

ON UNIQUENESS, STABILITY, AND POINTWISE ESTIMATES IN THE CAUCHY PROBLEM FOR COUPLED ELLIPTIC EQUATIONS*

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Abstract. Results of the kind cited in the title are obtained for the improperly posed Cauchy problem for a system of two coupled elliptic partial differential equations. We assume one stabilizing condition is imposed on one of the dependent variables. The results follow from an a priori inequality which is derived as a consequence of the logarithmic convexity of a suitable functional.

1. Introduction. Hadamard [2] showed that the Cauchy problem for elliptic partial differential equations is ill-posed in that a slight variation in the Cauchy data may result in a large variation in the solution function. Although thought to be unimportant, such problems have been encountered recently in potential theory, in geophysics, and elsewhere (see [8] and [4]). Pucci [9] and John [3] discovered that imposition of a suitable bound on the solution of the Cauchy problem for the Laplace equation would render the solution stable. In [5] Payne developed a method for obtaining error bounds for the solution of the Cauchy problem for the Laplace equation in n -dimensions under the assumption that the solution was uniformly bounded. He extended his technique in [6] to the Cauchy problem for the biharmonic equation where he assumed both the solution and its Laplacian were uniformly bounded.

In [10] the author showed that the second stabilizing condition in the Cauchy problem for the biharmonic equation and, more generally, in certain weakly coupled elliptic systems, could be removed. From an a priori inequality developed there, one is able to deduce the uniqueness and stability of the solution function as well as to obtain pointwise bounds for the solution and the square of its gradient. These results were later extended in [12] to a quasi-linear fourth-order elliptic equation whose principal part is the biharmonic operator. Since we expressed the equations as a system, each equation must have the same elliptic operator.

In this paper we consider the Cauchy problem for a more general coupled set of two elliptic equations involving different uniformly elliptic operators. As we only assume one a priori condition on one of the dependent variables, the system must be truly coupled. For otherwise the Cauchy problem for the system $\Delta u = v$, $\Delta v = 0$, where Δ is the Laplace operator and only v is assumed to be uniformly bounded, may have an unstable solution set.

We note that the results presented here, namely uniqueness, continuous dependence on the data, and pointwise estimates, answer affirmatively and more completely the

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conjecture made in [11]. Moreover, in view of [13], the pointwise estimates which one obtains are improvable, whereas in [11] they were not improvable.

2. Statement of the problem and results. Let D be a domain in Euclidean n -space with boundary B , a Lyapunov boundary, and let Σ be that portion of B on which Cauchy data is prescribed or, as presented below, measured within an allowable amount of error. We assume Σ is a C^1 surface.

Let

$$f(x) = \alpha, \quad 0 < \alpha \leq 1, \tag{2.1}$$

where $x = (x_1, \dots, x_n)$, be a family of (not necessarily closed) surfaces which intersect D and form, for each α , a closed region D_α whose boundary consists only of points of Σ , denoted Σ_α , and points of the surface $f(x) = \alpha$, denoted S_α .

We assume that f is a C^2 function in \bar{D}_1 such that

- (i) if $0 < \lambda < \mu \leq 1$, then $D_\lambda \subset D_\mu$,
- (ii) $|\text{grad } f| > \delta > 0$ in D_1 ,
- (iii) $\mathcal{L}_1 f \leq 0, \mathcal{L}_2 f \leq 0, |\mathcal{L}_1 f| \leq ca_0 \delta^2, |\mathcal{L}_2 f| \leq cb_0 \delta^2$, in D_1 ,

where c and δ are fixed positive constants. Here \mathcal{L}_1 and \mathcal{L}_2 are uniformly elliptic operators

$$\mathcal{L}_1 u = (a_{ij} u_{,i})_{,j}, \quad \mathcal{L}_2 v = (b_{ij} v_{,i})_{,j}, \tag{2.3}$$

where the repeated indices denote summation, the comma notation indicates partial differentiation, and the coefficients are C^1 functions which satisfy

