho(x))} \tag{30}$$

for $x \in T_\varrho(\Omega)$, and remember that $\mathbf{u}_\varrho(x) \equiv 0$ for $x \in \Omega - T_\varrho(\Omega)$. One merely observes that the right side of (30), and its first derivatives, involve only derivatives of T_ϱ and S_ϱ which are uniformly continuous and hence bounded, and also that $\mathcal{J}_\varrho$ is bounded away from zero. Noting, in addition, that $\mathcal{J}_\varrho(x)$ is uniformly bounded, one verifies the continuity in $\mathring{W}_2^1(\Omega)$ of the map $\mathbf{u} \to \mathbf{u}_\varrho$. Finally, it is enough to prove that $\|\mathbf{u}_\varrho - \mathbf{u}\|_1 \to 0$, as $\varrho \to 1$, for functions $\mathbf{u} \in C_0^1(\Omega)$. Clearly there is a constant C_δ such that $\max_\Omega |\nabla \mathbf{u}_\varrho| \leqslant C_\delta (\max_\Omega |\mathbf{u}| + \max_\Omega |\nabla \mathbf{u}|)$ holds for all $\mathbf{u} \in C_0^1(\Omega)$ and all $\varrho \in [\delta, 1]$. Now given a particular $\mathbf{u} \in C_0^1(\Omega)$, and a number $\varepsilon > 0$, choose a number σ which is so small that the measure of $N_\sigma = \{x\colon x \in \Omega \text{ and } |x - y| < \sigma$ for some $y \in \partial\Omega\}$ is less than $\varepsilon (\max_\Omega |\nabla \mathbf{u}|)^{-2}$ and also less than $\varepsilon C_\delta^{-2} (\max_\Omega |\mathbf{u}| + \max_\Omega |\nabla \mathbf{u}|)^{-2}$. Then writing

$$\|\nabla(\mathbf{u}_\varrho - \mathbf{u})\|^2 = \int_{N_\sigma} [\nabla(\mathbf{u}_\varrho - \mathbf{u})]^2\, dx + \int_{\Omega - N_\sigma} [\nabla(\mathbf{u}_\varrho - \mathbf{u})]^2\, dx,$$

it is easily seen that the first integral on the right is less than 4ε, for all $\varrho \in [\delta, 1]$, and that the second integral on the right converges to zero as $\varrho \to 1$.

LEMMA 6. *Let K be any cube which, along with its boundary Γ, is contained in Ω, and which has faces parallel to the coordinate planes of R^n. Let $\varrho \in [\delta, 1]$. Then, for every $\mathbf{u} \in C^1(\Omega)$, the outflux of $\mathbf{u}_\varrho$ across the boundary of $T_\varrho(K)$ equals the outflux of $\mathbf{u}$ across the boundary of K; that is,*

$$\int_{T_\varrho(\Gamma)} \mathbf{u}_\varrho \cdot \mathbf{n}\, ds = \int_\Gamma \mathbf{u} \cdot \mathbf{n}\, ds, \tag{31}$$

where $\mathbf{n}$ denotes the outer normal to each surface. Finally, for every $\mathbf{u} \in J_0^*(\Omega)$, we have $\mathbf{u}_\varrho \in J_0(\Omega)$.

Proof. For clarity we give the proof in several steps.

(i) Suppose that $\gamma(s)$ is a differentiable curve in Ω, with image $T_\varrho(\gamma(s))$ in $T_\varrho(\Omega)$. Let $x_0=\gamma(s_0)$. Then $\mathbf{u}(x_0)$ is tangent to γ at x_0 if and only if $\mathbf{u}_\varrho(T_\varrho(x_0))$ is tangent to $T_\varrho(\gamma)$ at $T_\varrho(x_0)$.

Let γ' denote $(d/ds)\gamma$. Then $\gamma'(s_0)$ is a vector tangent to γ at s_0, and, by the chain rule, $\nabla T_\varrho(x_0)\cdot\gamma'(s_0)$ is a vector tangent to $T_\varrho(\gamma)$ at $T_\varrho(x_0)$. Thus, referring to (29), $\mathbf{u}(x_0)$ is a multiple of $\gamma'(s_0)$ if and only if $\mathbf{u}_\varrho(T_\varrho(x_0))$ is a multiple of $\nabla T_\varrho(x_0)\cdot\gamma'(s_0)$.

(ii) Suppose that the vector field $\mathbf{u}$ is parallel to one coordinate axis, say $\mathbf{u}(x)=(0, ..., 0, u_i(x), 0, ..., 0)$. Let Σ be a surface in Ω which is parallel to the other coordinate axes. Then $\int_{T_\varrho(\Sigma)}\mathbf{u}_\varrho\cdot\mathbf{n}\,ds=\int_\Sigma\mathbf{u}\cdot\mathbf{n}\,ds$.

Let $A_{ji}(x)$ be the matrix of cofactors of $\mathcal{J}_\varrho(x)$, so that $\mathcal{J}_\varrho(x)=\sum_{j=1}^n(\partial T_{\varrho j}(x)/\partial x_i)A_{ji}(x)$. Letting $dx_i'=dx_1\,...\,dx_{i-1}dx_{i+1}\,...\,dx_n$, we have

$$\int_{T_\varrho(\Sigma)}\mathbf{u}_\varrho\cdot\mathbf{n}\,ds=\int_\Sigma\sum_{j=1}^n u_{\varrho j}(T_\varrho(x))\,A_{ji}(x)\,dx_i'$$

$$=\int_\Sigma\sum_{j=1}^n\frac{\dfrac{\partial T_{\varrho j}(x)}{\partial x_i}u_i(x)}{\mathcal{J}_\varrho(x)}A_{ji}(x)\,dx_i'=\int_\Sigma u_i(x)\,dx_i'=\int_\Sigma\mathbf{u}\cdot\mathbf{n}\,ds.$$

(iii) Equation (31) holds for every $\mathbf{u}\in C^1(\Omega)$.

First, suppose $\mathbf{u}$ is parallel to one coordinate axis as in (ii). Then all but two faces of K are generated by lines parallel to $\mathbf{u}$, and there is no flux of $\mathbf{u}$ across these faces. By (i) the corresponding "faces" of $T_\varrho(K)$ are generated by curves tangent to $\mathbf{u}_\varrho$, and hence there is no flux of $\mathbf{u}_\varrho$ across these "faces" of $T_\varrho(K)$. By (ii), the flux of $\mathbf{u}$ across the remaining two faces of K equals the flux of $\mathbf{u}_\varrho$ across the corresponding "faces" of $T_\varrho(K)$. An arbitrary vector field $\mathbf{u}\in C^1(\Omega)$ can be written as a sum of component vector fields which are each parallel to one coordinate axis. Since $\mathbf{u}_\varrho$ depends linearly on $\mathbf{u}$, we obtain (31).

(iv) For every $\mathbf{u}\in J_0^*(\Omega)$, we have $\mathbf{u}_\varrho\in J_0(\Omega)$.

In order to verify that $\nabla\cdot\mathbf{u}_\varrho=0$ in $T_\varrho(\Omega)$, and hence in all Ω, it is sufficient to check that $\int_{T_\varrho(K)}\nabla\cdot\mathbf{u}_\varrho\,dx=0$ for the image $T_\varrho(K)$ of every cube in Ω with faces parallel to the coordinate planes. Since $\mathbf{u}\in J_0^*(\Omega)$, there is a sequence $\{\mathbf{u}_n\}$ of functions in $C_0^\infty(\Omega)$ which converges to $\mathbf{u}$ in $\mathring{W}_2^1(\Omega)$. In view of Lemma 5 we have

$$\int_{T_\varrho(K)}\nabla\cdot\mathbf{u}_\varrho\,dx=\lim_{n\to\infty}\int_{T_\varrho(K)}\nabla\cdot\mathbf{u}_{n\varrho}\,dx=\lim_{n\to\infty}\int_{T_\varrho(\Gamma)}\mathbf{u}_{n\varrho}\cdot\mathbf{n}\,ds$$

$$=\lim_{n\to\infty}\int_\Gamma\mathbf{u}_n\cdot\mathbf{n}\,ds=\lim_{n\to\infty}\int_K\nabla\cdot\mathbf{u}_n\,dx=\int_K\nabla\cdot\mathbf{u}\,dx=0.$$

Now for $\varrho < 1$, $\mathbf{u}_\varrho$ has compact support in Ω, and since averaging convolutions preserve the solenoidal character of a vector field, we have $\mathbf{u}_\varrho \in J_0(\Omega)$. Since $\|\mathbf{u}_\varrho - \mathbf{u}\|_1 \to 0$ as $\varrho \to 1$, we have proved that $\mathbf{u} \in J_0(\Omega)$.

COROLLARY 1. *If Ω is a bounded domain with boundary of class C^2, then $J_0^*(\Omega) = J_0(\Omega) = J_1^*(\Omega) = J_1(\Omega)$, and consequently problems* (1)–(4) *and* (7)–(11) *possess at most one solution.*

Proof. For some positive σ, the neighborhood $N_\sigma = \{x: x \in \Omega \text{ and } |x-y| < \sigma \text{ for some } y \in \partial\Omega\}$ of $\partial\Omega$ is covered by nonintersecting normals to $\partial\Omega$. For each point $x \in \Omega - N_\sigma$, let $T_\varrho(x) = x$; and for each point $x \in N_\sigma$, of distance ξ from $\partial\Omega$, let $T_\varrho(x)$ be the point, on the same normal as x, of distance $\xi + \frac{1}{2}(1-\varrho)\sigma^{-2}(\sigma - \xi)^3$ from $\partial\Omega$. This map T_ϱ satisfies the hypothesis of Theorem 3.

Although it is evident that some domains do not satisfy the hypothesis of Theorem 3, the example of a star-like domain shows that not much regularity of $\partial\Omega$ is required. The following corollary generalizes Corollary 1 and applies to domains which have roughly the same degree of boundary regularity as star-like domains. Its simple proof is left to the reader.

COROLARY 2. *Let Ω be a bounded domain. Suppose there is an interior subdomain D of Ω with a class C^2 boundary $\partial D \subset \Omega$, such that the region $\Omega - D$ is covered by nonintersecting normals to ∂D. Suppose further that the normals do not intersect at points of $\partial\Omega$. Then $J_0^*(\Omega) = J_0(\Omega) = J_1^*(\Omega) = J_1(\Omega)$, and consequently problems* (1)–(4) *and* (7)–(11) *possess at most one solution.*

Remark. All presently known proofs of regularity up to the boundary for solutions of viscous flow problems are based on potential theoretic methods that trace back to Odqvist [28]. These results concerning regularity up to the boundary have been carried over to generalized solutions by some authors, [12] and [21]; however, a complete justification for this depends upon identifying the generalized solution of the Stokes equations with the classical solution. One may either prove Corollary 1, which implies uniqueness in a class of generalized solutions that includes the classical solution, or prove the generalized solution admits an integral representation, as suggested by Finn. In [21] the generalized and classical solutions are tacitly identified; in [12] the result of Corollary 1 is used and the author states his intention to prove it in a subsequent work. We mention that, if regularity up to the boundary is known or assumed for generalized solutions, our Corollary 1 follows by an argument of Lions [24, p. 67].

