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On Unitarity of Massive Gravity in Three Dimensions

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We examine a unitarity of a particular higher-derivative extension of general relativity in three space-time dimensions, which has been recently shown to be equivalent to the Pauli-Fierz massive gravity at the linearized approximation level, and explore a possibility of generalizing the model to higher space-time dimensions. We find that the model in three dimensions is indeed unitary in the tree-level, but the corresponding model in higher dimensions is not so due to the appearance of non-unitary massless spin-2 modes.

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In recent years, there has been a revival of interests of massive gravity models from various physical viewpoints. For instance, some people conjecture that the massless graviton might acquire mass via spontaneous symmetry breakdown of general coordinate reparametrization invariance, whose dynamical mechanism is sometimes called "gravitational Higgs mechanism".[1)–9)] This expectation naturally stems from brane world scenario where the presence of a brane breaks some of diffeomorphisms in the directions perpendicular to the brane spontaneously.[2),3)] Moreover, this study is also related to the recent development of string theory approach to quantum chromodynamics (QCD)[5)] since if we wish to apply a bosonic string theory to QCD, massless fields such as tachyonic scalar and spin 2 graviton in string theory, must become massive or be removed somehow because such the fields do not exist in QCD.

The other interest of massive gravity is relevant to the problem of counting the microscopic physical degrees of freedom existing in black holes through a holographic two-dimensional dual theory where the well-known topological massive gravity with the Chern-Simons term[10)] plays an important role.[11)]

It is well known that there is a unique way to add mass term to general relativity in a Lorentz-covariant manner without worrying the emergence of a non-unitary *ghost* in any space-time dimension whose theory is called the Pauli-Fierz massive gravity.[12)] However, there is at least one serious disadvantage in the Pauli-Fierz massive gravity. Namely, the massive gravity theory defined by Pauli and Fierz only makes sense as a free and linearized theory since the diffeomorphism-invariant mass term cannot be introduced into general relativity owing to an obvious identity $g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu$ so it seems to be difficult to construct a sensible interacting theory for the massive graviton.

One resolution for overcoming this difficulty is to introduce some matter fields in general relativity and then trigger the above-mentioned gravitational Higgs mecha-

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nism. However, recently, there has been an alternative progress for getting a sensible interacting massive gravity theory in three space-time dimensions without introducing matter fields such as scalar fields.[13] This model has been shown to be equivalent to the Pauli-Fierz massive gravity at the linearized approximation level. A key idea in this model is that one takes into consideration higher-derivative curvature terms in the Einstein-Hilbert action with the *wrong* sign in such a way that the trace part of the stress-energy tensor associated with those higher-derivative terms is proportional to the original higher-derivative Lagrangian.

The main aim of this paper is not olny to explore a possibility of generalizing this three-dimensional model to higher space-time dimensions but also to examine the unitarity of this particular higher-derivative extension of general relativity in three space-time dimensions by Bergshoeff et al.[13]

Since we wish to explore a possibility of generalization of three-dimensional massive gravity model to higher dimensions, we will start with a typical higher-derivative gravity model[14] without cosmological constant up to fourth-order in derivative in a general D space-time dimensions:[*]

$$S = \int d^D x \sqrt{-g} \left[-\frac{1}{\kappa^2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) \right], \tag{1}$$

where $\kappa^2 \equiv 16\pi G_D$ (G_D is the D-dimensional Newton's constant), α, β and γ are constants. One important remark is that the Einstein-Hilbert action, which is the first term having κ^2, has the *wrong* sign. This is a characteristic feature in the present formalism. The last term proportional to γ is nothing but the Gauss-Bonnet term, which is a surface term in four space-time dimensions. Einstein's equations are then given by

$$-\frac{1}{\kappa^2} G_{\mu\nu} + K_{\mu\nu} = 0, \tag{2}$$

where $G_{\mu\nu}$ is the conventional Einstein's tensor defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and the tensor $K_{\mu\nu}$ is defined as

$$\begin{aligned} K_{\mu\nu} &= (2\alpha + \beta)(g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu) R + \beta \nabla^2 G_{\mu\nu} \\ &\quad + 2\alpha R \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) + 2\beta \left(R_{\mu\rho\nu\sigma} - \frac{1}{4} g_{\mu\nu} R_{\rho\sigma} \right) R^{\rho\sigma} + 2\gamma \Big[R R_{\mu\nu} - 2 R_{\mu\rho\nu\sigma} R^{\rho\sigma} \\ &\quad + R_{\mu\rho\sigma\tau} R_\nu{}^{\rho\sigma\tau} - 2 R_{\mu\rho} R_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} (R^2_{\rho\sigma\tau\lambda} - 4 R^2_{\rho\sigma} + R^2) \Big], \end{aligned} \tag{3}$$

where ∇_μ is the usual covariant derivative and $\nabla^2 \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$.