$$\begin{aligned} a_{ij} &= a_{ji}, & a_0 \xi_i \xi_i &\leq a_{ij} \xi_i \xi_j \leq a_1 \xi_i \xi_i, \\ b_{ij} &= b_{ji}, & b_0 \xi_i \xi_i &\leq b_{ij} \xi_i \xi_j \leq b_1 \xi_i \xi_i, \end{aligned} \tag{2.4}$$

in D for positive constants $a_0, a_1, b_0,$ and b_1 and all real vectors $\xi = (\xi_1, \dots, \xi_n)$. We shall assume that $D_\alpha, 0 < \alpha \leq 1$, has nonzero volume and D_0 has zero volume.

The existence of such a family of surfaces and the usefulness in forming regions D_α which may include points that are not close to Σ but at which bounds are sought, was discussed in [7] and [13].

Consider the set of coupled equations

$$\begin{aligned} \mathcal{L}_1 u &= G(x, u, v, u_{,i}, v_{,i}) \\ \mathcal{L}_2 v &= H(x, u, v, v_{,i}) + u + h_{,i} u_{,i} \end{aligned} \tag{2.5}$$

where v is uniformly bounded in D and no a priori condition is imposed on u . We assume G and H satisfy uniform Lipschitz conditions in all but the x variables with the Lipschitz constant in the u argument of H strictly less than one and the vector valued function $h = (h_1, \dots, h_n)$ satisfies the condition prescribed in (3.9). These restrictions on the appearance of u in the second equation are a consequence of the omission of an a priori condition on u and the method used to obtain a suitable "replacement" condition via the system of equations. One could permit a suitably restricted function h_0 as coefficient of the linear term u in (2.5), but this would result in more complicated notation and restrictions on the appearance of u .

We ask that u and v be C^2 functions which satisfy (2.5) in D and

$$\int_{\Sigma} (u - u_0)^2 d\sigma \leq \pi_1, \quad \int_{\Sigma} (u_{,i} - u_i)(u_{,i} - u_i) d\sigma \leq \pi_2, \tag{2.6}$$

$$\int_{\Sigma} (v - v_0)^2 d\sigma \leq \pi_3, \quad \int_{\Sigma} (v_{,i} - v_i)(v_{,i} - v_i) d\sigma \leq \pi_4,$$

on Σ , where the quantities u_0, u_i, v_0, v_i are the respective measured values of u, u_i, v, v_i on Σ and π_1, π_2, π_3 , and π_4 are given bounds for the error in the measurement of the data. Since the data is usually determined by measurement, the above form of the initial data is more useful in applications.

Let

$$U = u - \phi, \quad V = v - \psi, \tag{2.7}$$

where ϕ and ψ are C^2 approximating functions. Then by (2.5) we have

$$\begin{aligned} \mathcal{L}_1 U &= G(x, u, v, u_{,i}, v_{,i}) - G(x, \phi, \psi, \phi_{,i}, \psi_{,i}) + \mathcal{G}(\phi, \psi), \\ \mathcal{L}_2 V &= H(x, u, v, v_{,i}) - H(x, \phi, \psi, \psi_{,i}) + U + h_i U_{,i} + \mathcal{H}(\phi, \psi), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \mathcal{G} &= \mathcal{G}(\phi, \psi) = G(x, \phi, \psi, \phi_{,i}, \psi_{,i}) - \mathcal{L}_1 \phi, \\ \mathcal{H} &= \mathcal{H}(\phi, \psi) = H(x, \phi, \psi, \psi_{,i}) + \phi + h_i \phi_{,i} - \mathcal{L}_2 \psi. \end{aligned}$$

By means of the Lipschitz assumption on G and H , it follows that

$$\begin{aligned} |\mathcal{L}_1 U| &\leq L_1 |U| + L_2 |V| + L_3 |U_{,i}| + L_4 |V_{,i}| + |\mathcal{G}|, \\ |\mathcal{L}_2 V| &\leq (L_5 + 1) |U| + L_6 |V| + L_7 |V_{,i}| + |h_i U_{,i}| + |\mathcal{H}|, \end{aligned} \tag{2.9}$$

for constants L_1, \dots, L_7 , where $|U_{,i}|$ denotes the magnitude of the gradient vector.