4. Uniqueness in exterion domains and in $R^n (n=2 \text{ or } 3)$

Our first step in treating questions of uniqueness in the case of an exterior domain, is to consider the special case of a domain Ω which is the exterior of a circle or a sphere. Later, at the end of this section, we prove uniqueness theorems for more general exterior domains by using results of sections 2 and 3 in conjunction with the theorems obtained first for these special cases. Because the form of the equations in spherical coordinates changes slightly with the number of spatial dimensions, we treat in detail only the cases of two and three-dimensional exterior domains; we believe the method works in any number of dimensions. Below, we will call a function $\mathbf{u} \in J_1^*(\Omega)$ which satisfies $\int_\Omega (\nabla \mathbf{u}: \nabla \boldsymbol{\phi} + \mathbf{u} \cdot \boldsymbol{\phi})\,dx = 0$ for all $\boldsymbol{\phi} \in D(\Omega)$ a generalized solution of problem (12)–(15). We begin by showing that the "pressure" gradient ∇p, for either problem (1)–(4) or problem (12)–(15), is square-summable in a neighborhood of infinity. We may assume without loss of generality that $\partial\Omega$ is the unit circle or sphere.

LEMMA 7. *Let $\Omega = \{x: x \in R^n \text{ and } |x| > 1\}$. If $\mathbf{u}$ is a generalized solution of problem* (1)–(4) *in Ω, and if p is a corresponding pressure function as found in Proposition* 5, *then $\int_{|x|>3} (\nabla p)^2 dx < \infty$. If $\mathbf{u}$ is a generalized solution of problem (12)–(15) in Ω, and if p is a corresponding "pressure" function as found in Proposition 3, then $\int_{|x|>3} (\nabla p)^2 dx < \infty$.*

Proof. Suppose first that $\mathbf{u}$ is a generalized solution of problem (1)–(4). Then $\mathbf{u}$ satisfies the hypotheses of Lemma 3 with $\mathbf{f}=0$. Adopting the notation of Lemma 3, let $\Omega_k'' = \{x: 3 < |x| < k\}$ and $\Omega_k' = \{x: 2 < |x| < 2k\}$, for integers $k \geq 4$. Define $\zeta_k(x)$ to be a function of $|x| = r$ by setting $\zeta_k(x) = 0$ for $1 \leq r \leq 2$, $\zeta_k(x) = (r-2)^2(7-2r)$ for $2 \leq r \leq 3$, $\zeta_k(x) = 1$ for $3 \leq r \leq k$, $\zeta_k(x) = k^{-3}(r-2k)^2(2r-k)$ for $k \leq r \leq 2k$, and $\zeta_k(x) = 0$ for $r \geq 2k$. Since for every k, $\max |\nabla \zeta_k| = 3/2$, we have

$$\|\mathbf{u}_{x_i x_j}\|_{\Omega_k''} \leq C_1 \|\nabla \mathbf{u}\| + \|(\Delta \zeta_k)\mathbf{u}\|_{\Omega_k'} \tag{32}$$

by Lemma 3, with a constant C_1 which is independent of k. It is easy to check that there exist constants C_2 and C_3, independent of k, such that $|\Delta \zeta_k| < C_2$ for all $2 \leq |x| \leq 3$, and such that $|\Delta \zeta_k| \leq C_3/4k^2 \leq C_3/r^2$ for all x satisfying $k \leq r \leq 2k$. Thus, if C_4 is the maximum of the two numbers $81 C_2^2$ and C_3^2, we have

$$\|(\Delta \zeta_k)\mathbf{u}\|_{\Omega_k'}^2 \leq \int_{2 \leq |x| \leq 3} C_2^2 \mathbf{u}^2\,dx + \int_{k \leq |x| \leq 2k} C_3^2 \mathbf{u}^2/r^4\,dx \leq C_4 \int_{2 \leq |x|} \mathbf{u}^2/r^4\,dx. \tag{33}$$

In the proof of Lemma 4, inequalities (28) were obtained for functions belonging to the completion of $C_0^\infty(\Omega)$ in norm $\|\nabla \cdot\|$. Thus the right sides of (33) and (32) are bounded by constants independent of k. Letting $k \to \infty$ we obtain $\int_{|x| \geq 3} u_{x_i x_j}^2 dx < \infty$. Since $\nabla p = \Delta \mathbf{u}$, this proves the first part of Lemma 7.

If $\mathbf{u}$ is a generalized solution of (12)–(15), then $\mathbf{u}$ satisfies the hypotheses of Lemma 3 with $\mathbf{f}=-\mathbf{u}$. Thus

$$\|\mathbf{u}_{x_i x_j}\|_{\Omega_k''} \leqslant C_1 \|\nabla \mathbf{u}\| + C_1 \|\mathbf{u}\| + \|(\Delta\zeta_k)\,\mathbf{u}\|_{\Omega_k'}. \tag{34}$$

Since $\mathbf{u}\in L^2(\Omega)$, the right side is obviously bounded by a constant which is independent of k. The argument is completed as before.

THEOREM 4. *If $\Omega=\{x\colon x\in R^2$ and $|x|>1\}$, then the only generalized solution of (1)–(4) is $\mathbf{u}=0$, and hence $J_0^*(\Omega)=J_0(\Omega)$.*

Proof. We introduce polar coordinates $x_1=r\cos\theta$, $x_2=r\sin\theta$. The radial and angular components of $\mathbf{u}$ are related to the cartesian components by

$$u_1=u_r\cos\theta-u_\theta\sin\theta,\quad u_2=u_r\sin\theta+u_\theta\cos\theta$$

and the polar expressions for the derivatives of a function p are

$$\frac{\partial p}{\partial x_1}=\frac{\partial p}{\partial r}\cos\theta-\frac{\partial p}{\partial \theta}\frac{\sin\theta}{r},\quad \frac{\partial p}{\partial x_2}=\frac{\partial p}{\partial r}\sin\theta+\frac{\partial p}{\partial \theta}\frac{\cos\theta}{r}.$$

Thus the Stokes equations $\Delta u_1=\partial p/\partial x_1$, $\Delta u_2=\partial p/\partial x_2$ become

$$\Delta(u_r\cos\theta-u_\theta\sin\theta)=\frac{\partial p}{\partial r}\cos\theta-\frac{\partial p}{\partial\theta}\frac{\sin\theta}{r}$$

$$\Delta(u_r\sin\theta+u_\theta\cos\theta)=\frac{\partial p}{\partial r}\sin\theta+\frac{\partial p}{\partial\theta}\frac{\cos\theta}{r}.$$

By eliminating $\partial p/\partial\theta$ in the obvious way, and using the polar expression for the Laplacian

$$\Delta\zeta=\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2},$$

one obtains

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{r^2}=\frac{\partial p}{\partial r}. \tag{35}$$

The equation $\nabla\cdot\mathbf{u}=0$ becomes in polar coordinates

$$\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}=0. \tag{36}$$

By using (36) we can eliminate from (35) the term involving u_θ, thus obtaining

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{3}{r}\frac{\partial u_r}{\partial r}+\frac{u_r}{r^2}=\frac{\partial p}{\partial r}. \tag{37}$$

Now since p is harmonic in the region $r>1$, it has a series expansion of the form

$$p(r,\theta)=\alpha_0+\tilde\alpha_0\log r+\sum_{n=1}^{\infty}\left[(\alpha_n r^{-n}+\tilde\alpha_n r^n)\cos n\theta+(\beta_n r^{-n}+\tilde\beta_n r^n)\sin n\theta\right]$$

which may be differentiated term by term. One readily finds that

$$\int_{3\leq|x|\leq R}\left(\frac{\partial p}{\partial r}\right)^2 r d\theta\, dr = \int_3^R \tilde{\alpha}_0^2 r^{-1}\, dr + \pi \sum_{n=1}^{\infty} \int_3^R (-n\alpha_n r^{-n-1} + n\tilde{\alpha}_n r^{n-1})^2 r\, dr$$
$$+ \pi \sum_{n=1}^{\infty} \int_3^R (-n\beta_n r^{-n-1} + n\tilde{\beta}_n r^{n-1})^2 r\, dr.$$

Since $\int_{|x|\geq 3} (\nabla p)^2 dx < \infty$, it follows that the coefficients of all positive powers of r and of $\log r$ in the expansion of p must be zero. Thus we have

$$\frac{\partial p}{\partial r} = \sum_{n=1}^{\infty} (-n\alpha_n \cos n\theta - n\beta_n \sin n\theta)\, r^{-n-1}. \tag{38}$$

Since $\mathbf{u} \in C^\infty(\Omega)$, the series for the radial component of $\mathbf{u}$,

$$u_r(r, \theta) = a_0(r) + \sum_{n=1}^{\infty} (a_n(r) \cos n\theta + b_n(r) \sin n\theta), \tag{39}$$

can be differentiated term by term. Substituting (38) and (39) into (37) gives differential equations for the coefficients of u_r:

$$\begin{aligned} a_0''(r) + 3r^{-1} a_0'(r) + r^{-2} a_0(r) &= 0 \\ a_n''(r) + 3r^{-1} a_n'(r) + r^{-2}(1-n^2) a_n(r) &= -n\alpha_n r^{-n-1} \\ b_n''(r) + 3r^{-1} b_n'(r) + r^{-2}(1-n^2) b_n(r) &= -n\beta_n r^{-n-1}. \end{aligned} \tag{40}$$

Since both u_r and u_θ vanish in a generalized sense on the circle $r = 1$, it follows formally from (36) that $\partial u_r/\partial r = 0$ on the circle $r = 1$, and therefore that the coefficients may be defined on the interval $1 \leq r < \infty$, and will satisfy at $r = 1$ the initial conditions:

$$a_0(1) = a_0'(1) = 0, \quad a_n(1) = a_n'(1) = 0, \quad b_n(1) = b_n'(1) = 0. \tag{41}$$

This argument is easily justified. Since $\mathbf{u}$ belongs to the completion of $C_0^\infty(\Omega)$ in norm $\|\nabla \cdot\|$, one has by a Poincaré type inequality, that the integral $\int_{|x|=r} \mathbf{u}^2 ds \to 0$ as $r \to 1$. Thus the coefficients $a_n(r)$ and $b_n(r)$ in the expansion of u_r each converge to zero as $r \to 1$, and so do the coefficients, say $A_n(r)$ and $B_n(r)$, in the expansion of u_θ. Now (36) implies that $a_0'(r) = -a_0(r) r^{-1}$, $a_n'(r) = -nB_n(r) r^{-1} - a_n(r) r^{-1}$, and $b_n'(r) = nA_n(r) r^{-1} - b_n(r) r^{-1}$. This proves (41).

Clearly $a_0(r) \equiv 0$. We will now show that if any one of the coefficients α_n or β_n in the expansion for the pressure is nonzero, then the corresponding coefficient $a_n(r)$ or $b_n(r)$ of u_r is a monotonically increasing or decreasing function of r, and consequently that $\|\nabla \mathbf{u}\|$ is not finite. To see that the Dirichlet integral of $\mathbf{u}$ will not be finite if one of the coefficients is strictly monotone, observe that $(\nabla \mathbf{u})^2 \geq (\partial u_r / r\partial\theta)^2$ and that

$$\int_1^\infty \int_0^{2\pi} \left(\frac{\partial u_r}{r\partial\theta}\right)^2 r\, d\theta\, dr = \pi \int_1^\infty \sum_{n=1}^{\infty} \left(\frac{n^2 a_n^2(r)}{r} + \frac{n^2 b_n^2(r)}{r}\right) dr.$$

The following lemma concerns equations of the form (40); it shows that coefficients of u_r corresponding to nonzero coefficients of p are strictly monotone. Since this is impossible, it follows that ∇p and hence $\mathbf{u}$ vanish identically. That $J_0^*(\Omega)=J_0^*(\Omega)$ follows from Proposition 4.