In the construction of a new type of massive gravity theory,[13] the tensor $K_{\mu\nu}$ plays a critical role and must satisfy the following condition:

- Its trace $K \equiv g^{\mu\nu} K_{\mu\nu}$ is proportional to the original higher-derivative Lagrangian.

[*] The space-time indices $\mu, \nu, \cdots$ run over $0, 1, \cdots, D-1$. We take the metric signature $(-, +, \cdots, +)$ and follow the notation and conventions of the textbook of MTW.[15]

In particular, this condition ensures that the scalar curvature can be set to zero in the trace part of the linearized Einstein's equations.

Taking trace of $K_{\mu\nu}$ gives rise to

$$K = \left[(2\alpha+\beta)(D-1) + \beta\left(1-\frac{D}{2}\right)\right]\nabla^2 R$$
$$+ 2\left(1-\frac{D}{4}\right)[\gamma R^2_{\mu\nu\rho\sigma} + (\beta-4\gamma)R^2_{\mu\nu} + (\alpha+\gamma)R^2]. \tag{4}$$

First, from this condition, the $\nabla^2 R$ term must vanish so that we have a relation between the constants α and β

$$\alpha = -\frac{D}{4(D-1)}\beta. \tag{5}$$

Then, the condition also requires three kinds of independent R^2 terms to satisfy

$$(\alpha+\gamma)\left[1-2c\left(1-\frac{D}{4}\right)\right] = 0,$$
$$(\beta-4\gamma)\left[1-2c\left(1-\frac{D}{4}\right)\right] = 0,$$
$$\gamma\left[1-2c\left(1-\frac{D}{4}\right)\right] = 0, \tag{6}$$

where c denotes a proportional constant. Of course, in three and four dimensions, three R^2 terms are not completely independent, so precisely speaking, the equations (6) are valid for $D > 4$. The cases of $D = 3,\ 4$ are separately considered later. It is obvious that all the equations in (6) are satisfied when

$$c = \frac{2}{4-D}. \tag{7}$$

If the equation (7) were not true, we would have a trivial solution $\alpha = \beta = \gamma = 0$, so we shall confine ourselves to the solution (7) in what follows. Consequently, the trace part of Einstein's equations (2) gives us

$$\frac{1}{\kappa^2}\left(1-\frac{D}{2}\right)R = K. \tag{8}$$

As the next step, we shall linearize Einstein's equations around a Minkowski flat space-time as usual by writing out $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Equation (8) together with the fact that K does not involve the linear term in $h_{\mu\nu}$ from the equation (5) produces

$$R^{\rm lin} = 0, \tag{9}$$

and with the help of this equation, Eq. (2) yields

$$\left(\Box - \frac{1}{\beta\kappa^2}\right)G^{\rm lin}_{\mu\nu} = 0, \tag{10}$$

where $R^{\rm lin}$ and $G^{\rm lin}_{\mu\nu}$ denote the linearized scalar curvature and Einstein's tensor, respectively. Moreover, we have defined $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. Let us note that the positivity of mass of the graviton requires us to take $\beta > 0$.

Now we are ready to show that with an appropriate choice of the constants α, β and γ, the action (1) becomes equivalent to the Pauli-Fierz massive gravity at the quadratic level. To do that, let us begin with an action[13)]

$$S_f = -\frac{1}{\kappa^2}\int d^D x\sqrt{-g}\left[R + f^{\mu\nu}G_{\mu\nu} + \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu} - f^2)\right], \tag{11}$$

where $f_{\mu\nu}$ is some symmetric tensor field with trace $f = g^{\mu\nu}f_{\mu\nu}$ and m^2 is a constant. Integrating out the auxiliary field $f_{\mu\nu}$, this action is reduced to the form

$$S_f = -\frac{1}{\kappa^2}\int d^D x\sqrt{-g}\left[R - \frac{1}{m^2}R^2_{\mu\nu} + \frac{1}{m^2}\frac{D}{4(D-1)}R^2\right]. \tag{12}$$

Note that in order to make this action (12) coincide with the original action (1), we have to impose constraints on the coefficients in the action (1)

$$\begin{aligned}
\alpha &= -\frac{D}{4(D-1)}\frac{1}{\kappa^2 m^2}, \\
\beta &= \frac{1}{\kappa^2 m^2}, \\
\gamma &= 0.
\end{aligned} \tag{13}$$

It is of interest to notice that not only the first and second constraints naturally lead to the previous relation (5), but also the second constraint is consistent with the mass positivity $\beta > 0$, which was mentioned below Eq. (10).