We now set

$$\begin{aligned} \epsilon_1 &= \int_{\Sigma} U^2 d\sigma, & \epsilon_2 &= \int_{\Sigma} V^2 d\sigma, & \epsilon_3 &= \int_{\Sigma} U_{,i} U_{,i} d\sigma, \\ \epsilon_4 &= \int_{\Sigma} V_{,i} V_{,i} d\sigma, & \epsilon_5 &= \iint_{D_1} \mathcal{G}^2 dx, & \epsilon_6 &= \iint_{D_1} \mathcal{H}^2 dx, \end{aligned} \tag{2.10}$$

where dx is the element of volume in D_1 . Furthermore, since v is assumed to be uniformly bounded in D , we have

$$\iint_D V^2 dx \leq M^2 \tag{2.11}$$

for some constant M .

Now consider the functional

$$F(\alpha) = \int_0^\alpha (\alpha - \eta) \iint_{D_\eta} [a_{ij} U_{,i} U_{,j} + U \mathcal{L}_1 U + b_{ij} V_{,i} V_{,j} + V \mathcal{L}_2 V] dx d\eta + k_i \epsilon_i \tag{2.12}$$

for $0 \leq \alpha \leq 1$, where the k_i are determinable constants. As is done in [13], one can derive the continued inequality

$$(k_i - \theta_i)\epsilon_i + \frac{1}{2} \iint_{D_{\alpha}} (\rho U^2 + \tau V^2) dx \leq F(\alpha) \leq \frac{c+1}{2} \cdot \iint_{D_{\alpha}} (\rho U^2 + \tau V^2) dx + (k_i + \theta_i)\epsilon_i, \tag{2.13}$$

where θ_i are computable constants and $\rho = a_{i,j,f,i}$ and $\tau = b_{i,j,f,i}$. We choose $k_i > \theta_i$ to ensure the nonnegativity of the lower side. Further, it follows by means of the logarithmic convexity argument (see [13]) that

$$F(\alpha) \leq K_0[F(0)]^d[F(\bar{\alpha})]^{1-d}, \quad 0 \leq \alpha \leq \bar{\alpha} < 1, \tag{2.14}$$

where K_0 is a constant and d is a fixed number in $0 < d < 1$.

In (2.14), $F(0)$ can be made small, but we must ascertain that $F(\bar{\alpha})$ can be suitably bounded so that the product does not become large. To this end we consider the following:

THEOREM: $\iint_{D_{\bar{\alpha}}} U^2 dx \leq C_0 M^2$, where C_0 is a computable constant.

By means of this theorem, (2.11), and the upper side of (2.13), we can deduce that $F(\bar{\alpha})$ is bounded in terms of M^2 and, consequently, by the lower side of (2.13) and (2.14), that

$$\iint_{D_{\alpha}} (U^2 + V^2) dx \leq K M^{2(1-d)} [k_i \epsilon_i]^d, \tag{2.15}$$

where K is a computable constant.

From (2.15) we deduce the uniqueness and continuous dependence on the data of the solution set u, v of the Cauchy problem for (2.5). Furthermore, using analogues of the pointwise inequalities derived in [1], which are in terms of the integrals in (2.15), we can obtain pointwise estimates for u and v and their derivatives at points P in $D_{\bar{\alpha}}$. As mentioned earlier, these estimates may be improved by the Ritz procedure.

Thus we need only demonstrate that the integral of U^2 over compact subsets of D_1 is bounded in terms of M^2 . This we do in the next section.