LEMMA 8. *Let $\phi(t)$ be the solution of the initial value problem $\phi''(t)+a(t)\phi'(t)-b(t)\phi(t)=c(t)$, $\phi(0)=\phi'(0)=0$, where it is assumed that $a(t)$, $b(t)$, and $c(t)$ are continuous functions of $t\geqslant 0$, and that $b(t)\geqslant 0$ for all $t\geqslant 0$. Then $\phi'(t)>0$ for all $t>0$, if $c(t)>0$ for all $t\geqslant 0$. Similarly, $\phi'(t)<0$ for all $t>0$, if $c(t)<0$ for all $t\geqslant 0$.*

Proof. Suppose that $c(t)>0$ for all $t\geqslant 0$. Since $\phi''(t)>0$ wherever $a(t)\phi'(t)<c(t)+b(t)\phi(t)$, and since ϕ and ϕ' are continuous functions which vanish at $t=0$, it follows that $\phi''(t)$ is positive on some initial t-interval $[0,\delta)$. Evidently ϕ' and ϕ are positive on $(0,\delta)$. We claim $\phi'(t)>0$ for all $t>0$. If not, there must be a first $t>0$, say t^*, at which $\phi'(t)=0$. Certainly $\phi(t^*)$ is positive, so $a(t^*)\phi'(t^*)<c(t^*)+b(t^*)\phi(t^*)$, and by continuity $a(t)\phi'(t)<c(t)+b(t)\phi(t)$ must hold in some neighborhood of t^*. Thus in some interval $[t^*-\varepsilon, t^*]$ we have $\phi'(t)>0$ and $\phi''(t)>0$. It follows that $\phi'(t^*)>0$.

THEOREM 5. *If $\Omega=\{x: x\in R^2 \text{ and } |x|>1\}$, then the only generalized solution of* (12)–(15) *is $\mathbf{u}=0$, and hence $J_1^*(\Omega)=J_1(\Omega)$.*

Proof. We introduce polar coordinates as in the proof of Theorem 4. The equations $\Delta u_1-u_1=\partial p/\partial x_1$, and $\Delta u_2-u_2=\partial p/\partial x_2$ become

$$\Delta(u_r\cos\theta-u_\theta\sin\theta)-(u_r\cos\theta-u_\theta\sin\theta)=\frac{\partial p}{\partial r}\cos\theta-\frac{\partial p}{\partial\theta}\frac{\sin\theta}{r},$$

$$\Delta(u_r\sin\theta+u_\theta\cos\theta)-(u_r\sin\theta+u_\theta\cos\theta)=\frac{\partial p}{\partial r}\sin\theta+\frac{\partial p}{\partial\theta}\frac{\cos\theta}{r}.$$

Multiplying the first of these equations through by $\cos\theta$, and the second equation through by $\sin\theta$, and then adding and using the polar expression for the Laplacian, we obtain

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\left(1+\frac{1}{r^2}\right)u_r=\frac{\partial p}{\partial r}. \tag{42}$$

The term involving u_θ can be eliminated, as before, by using the polar form (36) of $\nabla\cdot\mathbf{u}=0$. Thus

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{3}{r}\frac{\partial u_r}{\partial r}+\left(\frac{1}{r^2}-1\right)u_r=\frac{\partial p}{\partial r}. \tag{43}$$

Just as in the case of the Stokes equations, the "pressure" p and the radial component of $\mathbf{u}$ have series expansions (38) and (39). Substituting (38) and (39) into (43) gives differential equations for the coefficients of u_r:

$$\begin{aligned} &a_0''(r)+3r^{-1}a_0'(r)+(r^{-2}-1)a_0(r)=0,\\ &a_n''(r)+3r^{-1}a_n'(r)+r^{-2}(1-n^2-r^2)a_n(r)=-n\alpha_n r^{-n-1},\\ &b_n''(r)+3r^{-1}b_n'(r)+r^{-2}(1-n^2-r^2)b_n(r)=-n\beta_n r^{-n-1}. \end{aligned} \tag{44}$$

As in the case of the Stokes equations, these equations for the coefficients of u_r are of the form considered in Lemma 8, the coefficients are defined on $1\leqslant r<\infty$, and the coefficients satisfy initial conditions (41). Therefore both ∇p and $\mathbf{u}$ vanish identically, because otherwise the Dirichlet integral would not be finite. It follows from Proposition 2 that $J_1^*(\Omega)=J_1(\Omega)$.

THEOREM 6. *If* $\Omega=\{x\colon x\in R^3$ *and* $|x|>1\}$, *then the only generalized solution of* (1)–(4) *is* $\mathbf{u}=0$, *and hence* $J_0^*(\Omega)=J_0(\Omega)$.

Proof. We introduce spherical coordinates $x_1=r\cos\phi\sin\theta$, $x_2=r\sin\phi\sin\theta$, $x_3=r\cos\theta$. The radial and angular components of $\mathbf{u}$ are related to the cartesian components by

$$\begin{aligned} u_1&=u_r\sin\theta\cos\phi+u_\theta\cos\theta\cos\phi-u_\phi\sin\phi,\\ u_2&=u_r\sin\theta\sin\phi+u_\theta\cos\theta\sin\phi+u_\phi\cos\phi,\\ u_3&=u_r\cos\theta-u_\theta\sin\theta. \end{aligned}$$

The derivatives of a function p may be expressed as

$$\begin{aligned} \frac{\partial p}{\partial x_1}&=\frac{\partial p}{\partial r}\cos\phi\sin\theta+\frac{\partial p}{\partial\theta}\frac{\cos\phi\cos\theta}{r}-\frac{\partial p}{\partial\phi}\frac{\sin\phi}{r\sin\theta},\\ \frac{\partial p}{\partial x_2}&=\frac{\partial p}{\partial r}\sin\phi\sin\theta+\frac{\partial p}{\partial\theta}\frac{\sin\phi\cos\theta}{r}+\frac{\partial p}{\partial\phi}\frac{\cos\phi}{r\sin\theta},\\ \frac{\partial p}{\partial x_3}&=\frac{\partial p}{\partial r}\cos\theta\qquad-\frac{\partial p}{\partial\theta}\frac{\sin\theta}{r}. \end{aligned}$$

Multiplying the equation $\Delta u_1=\partial p/\partial x_1$ by $\sin\theta\cos\phi$, the equation $\Delta u_2=\partial p/\partial x_2$ by $\sin\theta$ $\sin\phi$, and the equation $\Delta u_3=\partial p/\partial x_3$ by $\cos\theta$, and then adding, we obtain

$$\frac{\partial^2u_r}{\partial r^2}+\frac{1}{r^2}\Delta^*u_r+\frac{2}{r}\frac{\partial u_r}{\partial r}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}-\frac{2u_r}{r^2}-\frac{2u_\theta\cot\theta}{r^2}=\frac{\partial p}{\partial r}. \tag{45}$$

In deriving (45) we have used the expression

$$\Delta\zeta=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\zeta}{\partial r}\right)+\frac{\partial^2\zeta}{\partial\theta^2}+\cot\frac{\partial\zeta}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2\zeta}{\partial\phi^2}\right]$$

for the Laplacian, and denoted by Δ^* the spherical part

$$\Delta^*\zeta = \frac{\partial^2 \zeta}{\partial \theta^2} + \cot\theta \frac{\partial \zeta}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 \zeta}{\partial \phi^2}$$

of the Laplacian. The equation $\nabla \cdot \mathbf{u} = 0$ can be written

$$\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{u_\theta \cot\theta}{r} = 0. \tag{46}$$

The terms involving u_θ and u_ϕ in (45) can be eliminated by multiplying equation (46) through by $2/r$ and adding the result to (45); one obtains

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\Delta^* u_r + \frac{4}{r}\frac{\partial u_r}{\partial r} + \frac{2u_r}{r^2} = \frac{\partial p}{\partial r}. \tag{47}$$

Since the pressure p is harmonic in the region $r > 1$, it has an expansion in spherical harmonics of the form

$$p(r, \theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (\tilde{\alpha}_{nm} r^n + \alpha_{nm} r^{-n-1})\, Y_{nm}(\theta, \phi). \tag{48}$$

Here the functions Y_{nm} are everywhere-regular eigenfunctions of Δ^* corresponding to eigenvalues $-n(n+1)$. That is

$$\Delta^* Y_{nm} = -n(n+1)\, Y_{nm}, \quad -n \leqslant m \leqslant n. \tag{49}$$

Since $\int_{|x|>3} (\partial p/\partial r)^2 dx < \infty$ by Lemma 7, and since the functions Y_{nm} are orthogonal on the sphere, it is easy to see that the coefficients $\tilde{\alpha}_{nm}$ must vanish for $n > 0$. Thus we have

$$\frac{\partial p}{\partial r} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (-n-1)\,\alpha_{nm} r^{-n-2} Y_{nm}(\theta, \phi). \tag{50}$$

Since $\mathbf{u} \in C^\infty(\Omega)$, we can expand u_r in spherical harmonics

$$u_r(r, \theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}(r)\, Y_{nm}(\theta, \phi), \tag{51}$$

and the derivatives of u_r may be found by term-by-term differentiation of the series. Substituting (50) and (51) into (47), and using (49), we obtain differential equations for the coefficients of u_r:

$$a''_{nm}(r) + 4r^{-1}a'_{nm}(r) + r^{-2}(2 - n(n+1))\,a_{nm}(r) = -(n+1)\,\alpha_{nm} r^{-n-2} \tag{52}$$

Since $\mathbf{u}$ vanishes in a generalized sense on the sphere $r = 1$, it follows formally from (46) that both $\partial u_r/\partial r$ and u_r vanish on the sphere $r = 1$. Just as in the proof of Theorem 4, one can prove that the coefficients are defined on the interval $1 \leqslant r < \infty$ and satisfy initial conditions

$$a_{nm}(1) = a'_{nm}(1) = 0. \tag{52}$$

It follows from Lemma 8 that each coefficient $a_{nm}(r)$, except $a_{00}(r)$, either vanishes identically (if $\alpha_{nm}=0$) or is strictly monotone (if $\alpha_{nm}\neq 0$). Now, as we may assume that the functions Y_{nm} are orthonormal over the unit sphere, we have

$$\int_{|x|>1} \frac{u_r^2}{r^2}\,dx = \int_1^\infty \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm}^2(r)\,dr. \tag{53}$$

Since $\int_\Omega \mathbf{u}^2/r^2 dx < \infty$ holds for functions $\mathbf{u}$ which belong to the completion in norm $\|\nabla\cdot\|$ of $C_0^\infty(\Omega)$, as was shown in the proof of Lemma 4, it follows that all the coefficients $a_{nm}(r)$, except possibly $a_{00}(r)$, vanish identically. But $a_{00}(r)$ also vanishes identically, because the conditions $\mathbf{u}=0$ on the sphere $r=1$, and $\nabla\cdot\mathbf{u}=0$ in Ω, together imply that

$$0 = \int_{|x|=1} \mathbf{u}\cdot\mathbf{n}\,ds = \int_{|x|=r} u_r\,ds = a_{00}(r)\sqrt{4\pi}\,r^2. \tag{54}$$

Thus $\mathbf{u}$ and ∇p vanish identically, and Proposition 4 implies that $J_0^*(\Omega) = J_0(\Omega)$.

THEOREM 7. *If* $\Omega = \{x\colon x\in R^3$ *and* $|x|>1\}$, *then the only generalized solution of* (12)–(15) *is* $\mathbf{u}=0$, *and hence* $J_1^*(\Omega)=J_1(\Omega)$.