Next, let us expand the metric around a flat Minkowski background $\eta_{\mu\nu}$ and keep only quadratic fluctuations in the action (11)

$$S_f = \frac{1}{\kappa^2}\int d^D x\left[\left(f^{\mu\nu} - \frac{1}{2}h^{\mu\nu}\right)\mathcal{O}_{\mu\nu,\rho\sigma}h^{\rho\sigma} - \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu} - f^2)\right], \tag{14}$$

where the operator $\mathcal{O}_{\mu\nu,\rho\sigma}$ can be expressed in terms of the spin projection operators

$$\mathcal{O}_{\mu\nu,\rho\sigma} = \Box\left[\frac{1}{2}P^{(2)} - \frac{D-2}{2}P^{(0,s)}\right]_{\mu\nu,\rho\sigma}, \tag{15}$$

where $P^{(2)}$ and $P^{(0,s)}$ are the spin-2 and spin-0 projection operators. Concretely, in the D-dimensions they take the form

$$\begin{aligned}
P^{(2)}_{\mu\nu,\rho\sigma} &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}, \\
P^{(0,s)}_{\mu\nu,\rho\sigma} &= \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma},
\end{aligned} \tag{16}$$

where the transverse operator $\theta_{\mu\nu}$ and the longitudinal operator $\omega_{\mu\nu}$ are defined as

$$\begin{aligned}
\theta_{\mu\nu} &= \eta_{\mu\nu} - \frac{1}{\Box}\partial_\mu\partial_\nu = \eta_{\mu\nu} - \omega_{\mu\nu}, \\
\omega_{\mu\nu} &= \frac{1}{\Box}\partial_\mu\partial_\nu.
\end{aligned} \tag{17}$$

It is worth while to stress that the structure of the operator $\mathcal{O}_{\mu\nu,\rho\sigma}$ is the same in any space-time dimension.

Here we wish to perform the path integration over $h_{\mu\nu}$ in the action (14). To do that, it is convenient to think of partition function

$$Z = \int \mathcal{D}h_{\mu\nu}\mathcal{D}f_{\mu\nu}e^{iS_f}. \tag{18}$$

One can rewrite the action (14) as

$$S_f = \frac{1}{\kappa^2}\int d^Dx\Big[-\frac{1}{2}(h-f)^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}(h-f)^{\rho\sigma} + \frac{1}{2}f^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}f^{\rho\sigma} - \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu}-f^2)\Big]. \tag{19}$$

Changing the variables from $h_{\mu\nu}$ to $k_{\mu\nu} \equiv h_{\mu\nu} - f_{\mu\nu}$, the partition function (18) reads

$$Z = \int \mathcal{D}k_{\mu\nu}\mathcal{D}f_{\mu\nu}e^{iS'_f}, \tag{20}$$

where S'_f is defined by

$$S'_f \equiv \frac{1}{\kappa^2}\int d^Dx\left[-\frac{1}{2}k^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}k^{\rho\sigma} + \frac{1}{2}f^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}f^{\rho\sigma} - \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu}-f^2)\right]. \tag{21}$$

In attempting to perform the path integration over $k^{\mu\nu}$, we find it impossible to do so since there is no inverse matrix of $\mathcal{O}_{\mu\nu,\rho\sigma}$. That is, because of the gauge invariance, the linearized diffeomorphisms, in the action (21), the operator $\mathcal{O}_{\mu\nu,\rho\sigma}$ has zero eigenvalues so that its inverse matrix, which is in essence the propagator of the massless graviton, does not generally exist. This is also clear from the observation that we need more spin projection operators $P^{(1)}, P^{(0,w)}, P^{(0,sw)}$ and $P^{(0,ws)}$ in addition to $P^{(2)}$ and $P^{(0,s)}$ in order to form a complete set of the spin projection operators in the space of second rank symmetric tensors. Thus, in order to make the operator $\mathcal{O}_{\mu\nu,\rho\sigma}$ invertible, we fix the gauge transformations by De Donder's gauge-fixing conditions. Then, the gauge-fixed action of (21) is of form