3. Proof of the theorem. We define the function s as

$$\begin{aligned} s(x) &= 1 && \text{in } \bar{D}_{\bar{\alpha}} \\ &= \frac{1 - f(x)}{1 - \bar{\alpha}} && \text{in } \bar{D}_1 - \bar{D}_{\bar{\alpha}}, \end{aligned} \tag{3.1}$$

where f satisfies (2.1) and (2.2). Clearly, s has the properties that $s_{,i} s_{,i} \leq M_1$, $|s_{,i} s_{,ij}| \leq M_2$ for constants M_1 and M_2 , and, in the big-oh notation, $\mathcal{L}_1 s^8 = O(s^6)$ and $\mathcal{L}_2 s^4 = O(s^2)$. Further, it follows that

$$\iint_{D_{\bar{\alpha}}} U^2 dx \leq \iint_{D_1} s^6 U^2 dx. \tag{3.2}$$

Before establishing the desired result, we note the following inequalities and identities which are important to the derivation. First, we have

$$(s_{,i} b_{ij} V_{,i})^2 \leq d_1 V_{,i} V_{,i} \leq \frac{d_1}{b_0} b_{ij} V_{,i} V_{,j}, \tag{3.3}$$

$$U_{,i} b_{ij} V_{,i} \leq d_2 U_{,i} U_{,i} + d_3 V_{,i} V_{,i} \leq \frac{d_2}{a_0} a_{ij} U_{,i} U_{,j} + \frac{d_3}{b_0} b_{ij} V_{,i} V_{,j}, \tag{3.4}$$

where d_1 , d_2 , and d_3 are fixed constants. These inequalities follow as a consequence of the bounded derivatives of s and coefficients of the operators and the ellipticity conditions on the operators \mathcal{L}_1 and \mathcal{L}_2 . Secondly, we have the identities

$$\iint_{D_1} s^8 a_{,i} U_{,i} U_{,i} dx = \int_{\Sigma_1} s^8 U \left(\frac{\partial U}{\partial \nu} \right)_a d\sigma - \iint_{D_1} s^8 U \mathcal{L}_1 U dx - \frac{1}{2} \int_{\Sigma_1} \left(\frac{\partial s^8}{\partial \nu} \right)_a U^2 d\sigma + \frac{1}{2} \iint_{D_1} (\mathcal{L}_1 s^8) U^2 dx, \tag{3.5}$$

$$\iint_{D_1} s^4 b_{,i} V_{,i} V_{,i} dx = \int_{\Sigma_1} s^4 V \left(\frac{\partial V}{\partial \nu} \right)_b d\sigma - \iint_{D_1} s^4 V \mathcal{L}_2 V dx - \frac{1}{2} \int_{\Sigma_1} \left(\frac{\partial s^4}{\partial \nu} \right)_b V^2 d\sigma + \frac{1}{2} \iint_{D_1} (\mathcal{L}_2 s^4) V^2 dx, \tag{3.6}$$

which follow on integrating by parts twice. Here $(\partial U / \partial \nu)_a = a_{i,i} U_{,i} n_i$, $(\partial V / \partial \nu)_b = b_{i,i} V_{,i} n_i$, for n_i , the i th component of the unit normal.

Now we observe that

$$\iint_{D_1} s^6 h_i U U_{,i} dx = \frac{1}{2} \int_{\Sigma_1} s^6 h_i n_i U^2 d\sigma - \frac{1}{2} \iint_{D_1} (s^6 h_i)_{,i} U^2 dx, \tag{3.7}$$

so that by (2.8) and (3.7) we can write

$$\iint_{D_1} s^6 U^2 dx \leq \iint_{D_1} s^6 U \mathcal{L}_2 V dx - \frac{1}{2} \int_{\Sigma_1} s^6 h_i n_i U^2 d\sigma - \iint_{D_1} s^6 U [H(x, u, v, v_{,i}) - H(x, \phi, \psi, \psi_{,i}) + \mathcal{I}(\phi, \psi)] dx, \tag{3.8}$$

provided that the vector function h satisfies

$$(s^6 h_i)_{,i} \leq 0, \quad x \text{ in } D_1. \tag{3.9}$$

Clearly, one such function h which satisfies this requirement is given by $h_i = f_{,i}$.