Proof. Our argument is similar to the proof of Theorem 6. Taking an appropriate linear combination of the equations $\Delta u_1 - u_1 = \partial p/\partial x_1$, $\Delta u_2 - u_2 = \partial p/\partial x_2$, $\Delta u_3 - u_3 = \partial p/\partial x_3$, and then adding to the result $2/r$ times the expression (46) for $\nabla\cdot\mathbf{u}=0$, one obtains

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\Delta^* u_r + \frac{4}{r}\frac{\partial u_r}{\partial r} + \left(\frac{2}{r^2} - 1\right) u_r = \frac{\partial p}{\partial r}. \tag{55}$$

Again $\partial p/\partial r$ and u_r admit the expansions (50) and (51). Substituting these expansions into (55) we obtain differential equations for the coefficients of u_r:

$$a''_{nm}(r) + 4r^{-1}a'_{nm}(r) + r^{-2}(2 - n(n+1) - r^2)a_{nm}(r) = -(n+1)\,\alpha_{nm} r^{-n-2}. \tag{56}$$

We show just as in the proof of Theorem 6 that the coefficients are defined for $1\leqslant r<\infty$, satisfy initial conditions (52), and must therefore vanish, because otherwise either the Dirichlet integral would not be finite or the condition $\nabla\cdot\mathbf{u}=0$ would be violated. It follows from Proposition 2 that $J_1^*(\Omega)=J_1(\Omega)$.

An exterior domain Ω is usually defined to be an open set of R^n which not only contains a complete neighborhood of infinity, but also has a nonempty complement Ω^c. For our next theorem the assumption that Ω^c is nonempty is unnecessary. Even if $\Omega = R^n$, we will still speak of generalized solutions of problems (1)–(4), (7)–(11), and (12)–(15), but with the understanding that the boundary conditions (3), (10), and (14) are in effect dropped. Thus problem (7)–(11) becomes the Cauchy problem for the Navier–Stokes equations if $\Omega = R^n$. There is one significant complication. If $\Omega \subset R^2$ and if the capacity of Ω^c is zero, then elements of the spaces $J_0^*(\Omega)$ and $J_0(\Omega)$ consist of equivalence classes of functions which differ by constants; see Deny and Lions [4] and Heywood [17]. Thus, in the case that $\Omega \subset R^2$ and the capacity of Ω^c is zero, it is our convention to speak of $\mathbf{u}=0$ as the unique solution of (1)–(4) if and only if every (locally square-summable function) solution of (1)–(4) belongs to the element (equivalence class) of $J_0^*(\Omega)$ which contains $\mathbf{u}=0$. This convention applies to Lemma 4 and to Proposition 4 as well as to the following theorem.

THEOREM 8. *Suppose that Ω is an open set of R^n, $n=2$ or 3, which contains a complete neighborhood of infinity, say $\{x: |x| > R\}$ for some sufficiently large R. Suppose that the boundary of Ω, if nonempty, is sufficiently regular so that the method of section* 3 (*see Corollary* 2) *can be applied to the "anular" region $\{x: x\in\Omega$ and $|x| < R+2\}$. Then $J_0^*(\Omega)=J_0(\Omega)$ and $J_1^*(\Omega)=J_1(\Omega)$, and consequently problems* (1)–(4), (7)–(11), *and* (12)–(15) *each possess at most one generalized solution in Ω.*

Proof. We will show that $J_0^*(\Omega)=J_0(\Omega)$; the proof that $J_1^*(\Omega)=J_1(\Omega)$ is similar (and simpler if $\Omega=R^2$). Let $\mathbf{u}\in J_0^*(\Omega)$ be given; if $\Omega\subset R^2$ and if the capacity of Ω^c is zero, let $\mathbf{u}$ be a function representing an equivalence class of $J_0^*(\Omega)$. We may assume $\mathbf{u}\in C^\infty(\Omega)$, for if not we let $\mathbf{v}$ be the element of $J_0(\Omega)$ such that $\mathbf{w}=\mathbf{u}-\mathbf{v}$ satisfies $\int_\Omega \nabla\mathbf{w}\colon \nabla\boldsymbol{\phi}\,dx=0$ for all $\boldsymbol{\phi}\in D(\Omega)$; then $\mathbf{w}\in C^\infty(\Omega)\cap J_0^*(\Omega)$, and if we show that $\mathbf{w}\in J_0(\Omega)$ we will have shown that $\mathbf{u}\in J_0(\Omega)$. Now it is possible to write $\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$ where $\mathbf{u}_1\in J_0^*(\Omega\cap\{x: |x|<R+2\})$ and $\mathbf{u}_2\in J_0^*(\{x: |x|>R+1\})$. We simply use the fact [21, p. 26] that it is possible to construct a smooth divergence free vector field $\mathbf{v}$ in the region $R+1\leqslant |x| \leqslant R+2$ which equals $\mathbf{u}$ on the sphere $|x|=R+1$, and which equals $\mathbf{0}$ on the sphere $|x|=R+2$. Then we set $\mathbf{u}_1=\mathbf{u}$ in $\Omega\cap\{x: |x|\leqslant R+1\}$, $\mathbf{u}_1=\mathbf{v}$ in $R+1\leqslant|x|\leqslant R+2$, and $\mathbf{u}_1=0$ in $|x|\geqslant R+2$. We set $\mathbf{u}_2=\mathbf{u}-\mathbf{u}_1$ in Ω. It is easy to see that $\mathbf{u}_1$ and $\mathbf{u}_2$ belong to the function spaces claimed above, by appealing to Lemma 4. Thus the theorem of section 3 implies that $\mathbf{u}_1\in J_0(\Omega\cap\{x:|x|<R+2\})$, and Theorems 4 and 6 of this section imply that $\mathbf{u}_2\in J_0\{x: |x|>R+1\})$. But these last two function spaces are both subspaces of $J_0(\Omega)$; thus $\mathbf{u}_1$, $\mathbf{u}_2\in J_0(\Omega)$, and so $\mathbf{u}\in J_0(\Omega)$.

5. Uniqueness in a half-space

In this section we prove uniqueness for the boundary value problems of viscous flow in a half-space $\Omega=\{x: x\in R^n$ and $x_1>0\}$. Beyond the evident interest in considering problems of flow in a half-space, our results here will find application in section 6, where they are combined with results of sections 2 and 3 to study questions of uniqueness pertaining to problems of flow through an aperture. The basic method of this section is a Fourier transform argument which the author learned from Marvin Shinbrot. In order to justify taking Fourier transforms, we will show that the functions to which the transform is applied are square-summable in every plane which is parallel to the boundary plane. Of course $\mathbf{u}$ and $\nabla \mathbf{u}$ are square-summable over such planes if $\mathbf{u}$ belongs to either $\overset{\circ}{W}{}_2^1(\Omega)$ or $W_0(\Omega)$; in the case of the space $W_0(\Omega)$ we observe that

$$\int_{0<x_1<a} \mathbf{u}^2\,dx \leqslant a^2 \int_{0<x_1<a} (\nabla \mathbf{u})^2\,dx \tag{57}$$

holds for every $a>0$, as follows from the inequality $\int_0^a f^2(x)\,dx \leqslant a^2 \int_0^a (f'(x))^2\,dx$ for functions $f \in C^1[0,a]$ which vanish at zero.

LEMMA 9. *Let* $\Omega = \{x: x \in R^n$ *and* $x_1 > 0\}$. *Suppose that* $\mathbf{u}$ *is a generalized solution in* Ω *of either problem* (1)–(4) *or problem* (12)–(15), *and that* p *is a corresponding "pressure" function as found in Propositions* 3 *and* 5. *Then for every* $\varepsilon > 0$, *the derivatives of* $\mathbf{u}$ *of second order and of higher orders belong to* $L^2(\Omega_\varepsilon)$, *where* $\Omega_\varepsilon = \{x: x_1 > \varepsilon\}$. *The derivatives of* p *of first order and of all higher orders also belong to* $L^2(\Omega_\varepsilon)$.

Proof. For every $a > \varepsilon$, let $\Omega_{\varepsilon,a} = \{x: \varepsilon < x_1 < a\}$. We will prove $\|\mathbf{u}_{x_i x_j}\|_{\Omega_\varepsilon} < \infty$ by showing that $\|\mathbf{u}_{x_i x_j}\|_{\Omega_{\varepsilon,a}}$ is bounded by a constant independent of a. Let us consider fixed values of a and ε. Adopting the notation of Lemma 3, let, for each positive integer k, Ω_k'' be the cylinder $\Omega_k'' = \{x: \varepsilon < x_1 < a$ and $x_2^2 + \dots + x_n^2 < k^2\}$. Let Ω_k' be the larger cylinder $\Omega_k' = \{x: \varepsilon/2 < x_1 < 2a$ and $x_2^2 + \dots + x_n^2 \leqslant (k+1)^2\}$. Let ζ_k be a function which satisfies the hypotheses of Lemma 3 with respect to Ω_k'' and Ω_k'. We can construct ζ_k so that, for $x_2^2 + \dots + x_n^2 < k^2$, it depends only on x_1 and satisfies $\max_{a<x_1<2a} |\Delta\zeta_k| \leqslant C_1/a^2$; for instance, let $\zeta_k(x) = a^{-3}(x_1 - 2a)^2(2x_1 - a)$ for $x_2^2 + \dots + x_n^2 \leqslant k^2$ and $a \leqslant x_1 \leqslant 2a$. Further, we can assume that C_ζ is a bound for not only $|\nabla\zeta_k|$ but also $|\Delta\zeta_k|$ in all of Ω_k'; note that C_ζ depends on ε, but not on k, or on $a > \varepsilon$.

Now suppose $\mathbf{u}$ is a generalized solution of problem (1)–(4). Lemma 3 gives $\|\mathbf{u}_{x_i x_j}\|_{\Omega_k''} \leqslant C_\zeta \|\nabla \mathbf{u}\| + \|(\Delta\zeta_k)\mathbf{u}\|_{\Omega_k'}$. To show that $\|\mathbf{u}_{x_i x_j}\|_{\Omega_{\varepsilon,a}}$ is bounded by a constant independent of a, we only need to show that as $k \to \infty$, $\|(\Delta\zeta_k)\mathbf{u}\|_{\Omega_k'}$ remains less than a constant independent of a. We have

$$\|(\Delta\zeta_k)\mathbf{u}\|^2 = \int\limits_{\substack{\varepsilon/2<x_1<2a \\ k^2<x_2^2+\dots+x_n^2<(k+1)^2}} (\Delta\zeta_k)^2\mathbf{u}^2\,dx + \int\limits_{\substack{\varepsilon/2<x_1<\varepsilon \\ x_2^2+\dots+x_n^2<k^2}} (\Delta\zeta_k)^2\mathbf{u}^2\,dx + \int\limits_{\substack{a<x_1<2a \\ x_2^2+\dots+x_n^2<k^2}} (\Delta\zeta_k)^2\,\mathbf{u}^2\,dx. \tag{58}$$

The first integral on the right side of (58) tends to zero as $k \to \infty$, as may be seen from inequality (57) with a replaced by $2a$. The second integral on the right side of (58) is bounded by $C_\zeta^2\varepsilon^2\|\nabla\mathbf{u}\|^2$; remember that C_ζ depends upon ε but not on k or a. The third integral on the right side of (58) is bounded by $(C_1a^{-2})^2 4a^2\|\nabla\mathbf{u}\|^2$; it actually tends to zero as $a\to\infty$. This completes the proof that $\|\mathbf{u}_{x_ix_j}\|_{\Omega_\varepsilon}<\infty$. If $\mathbf{u}$ is a solution of problem (12)–(15) one gets the same result, but the argument is of course simpler. The estimates for higher order derivatives are obtained in a similar way. For instance, if $\mathbf{u}$ is a generalized solution of (1)–(4) we know that $\mathbf{u}_{x_i}$, $\mathbf{u}_{x_ix_j}\in L^2(\Omega_\varepsilon)$ for every $\varepsilon>0$, that $\nabla\cdot\mathbf{u}_{x_i}=0$, and that $\int\nabla\mathbf{u}_{x_i}\colon\nabla\phi\,dx=0$ for every $\phi\in D(\Omega)$. Thus we can argue as before, except that it is not necessary to appeal now to inequality (57). The estimates for derivatives of p follow from the relations $\Delta\mathbf{u}=\nabla\varrho$ in the case of problem (1)–(4), and $\Delta\mathbf{u}-\mathbf{u}=\nabla p$ in the case of problem (12)–(15).

THEOREM 9. *If $\Omega=\{x\colon x\in R^n$ and $x_1>0\}$, then the only generalized solution of problem (1)–(4) is $\mathbf{u}=0$, and hence $J_0^*(\Omega)=J_0(\Omega)$.*

Proof. Let $x=(x_1,\tilde{x})$ where $\tilde{x}=(x_2,\dots,x_n)$, and let $\hat{f}=\hat{f}(x_1,\xi)=\int_{R^{n-1}}e^{-i\langle\tilde{x},\xi\rangle}f(x_1,\tilde{x})\,d\tilde{x}$ where $\xi=(\xi_2,\dots,\xi_k)$. Now since p is harmonic, $\Delta(\partial p/\partial x_i)=0$; the Fourier transform of this equation is

$$\frac{\partial^2}{\partial x_1^2}\widehat{\frac{\partial p}{\partial x_i}}-|\xi|^2\widehat{\frac{\partial p}{\partial x_i}}=0. \tag{59}$$

Thus $(\partial p/\partial x_i)^\wedge=\alpha_i(\xi)e^{-|\xi|x_1}+\beta_i(\xi)e^{|\xi|x_1}$. It follows from Parseval's formula and Lemma 9 that

$$\int_\varepsilon^\infty\int_{R^{n-1}}\left|\widehat{\frac{\partial p}{\partial x_i}}\right|^2d\xi\,dx_1=(2\pi)^{n-1}\int_{\Omega_\varepsilon}\left|\frac{\partial p}{\partial x_i}\right|^2dx<\infty.$$

We see that $\beta_i(\xi)$ vanishes almost everywhere. By taking the Fourier transform of $\Delta u_i=\partial p/\partial x_i$ one obtains

$$\frac{\partial^2\hat{u}_i}{\partial x_1^2}-|\xi|^2\hat{u}_i=\widehat{\frac{\partial p}{\partial x_i}}=\alpha_i(\xi)\,e^{-|\xi|x_1}, \tag{60}$$

and this may be solved by the method of variation of parameters; we find (for $|\xi|\neq0$) that

$$\hat{u}_i=a_i(\xi)\,e^{-|\xi|x_1}+b_i(\xi)\,e^{|\xi|x_1}-\frac{\alpha_i(\xi)}{2|\xi|}x_1e^{-|\xi|x_1}. \tag{61}$$

Upon differentiating (61) with respect to x_1, one obtains

$$\frac{\partial}{\partial x_1}\hat{u}_i=-a_i(\xi)|\xi|e^{-|\xi|x_1}+b_i(\xi)|\xi|e^{|\xi|x_1}-\frac{\alpha_i(\xi)}{2|\xi|}e^{-|\xi|x_1}+\frac{\alpha_1(\xi)}{2}x_1e^{-|\xi|x_1}. \tag{62}$$

Using Parseval's formula again, we have

$$\int_0^\infty \int_{R^{n-1}} \left| \frac{\partial}{\partial x_1} \hat{u}_i \right|^2 d\xi \, dx_1 = (2\pi)^{n-1} \left\| \frac{\partial}{\partial x_1} u_i \right\|^2 < \infty,$$

from which it follows that $b_i(\xi)$ vanishes identically. Now it can be seen that $a_i(\xi)$ must vanish in (61) because $u_i(0, \tilde{x}) \equiv 0$ implies that $\hat{u}_i(0, \xi) \equiv 0$; more precisely one argues that

$$\int_{R^{n-1}} |\hat{u}_i(x_1, \xi)|^2 d\xi = (2\pi)^{n-1} \int_{R^{n-1}} |u_i(x_1, \tilde{x})|^2 d\tilde{x} \to 0 \quad \text{as} \quad x_1 \to 0.$$

Finally, we show that the $\alpha_i(\xi)$ must all vanish, and therefore so also must $\mathbf{u}$, because of (61). The Fourier transform of $\nabla \cdot \mathbf{u} = 0$ is

$$\frac{\partial}{\partial x_1} \hat{u}_1 + i\xi_2 \hat{u}_2 + \ldots + i\xi_n \hat{u}_n = 0, \tag{63}$$

and thus $(\partial/\partial x_1)\hat{u}_1(0, \xi) \equiv 0$. Therefore it is implied by (62) that at least $\alpha_1(\xi)$ vanishes identically. This means that $(\partial p/\partial x_i)^\wedge$, and so also $\partial p/\partial x_1$, vanish. We have just shown that p, being independent of x_1, is a harmonic function of the variables $\tilde{x} \in R^{n-1}$. Since, by Lemma 9, the Dirichlet integral of p as a function of $\tilde{x}$ must be finite, it follows that p is a constant. Thus the $\alpha_i(\xi)$ all vanish.

THEOREM 10. *If* $\Omega = \{x\colon x \in R^n$ *and* $x_1 > 0\}$, *then the only generalized solution of problem* (12)–(15) *is* $\mathbf{u} = 0$, *and hence* $J_1^*(\Omega) = J_1(\Omega)$.

Proof. Just as in the proof of Theorem 9, we argue on the basis of Lemma 9 that the Fourier transforms of derivatives of p are of the form $(\partial p/\partial x_i)^\wedge = \alpha_i(\xi) e^{-|\xi| x_1}$. By taking the Fourier transform of $\Delta u_i - u_i = \partial p/\partial x_i$ we obtain

$$\frac{\partial^2 \hat{u}_i}{\partial x_1^2} - (|\xi|^2 + 1)\hat{u}_i = \widehat{\frac{\partial p}{\partial x_i}} = \alpha_i(\xi) e^{-|\xi| x_1}. \tag{64}$$

The general solution of (64), found by the method of variation of parameters, is

$$\hat{u}_i = a_i(\xi) e^{-\sqrt{\xi^2+1}\, x_1} + b_i(\xi) e^{\sqrt{\xi^2+1}\, x_1} + \alpha_i(\xi) \gamma_i(\xi) e^{-|\xi| x_1}, \tag{65}$$

where

$$\gamma_i(\xi) = \frac{-1}{|\xi| + \sqrt{\xi^2+1}} \left[\frac{1}{|\xi| + \sqrt{\xi^2+1}} + \frac{1}{\sqrt{\xi^2+1} - |\xi|} \right].$$

Since $\|u\|_\Omega < \infty$, it follows from Parseval's formula that the $b_i(\xi)$ must vanish identically. The Fourier transform of $\nabla \cdot \mathbf{u} = 0$ again yields (63), from which we conclude that

$(\partial/\partial x_1)\hat{u}_1(0,\xi)\equiv 0$. In view of (65), the pair of equations $\hat{u}_1(0,\xi)\equiv 0$ and $(\partial/\partial x_1)\hat{u}_1(0,\xi)\equiv 0$ together imply that

$$a_1(\xi)+\gamma_1(\xi)\,\alpha_1(\xi)=0$$
$$-\sqrt{\xi^2+1}\,a_1(\xi)-\gamma_1(\xi)\,|\xi|\,\alpha_1(\xi)=0. \tag{66}$$

The matrix of coefficients is nonsingular, and therefore both $a_1(\xi)$ and $\alpha_1(\xi)$ vanish identically. Now, as in the proof of Theorem 9, we see that p is a harmonic function with finite Dirichlet integral in the variables $\tilde{x}\in R^{n-1}$. Thus p is a constant, the $\alpha_i(\xi)$ must vanish identically, and so by (65) $\mathbf{u}$ must also vanish identically.

We remark that the result just proved implies uniqueness for the initial boundary value problem (7)–(11) in virtue of Proposition 2.

6. Flow through an aperture—Auxiliary conditions

In this section we study problems of flow through an aperture, as described in the introduction. For the most part we consider the case of a single rigid wall which, except for apertures in the wall, divides the space of fluid into two parts. More complicated situations will be only briefly mentioned. Assumptions about the number of apertures or about the shape of the apertures are not very important, but we give complete uniqueness proofs only in cases of smoothly bounded apertures. We shall assume that

$$\Omega=\{x\in R^n\colon x_1\neq 0 \quad\text{or}\quad (x_2,x_3,\dots,x_n)\in S\},\quad\text{where} \tag{67}$$
$$S \text{ is a bounded open subset of the } x_2, x_3, \dots, x_n\text{-plane,}$$

and make further assumptions about S as needed.

LEMMA 10. *Let Ω be defined by* (67). *Let $\mathbf{u}$ belong to either $J_0(\Omega)$ or $J_1(\Omega)$. Then $\int_S\mathbf{u}\cdot\mathbf{n}\,ds=0$, where $\mathbf{n}$ denotes the unit normal vector to the surface S, and where $ds=dx_2dx_3\dots dx_n$.*

Proof. We need only consider $\mathbf{u}\in J_0(\Omega)$, because $J_1(\Omega)\subset J_0(\Omega)$. Let $\{\boldsymbol{\phi}_k\}$ be a sequence of functions in $D(\Omega)$ which converges to $\mathbf{u}$ in norm $\|\nabla\cdot\|$. Then $\{\boldsymbol{\phi}_k\}$ also converges to $\mathbf{u}$ in the L^2-norm of a bounded open subset of Ω which contains the surface S; see for instance [27, p. 20]. Thus $\lim_{k\to\infty}\int_S\boldsymbol{\phi}_k\cdot\mathbf{n}\,ds=\int_S\mathbf{u}\cdot\mathbf{n}\,ds$; see [27, p. 15]. However, since each $\boldsymbol{\phi}_k$ has compact support, we have $\int_S\boldsymbol{\phi}_k\cdot\mathbf{n}\,ds=\int_{x_1<0}\nabla\cdot\boldsymbol{\phi}_k\,dx=0$.

LEMMA 11. *Let Ω be defined by* (67). *Assume* (*without any real loss of generality*) *that S contains the unit disc $x_2^2+\dots+x_n^2<1$. If $\Omega\subset R^n$ with $n>2$, then there exists a vector field $\mathbf{b}(x)$ which belongs to both $J_0^*(\Omega)$ and $J_1^*(\Omega)$, and which satisfies $\int_S\mathbf{b}\cdot\mathbf{n}\,ds=1$. If $\Omega\subset R^2$, then there exists a vector field $\mathbf{b}(x)\in J_0^*(\Omega)$ which satisfies $\int_S\mathbf{b}\cdot\mathbf{n}\,ds=1$. In either case, $\mathbf{b}\in C^\infty(\Omega)$ and the* *support of* $\mathfrak{b}$ *is confined to the set* $(x_2^2+\ldots+x_n^2)^{\frac{1}{2}}\leq 2^{-\frac{1}{2}}+|x_1|$. *Finally, if* $\Omega\subset R^n$, *then* $|\mathfrak{b}(x)|<C|x|^{-n+1}$, $|\nabla b(x)|<C|x|^{-n}$, *and* $|\Delta\mathfrak{b}(x)|<C|x|^{-n-1}$.