By the definition of the operator \mathcal{L}_2 , integration by parts, and the arithmetic mean-geometric mean inequality (abbreviated $A-G$ inequality), we see that

$$\begin{aligned} \iint_{D_1} s^6 U \mathcal{L}_2 V dx &= \int_{\Sigma_1} s^6 U \left(\frac{\partial V}{\partial \nu} \right)_b d\sigma - \iint_{D_1} s^6 U_{,i} b_{ij} V_{,i} dx - \iint_{D_1} 6s^6 s_{,i} U b_{ij} V_{,i} dx \\ &\leq \int_{\Sigma_1} s^6 U \left(\frac{\partial V}{\partial \nu} \right)_b d\sigma - \iint_{D_1} s^6 U_{,i} b_{ij} V_{,i} dx \\ &\quad + 3\gamma_1 \iint_{D_1} s^6 U^2 dx + 3\gamma_1^{-1} \iint_{D_1} s^4 (s_{,i} b_{ij} V_{,i})^2 dx, \end{aligned}$$

where γ_1 is some positive constant to be determined later. Thus, using (3.3) and (3.4), we find

$$\begin{aligned} \iint_{D_1} s^6 U \mathcal{L}_2 V dx &\leq \int_{\Sigma_1} s^6 U \left(\frac{\partial V}{\partial \nu} \right)_b d\sigma + \frac{d_2}{a_0} \gamma_2 \iint_{D_1} s^8 a_{i,i} U_{,i} U_{,i} dx \\ &\quad + \frac{d_3}{b_0 \gamma_2} \iint_{D_1} s^4 b_{i,i} V_{,i} V_{,i} dx + 3\gamma_1 \iint_{D_1} s^6 U^2 dx + \frac{3d_1}{b_0 \gamma_1} \iint_{D_1} s^4 b_{i,i} V_{,i} V_{,i} dx \end{aligned} \tag{3.10}$$

for γ_2 some positive constant.

Using the Lipschitz condition on H , the A - G inequality, and the ellipticity conditions, we can bound the third term in (3.8) by

$$\begin{aligned}
 & - \iint_{D_1} s^6 U [H(x, u, v, v_{,i}) - H(x, \phi, \psi, \psi_{,i}) + \mathfrak{C}(\phi, \psi)] dx \\
 & \leq L_5 \iint_{D_1} s^6 U^2 dx + \frac{1}{2} \gamma_3 \iint_{D_1} s^6 U^2 dx \\
 & \quad + \frac{3}{2\gamma_3} \iint_{D_1} s^6 [L_6^2 V^2 + \mathfrak{C}^2] dx + \frac{3L_7^2}{2\gamma_3 b_0} \iint_{D_1} s^4 b_{,i} V_{,i} V_{,i} dx, \quad (3.11)
 \end{aligned}$$

where γ_3 is some positive constant.

Now combining (3.10) and (3.11) with (3.8) and collecting terms, we obtain

$$\begin{aligned}
 \iint_{D_1} s^6 U^2 dx & \leq O(M^2) + (L_5 + 3\gamma_1 + \frac{1}{2}\gamma_3) \iint_{D_1} s^6 U^2 dx + \gamma_4 \iint_{D_1} s^8 a_{,i} U_{,i} U_{,i} dx \\
 & \quad + k_1 \iint_{D_1} s^4 b_{,i} V_{,i} V_{,i} dx, \quad (3.12)
 \end{aligned}$$

where $\gamma_4 = d_2 \gamma_2 (a_0)^{-1}$ and k_1 is a computable constant. Here we have collected all surface integrals over Σ_1 and terms involving V^2 and \mathfrak{C}^2 in the first term, as each can be bounded in terms of M^2 .

We now seek appropriate bounds on the latter two terms of (3.12). To accomplish this we will eventually need to consider the terms simultaneously, as each leads to the other after integrating by parts and using (2.9).