Proof. A suitable function $\mathfrak{b}(x)$ may be constructed as follows. Let θ be the angle between the positive x_1-axis and the ray joining a point x with the origin. Let $\tilde{\mathfrak{b}}(x)$ be defined by

$$\tilde{\mathfrak{b}}(x)=\begin{cases}(\cos 2\theta)^2|x|^{-n}(x_1,x_2,\ldots,x_n) & \text{for}\quad 0\leq\theta\leq\frac{1}{4}\pi\\ 0 & \text{for}\quad \frac{1}{4}\pi\leq\theta\leq\frac{3}{4}\pi\\ -(\cos 2\theta)^2|x|^{-n}(x_1,x_2,\ldots,x_n) & \text{for}\quad \frac{3}{4}\pi\leq\theta\leq\pi.\end{cases}$$

Let $\omega(x)$ be an averaging kernel; assume that $\omega\in C_0^\infty(|x|<\frac{1}{2})$ and that $\int_{|x|<\frac{1}{2}}\omega(x)\,dx=1$. We define $\mathfrak{b}(x)$ by setting $\mathfrak{b}(x)=\beta\int_{|y|<\frac{1}{2}}\tilde{\mathfrak{b}}(x+y)\omega(y)\,dy$, where β is a normalizing constant chosen so that $\int_S\mathfrak{b}\cdot\mathfrak{n}\,ds=1$. All of the assertions of the lemma are immediately apparent.

LEMMA 12. *Let* Ω *be defined by* (67), *with* S *the unit disc* $x_2^2+\ldots+x_n^2<1$. *If* $\mathfrak{u}\in J_1^*(\Omega)$ *satisfies* $\int_S\mathfrak{u}\cdot\mathfrak{n}\,ds=0$, *then* $\mathfrak{u}\in J_1(\Omega)$. *If* $\mathfrak{u}\in J_0^*(\Omega)$ *satisfies* $\int_S\mathfrak{u}\cdot\mathfrak{n}\,ds=0$, *then* $\mathfrak{u}\in J_0(\Omega)$. *Thus two functions* $\mathfrak{u}$, $\tilde{\mathfrak{u}}\in J_1^*(\Omega)$ *belong to the same coset of* $J_1^*(\Omega)/J_1(\Omega)$ *if and only if* $\int_S\mathfrak{u}\cdot\mathfrak{n}\,ds=\int_S\tilde{\mathfrak{u}}\cdot\mathfrak{n}\,ds$, *and two functions* $\mathfrak{u}$, $\tilde{\mathfrak{u}}\in J_0^*(\Omega)$ *belong to the same coset of* $J_0^*(\Omega)/J_0(\Omega)$ *if and only if* $\int_S\mathfrak{u}\cdot\mathfrak{n}\,ds=\int_S\tilde{\mathfrak{u}}\cdot\mathfrak{n}\,ds$.

Proof. Suppose that $\mathfrak{u}\in J_1^*(\Omega)$, and that $\int_S\mathfrak{u}\cdot\mathfrak{n}\,ds=0$. Because the total flux of $\mathfrak{u}$ across S vanishes, it is possible to construct a divergence free vector field $\mathfrak{v}\in\mathring{W}_2^1(D)$, where $D=\{x\colon |x|<1\}$, which equals $\mathfrak{u}$ on S in the sense of traces. In fact one can find a suitable function $\mathfrak{v}$, by finding separately its restriction $\mathfrak{v}^+$ to $D^+=\{x\colon x_1>0 \text{ and } |x|<1\}$, and its restriction $\mathfrak{v}^-$ to $D^-=\{x\colon x_1<0 \text{ and } |x|<1\}$, using for instance the method given in [21, p. 26]. Clearly $\mathfrak{v}\in J_1(D)$, by the result of section 3. Now consider the restriction $\mathfrak{u}^+$ of $\mathfrak{u}$ to $\Omega^+=\{x\colon x_1>0\}$, and the restriction $\mathfrak{u}^-$ of $\mathfrak{u}$ to $\Omega^-=\{x\colon x_1<0\}$. Clearly $\mathfrak{u}^+-\mathfrak{v}^+$ belongs to $J_1^*(\Omega^+)$, if we set $\mathfrak{v}^+$ equal to zero outside D^+. Thus by Theorem 10, we have $\mathfrak{u}^+-\mathfrak{v}^+\in J_1(\Omega^+)$. Similarly $\mathfrak{u}^--\mathfrak{v}^-\in J_1(\Omega^-)$. Now $J_1(\Omega^+)$, $J_1(\Omega^-)$, and $J_1(D)$ are all subspaces of $J_1(\Omega)$. Therefore $\mathfrak{u}=(\mathfrak{u}^+-\mathfrak{v}^+)+(\mathfrak{u}^--\mathfrak{v}^-)+\mathfrak{v}\in J_1(\Omega)$. The proof for the spaces $J_0^*(\Omega)$ and $J_0(\Omega)$ is exactly the same.

THEOREM 11. *Let* Ω *be defined by* (67), *with* S *the unit disc* $x_2^2+\ldots+x_n^2<1$. *Then for any prescribed number* F *there is a unique generalized solution* $\mathfrak{u}$ *of problem* (1)–(5).

Proof. Let $\mathfrak{b}$ be the vector field constructed in Lemma 11. Clearly $\int_\Omega\nabla\mathfrak{b}\colon\nabla\boldsymbol{\phi}\,dx$ defines a bounded linear functional on $\boldsymbol{\phi}\in J_0(\Omega)$. Let $\mathfrak{v}$ be the unique element of $J_0(\Omega)$ such that $\int_\Omega\nabla\mathfrak{v}\colon\nabla\boldsymbol{\phi}\,dx=-\int_\Omega\nabla\mathfrak{b}\colon\nabla\boldsymbol{\phi}\,dx$ holds for all $\boldsymbol{\phi}\in J_0(\Omega)$. Then $\mathfrak{b}+\mathfrak{v}$ is a generalized solution of problem (1)–(4) and $\int_S(\mathbf{b}+\mathbf{v})\cdot\mathbf{n}ds=1$, as follows from Lemmas 10 and 11. Thus $\mathbf{u}=F\cdot(\mathbf{b}+\mathbf{v})$ is a solution of (1)–(5). If $\mathbf{u}$ and $\bar{\mathbf{u}}$ are two solutions of problem (1)–(5), then they belong to the same coset of $J_0^*(\Omega)/J_0(\Omega)$ by Lemma 12, and consequently are identical by Proposition 4.

The next theorem is particularly important because it implies, that by modeling flow through an aperture on the basis of the Stokes equations, one can predict the net flux through an aperture from knowledge of the pressure drop from one side of the wall to the other.

THEOREM 12. *Let Ω be defined by* (67), *with S the unit disc $x_2^2+\ldots+x_n^2<1$. Assume $n>2$. Let $\mathbf{u}$ be a generalized solution of* (1)–(4) *in Ω, and let p be a corresponding pressure function as found in Proposition* 5. *Then there exist constants p_1 and p_2 such that* (6) *is satisfied in the sense that*

$$\int_{x_1<-1}\frac{(p(x)-p_1)^2}{|x|^2}dx<\infty \quad\text{and}\quad \int_{x_1>1}\frac{(p(x)-p_2)^2}{|x|^2}dx<\infty. \tag{68}$$

If $p_1=p_2$, then $\mathbf{u}\equiv 0$. For every prescribed pressure drop p_2-p_1 there exists a unique corresponding solution $\mathbf{u}$ of problem (1)–(4).

Proof. Modifying Lemma 9 very slightly we see that $\int_{x_1<-1}(\nabla p)^2dx<\infty$ and that $\int_{x_1>1}(\nabla p)^2dx<\infty$. Now let

$$\tilde{p}(x)=\begin{cases}p(x) & \text{for } x_1\geqslant 1\\ p(2-x_1,x_2,\ldots,x_n) & \text{for } x_1<1.\end{cases}$$

Clearly $\tilde{p}$ is defined and continuous in R^n and has a finite Dirichlet integral. By the proof of Lemma 4 (the inequality of Finn), we thus have $\int_{R^n}[(\tilde{p}(x)-c)^2/|x|^2]dx<\infty$ for some constant c. Since $\tilde{p}(x)=p(x)$ for $x_1>1$, we have proved the second inequality of (68) with $p_2=c$. The proof of the first inequality is similar.

Now suppose that $p_1=p_2$. Without loss of generality we may take $p_1=p_2=0$. Let $\mathbf{b}$ be the vector field constructed in Lemma 11. For some number α we have $\int_S\alpha\mathbf{b}\cdot\mathbf{n}ds=\int_S\mathbf{u}\cdot\mathbf{n}ds$. Since, for this α, $\alpha\mathbf{b}$ and $\mathbf{u}$ belong to the same coset of $J_0^*(\Omega)/J_0(\Omega)$, we have $\mathbf{u}=\alpha\mathbf{b}+\mathbf{v}$ for some $\mathbf{v}\in J_0(\Omega)$. By Proposition 5, $\mathbf{u}$ and p are smooth in Ω and satisfy $\Delta\mathbf{u}=\nabla p$. Therefore

$$\int_{|x|<R}\Delta\mathbf{u}\cdot\mathbf{b}\,dx=\int_{|x|<R}\nabla p\cdot\mathbf{b}\,dx \tag{69}$$

for every number R; remember that $\mathbf{b}$ is smooth and has support confined to $(x_2^2+\ldots+x_n^2)^{\frac{1}{2}}<$

$2^{-\frac{1}{2}}+|x_1|$. We can integrate (69) by parts to obtain

$$-\int_{|x|<R}\nabla\mathbf{u}:\nabla\mathbf{b}\,dx+\int_{|x|=R}\mathbf{b}\cdot\nabla\mathbf{u}\cdot\mathbf{n}\,ds=-\int_{|x|<R}p(\nabla\cdot\mathbf{b})\,dx+\int_{|x|=R}p\mathbf{b}\cdot\mathbf{n}\,ds. \tag{70}$$

Now since both $\mathbf{b}$ and $\nabla\mathbf{u}$ are square-summable over Ω, the surface integral on the left tends to zero as $R\to\infty$. The first integral on the right vanishes. The surface integral on the right tends to zero as $R\to\infty$ because

$$\left|\int_{|x|=R}p\mathbf{b}\cdot\mathbf{n}\,ds\right|\leqslant\left(\int_{|x|=R}\frac{p^2}{|x|^2}ds\right)^{1/2}\left(\int_{|x|=R}\left(\frac{C}{|x|^{n-2}}\right)^2ds\right)^{1/2}. \tag{71}$$

Here we have used the fact that $|\mathbf{b}(x)|<C|x|^{-n+1}$, and applied the Schwarz inequality. The second factor on the right of (71) remains at least bounded as $R\to\infty$; the first factor tends to zero because $\int_\Omega(p^2/|x|^2)dx<\infty$. Thus we have proved that $\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{b}\,dx=0$. Since $\mathbf{u}$ is a generalized solution of (1)–(4) and since $\mathbf{v}\in J_0(\Omega)$, we have $\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}\,dx=0$. Therefore $\int_\Omega(\nabla\mathbf{u})^2dx=\int_\Omega\nabla\mathbf{u}:\nabla(\alpha\mathbf{b}+\mathbf{v})\,dx=0$. We have proved that if the pressure drop is zero, then $\mathbf{u}$ vanishes. Since there exist nontrivial solutions of (1)–(4) in Ω with corresponding nonzero pressure drops, it follows from the linearity of problem (1)–(4) that there is a linear (hence one-to-one) correspondence between values of F in condition (5) and values of p_2-p_1 in condition (6).

It seems likely that Lemma 12 and hence Theorems 11 and 12 remain valid even if S is an arbitrary open bounded subset of the $x_2, \ldots, x_n$-plane. It would, however, require a considerable technical effort to prove this. In the next theorem we consider the case of a single wall with several possibly noncircular apertures.

THEOREM 13. *Let Ω be defined by* (67), *where S consists of a finite number of smoothly bounded disjoint open subsets of the $x_2, \ldots, x_n$-plane. Then all the conclusions of Lemmas* 10, 11 (*modified in an obvious way*), *and* 12, *and of Theorems* 11 *and* 12 *hold.*

Proof. Clearly we only need to be concerned about the conclusion of Lemma 12. If S consists of a single noncircular aperture, one can repeat the proof of Lemma 12, but with D defined to be the cylinder $D=\{x\colon -1<x_1<1 \text{ and } (x_2, \ldots, x_n)\in S\}$ instead of a sphere. The reader is referred again to [21, p. 26] for a method of constructing $\mathbf{v}\in J_1^*(D)$. Now suppose that S consists of two smoothly bounded regions, S_1 and S_2, of the $x_2, \ldots, x_n$-plane. Let $\mathbf{u}\in J_1^*(\Omega)$ be given, and let it satisfy $\int_S\mathbf{u}\cdot\mathbf{n}\,ds=0$. One can construct a tubular domain $T\subset\subset\Omega$ which links the two apertures and which has a smooth boundary ∂T. Let $\Gamma_1=T\cap S_1$ and let $\Gamma_2=T\cap S_2$. Now a vector field $\mathbf{w}\in J_1^*(T)$ can be constructed such that $\int_{\Gamma_1}\mathbf{w}\cdot\mathbf{n}\,ds=\int_{S_1}\mathbf{u}\cdot\mathbf{n}\,ds$. Clearly $\int_{\Gamma_2}\mathbf{w}\cdot\mathbf{n}\,ds=\int_{S_2}\mathbf{u}\cdot\mathbf{n}\,ds$ also holds. Now as in the case of a single aperture, let $D_i=\{x\colon -1<x_1<1 \text{ and } (x_2, \ldots, x_n)\in S_i\}$, and let $\mathbf{v}_i\in J_1^*(D_i)$ be constructed to satisfy $\mathbf{v}_i = \mathbf{u} - \mathbf{w}$ on S_i, $i=1$ or 2. Clearly $\mathbf{u}^+ - \mathbf{w}^+ - \mathbf{v}_1^+ - \mathbf{v}_2^+ \in J_1^*(\Omega^+)$ and $\mathbf{u}^- - \mathbf{w}^- - \mathbf{v}_1^- - \mathbf{v}_2^- \in J_1^*(\Omega^-)$, where we denote restrictions as in the proof of Lemma 12. By Theorems 3 and 10 we know that $J_1^*(T) \subset J_1(T)$, $J_1^*(D_i) \subset J_1(D_i)$, $J_1^*(\Omega^+) \subset J_1(\Omega^+)$ and $J_1^*(\Omega^-) \subset J_1(\Omega^-)$. Thus $\mathbf{u} \in J_1(\Omega)$, because it can be written as a sum of functions, each of which belongs to a subspace of $J_1(\Omega)$. This method extends to any finite number of apertures by an obvious induction argument. The proof for the spaces $J_0^*(\Omega)$ and $J_0(\Omega)$ is exactly the same.

The following statement is a corollary principally of Theorem 12.

THEOREM 14. *Suppose that $\Omega \subset R^n$, with $n > 2$, is defined as in Theorem 13; and consider various solutions of problem (1)–(4). Suppose that for a given pressure drop $p_2 - p_1$, there is a net flux F through the aperture (or apertures) S, as determined by Theorem 13. Then, if the pressure drop is held constant, and if the aperture S is replaced by the similarly shaped aperture $S_\varrho = \{\varrho x: x \in S\}$, $\varrho > 0$, the net flux through S_ϱ will be $\varrho^n F$. Thus, in the case of a three dimensional flow, the net flux through an aperture is proportional to the cube of its diameter.*

Proof. Let $\mathbf{u}(x)$ be the flow with net flux F through S, for the given pressure drop $p_2 - p_1$, and let $p(x)$ be a corresponding pressure. Thus $\Delta \mathbf{u} = \nabla p$. Let $\mathbf{v}$ and q be defined in $\Omega_\varrho = \{\varrho x: x \in \Omega\}$ by setting $\mathbf{v}(x) = \varrho \mathbf{u}(x/\varrho)$ and $q(x) = p(x/\varrho)$. Since $\Delta \mathbf{v}(x) = (1/\varrho)\mathbf{u}(x/\varrho)$, and since $\nabla q(x) = (1/\varrho)\nabla p(x/\varrho)$, we have $\Delta \mathbf{v} = \nabla q$. Thus $\mathbf{v}$ is a solution of (1)–(4) in Ω_ϱ. The pressure drop of q is clearly the same as that of p, and $\int_{S_\varrho} \mathbf{v} \cdot \mathbf{n}\, ds = \int_S \varrho \mathbf{u} \cdot \mathbf{n} \varrho^{n-1} ds = \varrho^n F$.

THEOREM 15. *Suppose that $\Omega \subset R^n$ is defined as in Theorem 13. Then there exists at most one generalized solution of the initial boundary value problem for the Navier-Stokes equations (7)–(11), which satisfies the auxiliary flux condition (16).*

Proof. If two solutions $\mathbf{u}$ and $\bar{\mathbf{u}}$ both satisfy (16), then $\int_S (\mathbf{u}(t) - \bar{\mathbf{u}}(t)) \cdot \mathbf{n}\, ds = 0$ for every t. Thus $\mathbf{u}(t) - \bar{\mathbf{u}}(t) \in J_1(\Omega)$ for every t, which implies that $\mathbf{u}$ and $\bar{\mathbf{u}}$ belong to the same coset of $L^2(0, T; J_1^*(\Omega))/L^2(0, T; J_1(\Omega))$. Hence $\mathbf{u} = \bar{\mathbf{u}}$ by Proposition 1.

THEOREM 16. *Suppose that $\Omega \subset R^3$ is defined as in Theorem 13. Let data $\mathbf{f}(x, t)$, $\mathbf{a}(x)$, and $F(t)$ be prescribed for problem (7)–(11), (16). Suppose that $\mathbf{a}(x) \in J_1^*(\Omega) \cap W_2^2(\Omega)$, that $\mathbf{f}, \mathbf{f}_t \in L^2(\Omega \times (0, T))$, and that $F(t)$ is a continuously differentiable function of t which satisfies $F(0) = \int_S \mathbf{a} \cdot \mathbf{n}\, ds$. Then there exists a number $T' > 0$, such that a generalized solution of (7)–(11), (16) exists in $\Omega \times (0, T')$.*

Proof. We seek a solution of (7)–(11) in the form $\mathbf{u}(x, t) = \mathbf{v}(x, t) + F(t)\mathbf{b}(x)$, where $\mathbf{b}(x)$ is the vector field constructed in Lemma 11, and $\mathbf{v}(x, t) \in L^2(0, T; J_1(\Omega))$. Clearly Lemma 10 implies that any such solution will satisfy the flux condition (16). Now $\mathbf{u}$ is a generalized solution of (7)–(11) if and only if $\mathbf{v}$ satisfies the initial condition $\mathbf{v}(x, 0) = \mathbf{a}(x) - F(0)\,\mathbf{b}(x)$ and also a generalized form of the differential equation

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + F\mathbf{b}\cdot\nabla\mathbf{v} + F\mathbf{v}\cdot\nabla\mathbf{b} = -\nabla q + \Delta\mathbf{v} - \mathbf{g},$$

where $\mathbf{g} = F'\mathbf{b} + F^2\mathbf{b}\cdot\nabla\mathbf{b} - F\Delta\mathbf{b} - \mathbf{f}$, and where q is an appropriate "pressure" function. A solution $\mathbf{v}$ can be found by using the method of Galerkin approximation as developed by Hopf [18], and by using estimates due to Kiselev and Ladyzhenskaya [20] and further developed by Serrin [33] and by Heywood [15]. If the data for problem (7)–(11), (16) is small in an approriate sense, and if $\mathbf{a}(x) = 0$, then Lemmas 6, 9, and 11 of [15] can be used without modification to obtain $\mathbf{v}$, and hence $\mathbf{u}$, in a finite time interval $(0, T')$. With a very slight modification of these lemmas, we need not assume $\mathbf{a}(x) = 0$. The assumption that the data be small is not necessary either; however, the estimates of [15] were obtained for the purpose of treating questions of stability and therefore apply only to small data.

We shall now consider briefly the problem of steady Navier-Stokes flow through an aperture. To be specific, assume that $\Omega \subset R^3$ is defined by (67), and consider the problem of finding a solution $\mathbf{u}(x)$, $p(x)$ of

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \Delta\mathbf{u} \quad \text{in} \quad \Omega, \tag{72}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

$$\mathbf{u}(x) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{4}$$

$$\int_S \mathbf{u}\cdot\mathbf{n}\, ds = F, \tag{5}$$

where F is a prescribed number.

Definition. Let Ω be an arbitrary open set of R^n. We call $\mathbf{u}$ a generalized solution of (72), (2), (3), (4) if and only if $\mathbf{u} \in J_0^*(\Omega)$ and

$$\int_\Omega (\nabla\mathbf{u} : \nabla\boldsymbol{\phi} + \mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi})\, dx = 0 \tag{73}$$

holds for every $\boldsymbol{\phi} \in D(\Omega)$.

THEOREM 17. *Let $\Omega \subset R^3$ be defined by* (67). *Then there exists a generalized solution of* (72), (2), (3), (4) *in Ω, which satisfies the auxiliary flux condition* (5), *provided F is sufficiently small so that for some $\gamma < 1$,*

$$\left| F \int_\Omega \mathbf{v}\cdot\nabla\mathbf{b}\cdot\mathbf{v}\, dx \right| \leqslant \gamma \int_\Omega (\nabla\mathbf{v})^2\, dx \tag{74}$$

holds for all $\mathbf{v} \in J_0(\Omega)$, where $\mathbf{b}$ is the function constructed in Lemma 11.

The reader is reminded that the inequality $\int_\Omega (\mathbf{v}^2/|x|^2)dx \leqslant 4\int_\Omega (\nabla \mathbf{v})^2 dx$ was given in the proof of Lemma 4 for functions $\mathbf{v} \in W_0(\Omega)$, if $\Omega \subset R^3$; and that $\nabla \mathfrak{b}(x)$ is smooth and decays like $|x|^{-3}$ at infinity. Therefore condition (74) does not imply that $F=0$.

Proof. Let $\tilde{\mathfrak{b}}(x) = F\mathfrak{b}(x)$. We will find a generalized solution of (72), (2), (3), (4) of the form $\mathbf{u} = \mathbf{v} + \tilde{\mathfrak{b}}$ where $\mathbf{v} \in J_0(\Omega)$. Clearly Lemma 10 implies that any such solution must satisfy condition (5).