From (3.5) we see that

$$\begin{aligned}
 & \iint_{D_1} s^8 a_{,i} U_{,i} U_{,i} dx \\
 & \leq O(M^2) + \iint_{D_1} s^8 |U \mathfrak{L}_1 U| dx + \frac{1}{2} k_2 \iint_{D_1} s^6 U^2 dx \\
 & \leq O(M^2) + L_1 \iint_{D_1} s^6 U^2 dx + \frac{1}{2} \iint_{D_1} s^6 U^2 dx + \frac{1}{2} \iint_{D_1} s^{10} [L_2 |V| + |\mathfrak{G}|]^2 dx \\
 & \quad + \frac{L_3^2}{2a_0} \iint_{D_1} s^6 U^2 dx + \frac{a_0}{2} \iint_{D_1} s^8 U_{,i} U_{,i} dx + \frac{L_4^2}{2\gamma_5 b_0} \iint_{D_1} s^6 U^2 dx \\
 & \quad + \frac{\gamma_5 b_0}{2} \iint_{D_1} s^{10} V_{,i} V_{,i} dx + \frac{1}{2} k_2 \iint_{D_1} s^6 U^2 dx,
 \end{aligned}$$

where k_2 is a computable constant and γ_5 is a positive constant to be chosen later. Using the ellipticity conditions on the sixth and eighth terms and collecting terms, we have

$$\iint_{D_1} s^8 a_{,i} U_{,i} U_{,i} dx \leq O(M^2) + k_3 \iint_{D_1} s^6 U^2 dx + \gamma_5 \iint_{D_1} s^4 b_{,i} V_{,i} V_{,i} dx, \quad (3.13)$$

where k_3 is a computable constant.

In a similar manner, from (3.6) it follows that

$$\begin{aligned} \iint_{D_1} s^4 b_{ij} V_{,i} V_{,j} dx &\leq O(M^2) + \iint_{D_1} s^4 |V \mathcal{L}_2 V| dx \\ &\leq O(M^2) + \frac{1}{2} \gamma_6 \iint_{D_1} s^6 U^2 dx + \frac{1}{2} b_0 \iint_{D_1} s^4 V_{,i} V_{,i} dx \\ &\quad + \frac{1}{2} a_0 \gamma_7 \iint_{D_1} s^8 U_{,i} U_{,i} dx, \end{aligned}$$

where γ_6 and γ_7 are constants to be chosen. Consequently, by the ellipticity conditions, we deduce that

$$\iint_{D_1} s^4 b_{ij} V_{,i} V_{,j} dx \leq O(M^2) + \gamma_6 \iint_{D_1} s^6 U^2 dx + \gamma_7 \iint_{D_1} s^8 a_{ij} U_{,i} U_{,j} dx. \tag{3.14}$$

If we multiply (3.13) by γ_4 and (3.14) by k_1 , add, and choose $\gamma_6 = k_1(2\gamma_4)^{-1}$ and $\gamma_7 = \gamma_4(2k_1)^{-1}$, then

$$\gamma_4 \iint_{D_1} s^8 a_{ij} U_{,i} U_{,j} dx + k_1 \iint_{D_1} s^4 b_{ij} V_{,i} V_{,j} dx \leq O(M^2) + 2(\gamma_4 k_3 + \gamma_6 k_1) \iint_{D_1} s^6 U^2 dx. \tag{3.15}$$

Thus, from (3.12) and (3.15), we conclude

$$\iint_{D_1} s^6 U^2 dx \leq O(M^2) + (L_5 + 3\gamma_1 + \frac{1}{2}\gamma_3 + 2\gamma_4 k_3 + 2\gamma_6 k_1) \iint_{D_1} s^6 U^2 dx.$$

Consequently, for $L_5 < 1$ and $\gamma_1, \gamma_3, \gamma_4$, and γ_6 chosen sufficiently small, we have

$$\iint_{D_1} s^6 U^2 dx \leq O(M^2); \tag{3.16}$$

i.e., from (3.2) and (3.16), the conclusion of the theorem follows.

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