Our argument now follows closely one given by Ladyzhenskaya [21, p. 116]. Let $\Omega_n = \Omega \cap \{x: |x| < n\}$. Our first step is to find $\mathbf{v}_n \in J_0(\Omega_n)$ such that

$$\int_\Omega \{\nabla \mathbf{v}_n : \nabla \boldsymbol{\phi} + \nabla \tilde{\mathfrak{b}} : \nabla \boldsymbol{\phi} - (\mathbf{v}_n + \tilde{\mathfrak{b}}) \cdot \nabla \boldsymbol{\phi} \cdot (\mathbf{v}_n + \tilde{\mathfrak{b}})\}\, dx = 0 \tag{75}$$

for all $\boldsymbol{\phi} \in D(\Omega_n)$. Through this part of the argument we will suppress the subscript n from $\mathbf{v}_n$ and write simply $\mathbf{v}$. Now

$$\int_\Omega \{-\nabla \tilde{\mathfrak{b}} : \nabla \boldsymbol{\phi} + (\mathbf{v} + \tilde{\mathfrak{b}}) \cdot \nabla \boldsymbol{\phi} \cdot (\mathbf{v} + \tilde{\mathfrak{b}})\}\, dx$$

defines a bounded linear functional on $\boldsymbol{\phi} \in J_0(\Omega_n)$, because of the inequality $\|\mathbf{v}\|_{L^4(\Omega_n)} \leqslant C_n \|\nabla \mathbf{v}\|$ which holds for $\mathbf{v} \in W_0(\Omega_n)$. Thus, for every $\mathbf{v} \in J_0(\Omega_n)$, there exists an unique element $A\mathbf{v} \in J_0(\Omega_n)$ such that

$$\int_\Omega \nabla (A\mathbf{v}) : \nabla \boldsymbol{\phi}\, dx = \int_\Omega \{-\nabla \tilde{\mathfrak{b}} : \nabla \boldsymbol{\phi} + (\mathbf{v} + \tilde{\mathfrak{b}}) \cdot \nabla \boldsymbol{\phi} \cdot (\mathbf{v} + \tilde{\mathfrak{b}})\}\, dx$$

holds for all $\boldsymbol{\phi} \in J_0(\Omega_n)$. Clearly $\mathbf{v} \in J_0(\Omega_n)$ satisfies (75) if and only if

$$\mathbf{v} = A\mathbf{v}. \tag{76}$$

The operator A is compact in $J_0(\Omega_n)$; if $\{\mathbf{v}^k\}$ is weakly convergent in $J_0(\Omega_n)$, then $\{\mathbf{v}^k\}$ converges strongly in $L^4(\Omega_n)$, and by a short computation

$$\int_\Omega \{\nabla (A\mathbf{v}^k - A\mathbf{v}^l) : \nabla \boldsymbol{\phi}\}\, dx \leqslant C \|\mathbf{v}^k - \mathbf{v}^l\|_{L^4(\Omega_n)} \|\nabla \boldsymbol{\phi}\|.$$

Therefore, setting $\boldsymbol{\phi} = A\mathbf{v}^k - A\mathbf{v}^l$, one obtains $\|\nabla (A\mathbf{v}^k - A\mathbf{v}^l)\| \to 0$ as $k, l \to \infty$.

Now a solution of (76) is assured by the Leray–Schauder fixed point theorem, if one can show that all possible solutions of $\mathbf{v} = \lambda A\mathbf{v}$, for $\lambda \in [0, 1]$, must satisfy $\|\nabla \mathbf{v}\| < C^*$ for some constant C^*. Any such solution, with corresponding λ, satisfies

$$\int_\Omega \{\nabla \mathbf{v} : \nabla \boldsymbol{\phi} + \lambda \nabla \tilde{\mathfrak{b}} : \nabla \boldsymbol{\phi} - \lambda (\mathbf{v} + \tilde{\mathfrak{b}}) \cdot \nabla \boldsymbol{\phi} \cdot (\mathbf{v} + \tilde{\mathfrak{b}})\}\, dx = 0$$

for all $\boldsymbol{\phi} \in J_0(\Omega)$. Setting $\boldsymbol{\phi} = \mathbf{v}$, and integrating one term by parts, one obtains

$$\|\nabla \mathbf{v}\|^2 \leqslant \lambda \|\nabla \tilde{\mathbf{b}}\| \cdot \|\nabla \mathbf{v}\| + \lambda \|\tilde{\mathbf{b}}\|^2_{L^4(\Omega)} \|\nabla \mathbf{v}\| + \lambda \left| \int_\Omega \mathbf{v} \cdot \nabla \tilde{\mathbf{b}} \cdot \mathbf{v} \, dx \right|.$$

Using (74), this implies

$$\|\nabla \mathbf{v}\| \leqslant (1-\gamma)^{-1} (\|\nabla \tilde{\mathbf{b}}\| + \|\tilde{\mathbf{b}}\|^2_{L^4(\Omega)}) = C^*. \tag{77}$$

We have shown the existence of $\mathbf{v}_n \in J_0(\Omega)$ satisfying (75) for all $\boldsymbol{\phi} \in D(\Omega_n)$. The bound (77) is independent of n. Thus, if we extend the domain of each $\mathbf{v}_n$ to all of Ω by setting $\mathbf{v}_n$ equal to zero in $\Omega - \Omega_n$, the sequence $\{\mathbf{v}_n\}$ must have a weakly convergent subsequence with limit $\mathbf{v} \in J_0(\Omega)$. It is easy to see that

$$\int_\Omega \{\nabla \mathbf{v} : \nabla \boldsymbol{\phi} + \nabla \tilde{\mathbf{b}} : \nabla \boldsymbol{\phi} - (\mathbf{v} + \tilde{\mathbf{b}}) \cdot \nabla \boldsymbol{\phi} \cdot (\mathbf{v} + \tilde{\mathbf{b}})\} \, dx = 0$$

for all $\boldsymbol{\phi} \in D(\Omega)$. Thus $\mathbf{u} = \mathbf{v} + \tilde{\mathbf{b}}$ is a generalized solution of (72), (2), (3), (4) which satisfies also (5).

Remarks. The proofs just given of the existence of multiple solutions of the various boundary value problems are applicable to a wider class of domains than those defined by (67). For instance, the conclusions of Theorems 11, 16, and 17 are valid for the domain $\Omega = \{x \in R^3: x_2^2 + x_3^2 < 1 + x_1^2\}$. As a further example, let Ω be a domain formed by dividing R^3 into four subregions by two intersecting plane walls and then joining the subregions by apertures in the walls. For this domain it can be easily shown that problem (1)–(4) possesses three linearly independent solutions. In order to prove uniqueness theorems for either of these domains one would need certain preparatory results analogous to those we proved for domains of the form (67) by treating half-spaces in section 5.

The integrability conditions set for solutions in this paper are not appropriate for some problems. For instance, one would need entirely different methods to study flow through an infinite tube, say, the initial boundary value problem in $\Omega = \{x \in R^3: x_2^2 + x_3^2 < 1\}$ with the auxiliary flux condition (16). It may be even more interesting to consider nonstationary flow through a slit, i.e., flow in the domain $\{x \in R^2: x_1 \neq 0 \text{ or } |x_2| < 1\}$. Even though we have shown (Theorem 11) that steady flow through a slit exists and possesses a finite Dirichlet integral, it can be easily seen that any nonstationary flow through a slit must possess an infinite energy integral and therefore be excluded from the solution classes studied in this paper. Perhaps a theory of nonstationary flow through a slit can be based on the Dirichlet integral along lines developed in [16]. It seems likely that $J_1^*(\Omega) = J_1(\Omega)$ for every two-dimensional domain Ω, in which case problem (7)–(11) possesses a unique finite energy solution in every two-dimensional domain; see Lions and Prodi [22].

References

[1]. Babenko, K. I., On stationary solutions of the problem of flow past a body of a viscous incompressible fluid. *Math. USSR-Sb.*, 20 (1973), No. 1, 1–25.

[2]. Buck, R. C., *Advanced Calculus*. McGraw-Hill Book Co., (1956), New York.

[3]. Chang, I-Dee & Finn, R., On the solutions of a class of equations occurring in continuum mechanics, with application to the Stokes paradox. *Arch. Rat. Mech. Anal*, 7 (1961), 388–401.

[4]. Deny, J. & Lions, J. L., Les espaces du type de Beppo Levi. *Ann. Inst. Fourier* (*Grenoble*), 5 (1955), 305–370.

[5]. Dolidze, D. E., A non-linear boundary value problem for unsteady motion of a viscous fluid. *Prikl. Mat. Meh.* (Akad. Nauk SSSR), 12 (1948), 165–180.

[6]. Finn, R. & Noll, W., On the uniqueness and non-existence of Stokes flows. *Arch. Rat. Mech. Anal.*, 1 (1957), 97–106.

[7]. Finn, R., On the steady-state solutions of the Navier–Stokes equations. *Acta Math.*, 3 (1961), 197–244.

[8]. —— On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. *Arch. Rat. Mech. Anal.*, 19 (1965), 363–406.

[9]. Foà, E., *L'industria. Milan*, 43 (1929), 426.

[10]. Ford, G. G., M.A. Thesis. University of British Columbia (1976).

[11]. Ford, G. G. & Heywood, J. G. (in preparation).

[12]. Fujita, H., On the existence and regularity of the steady-state solutions of the Navier-Stokes equation. *J. Fac. Sci. Univ. Tokyo* Sect. I. 9 (1961), 59–102.

[13]. Graffi, D., Sul teorema di unicità nella dinamica dei fluidi. *Ann. Mat. Pura Appl.*, 50 (1960), 379–388.

[14]. Heywood, J. G., On stationary solutions of the Navier-Stokes equations as limits of nonstationary solutions. *Arch. Rat. Mech. Anal.*, 37 (1970), 48–60.

[15]. —— The exterior nonstationary problem for the Navier-Stokes equations. *Acta Math.* 129 (1972), 11–34.

[16]. —— On nonstationary Stokes flow past an obstacle. *Indiana Univ. Math. J.*, 24 (1974), 271–284.

[17]. —— On some paradoxes concerning two-dimensional Stokes flow past an obstacle. *Indiana Univ. Math. J.*, 24 (1974), 443–450.

[18]. Hopf, E., Über die Anfangswertaufgabe für die Hydrodynamischen Grundgleichungen. *Math. Nachr.* 4 (1951), 213–231.

[19]. Ito, S., The existence and the uniqueness of regular solution of non-stationary Navier–Stokes equation. *J. Fac. Sci. Univ. Tokyo*, Sect. I, 9 (1961), 103–140.

[20]. Kiselev, A. A. & Ladyzhenskaya, O. A., On the existence and uniqueness of the solution of the nonstationary problem for a viscous incompressible fluid. *Izv. Akad. Nauk SSSR*, 21 (1957), 655–680.

[21]. Ladyzhenskaya, O. A., *The Mathematical Theory of Viscous Incompressible Flow*. Second Edition. Gordon and Breach (1969), New York.

[22]. Lions, J. L. & Prodi, G., Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2. *C.R. Acad. Sci. Paris*, 248 (1959), 3519–3521.

[23]. Lions, J. L., Sur la régularité et l'unicité des solutions turbulentes des équations de Navier-Stokes. Rend. Sem. Mat. Padova 30 (1960), 16–23.

[24]. —— *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod (1969), Paris.

[25]. Ma, C. M., Ph.D. Thesis. University of British Columbia (1975).

[26]. Meyers, N. G. & Serrin, J., "$H = W$". *Proc. Nat. Acad. Sci. USA*, 51 (1964), 1055–1056.

[27]. Nečas, J., *Les méthodes directes en théorie des équations elliptiques*. Masson et C^{ie}, Editeurs, (1967), Paris.

[28]. Odqvist, F. K. G., *Die Randwetaufgaben der Hydrodynamik zäher Flüssigkeiten*. P. A. Norstedt and Söner (1928), Stockholm. See also *Math. Z.*, 32 (1930), 329–375.

[29]. Oseen, C. W., *Neuere Methoden und Ergebnisse in der Hydrodynamik*. Akademische Verlagsgesellschaft M.G.H. (1927), Leipzig.

[30]. Prodi, G., Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale. *Rend. Sem. Mat. Univ. Padova*, 30 (1960), 1–15.

[31]. —— Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl.*, 48 (1959), 173–182.

[32]. Serrin, J., *Mathematical Principles of Classical Fluid Mechanics*. Handbuch der Physik, vol. VIII/1. Springer-Verlag (1959).

[33]. —— The initial value problem for the Navier-Stokes equations. *Nonlinear Problems*. Edited by R. E. Langer. The Univ. of Wisconsin Press (1963), Madison.

[34]. Shinbrot, M., *Lectures on Fluid Mechanics*. Gordon and Breach (1973), New York.

Received July 7, 1975