$$\begin{aligned}\hat{S}_f &\equiv \frac{1}{\kappa^2}\int d^Dx\Big[-\frac{1}{2}k^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}k^{\rho\sigma} + \frac{1}{2\alpha}\Big(\partial_\nu k_\mu{}^\nu - \frac{1}{2}\partial_\mu k\Big)^2 \\ &\quad + \frac{1}{2}f^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}f^{\rho\sigma} - \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu}-f^2)\Big] \\ &= \frac{1}{\kappa^2}\int d^Dx\Big[-\frac{1}{2}k^{\mu\nu}\hat{\mathcal{O}}_{\mu\nu,\rho\sigma}k^{\rho\sigma} + \frac{1}{2}f^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}f^{\rho\sigma} - \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu}-f^2)\Big],\end{aligned} \tag{22}$$

where α is a gauge parameter and the new operator $\hat{\mathcal{O}}_{\mu\nu,\rho\sigma}$ is defined through a complete set of the spin projection operators

$$\hat{\mathcal{O}}_{\mu\nu,\rho\sigma} = \Box\Big[\frac{1}{2}P^{(2)} + \frac{1}{2\alpha}P^{(1)} + \frac{-2(D-2)\alpha+D-1}{4\alpha}P^{(0,s)} + \frac{1}{4\alpha}P^{(0,w)}$$

$$- \frac{\sqrt{D-1}}{4\alpha} P^{(0,sw)} - \frac{\sqrt{D-1}}{4\alpha} P^{(0,ws)} \Big]_{\mu\nu,\rho\sigma}, \tag{23}$$

where $P^{(1)}, P^{(0,w)}, P^{(0,sw)}$ and $P^{(0,ws)}$ are defined as

$$\begin{aligned} P^{(1)}_{\mu\nu,\rho\sigma} &= \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}), \\ P^{(0,w)}_{\mu\nu,\rho\sigma} &= \omega_{\mu\nu}\omega_{\rho\sigma}, \\ P^{(0,sw)}_{\mu\nu,\rho\sigma} &= \frac{1}{\sqrt{D-1}}\theta_{\mu\nu}\omega_{\rho\sigma}, \\ P^{(0,ws)}_{\mu\nu,\rho\sigma} &= \frac{1}{\sqrt{D-1}}\omega_{\mu\nu}\theta_{\rho\sigma}. \end{aligned} \tag{24}$$

Note that all the spin projection operators $\{P^{(2)}, P^{(1)}, P^{(0,s)}, P^{(0,w)}, P^{(0,sw)}, P^{(0,ws)}\}$ satisfy the orthogonality relations

$$\begin{aligned} P^{(i,a)}_{\mu\nu,\rho\sigma} P^{(j,b)}_{\rho\sigma,\lambda\tau} &= \delta^{ij}\delta^{ab} P^{(i,a)}_{\mu\nu,\lambda\tau}, \\ P^{(i,ab)}_{\mu\nu,\rho\sigma} P^{(j,cd)}_{\rho\sigma,\lambda\tau} &= \delta^{ij}\delta^{bc} P^{(i,a)}_{\mu\nu,\lambda\tau}, \\ P^{(i,a)}_{\mu\nu,\rho\sigma} P^{(j,bc)}_{\rho\sigma,\lambda\tau} &= \delta^{ij}\delta^{ab} P^{(i,ac)}_{\mu\nu,\lambda\tau}, \\ P^{(i,ab)}_{\mu\nu,\rho\sigma} P^{(j,c)}_{\rho\sigma,\lambda\tau} &= \delta^{ij}\delta^{bc} P^{(i,ac)}_{\mu\nu,\lambda\tau}, \end{aligned} \tag{25}$$

with $i, j = 0, 1, 2$ and $a, b, c, d = s, w$ and the tensorial relation

$$[P^{(2)} + P^{(1)} + P^{(0,s)} + P^{(0,w)}]_{\mu\nu,\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}). \tag{26}$$

Using these relations, it is straightforward to derive the inverse of the matrix $\hat{\mathcal{O}}_{\mu\nu,\rho\sigma}$

$$\begin{aligned} \hat{\mathcal{O}}^{-1}_{\mu\nu,\rho\sigma} = \frac{1}{\Box}\Big[& 2P^{(2)} + 2\alpha P^{(1)} - \frac{2}{D-2}P^{(0,s)} - \frac{2\{-2(D-2)\alpha + D - 1\}}{D-2}P^{(0,w)} \\ & - \frac{2\sqrt{D-1}}{D-2}P^{(0,sw)} - \frac{2\sqrt{D-1}}{D-2}P^{(0,ws)}\Big]_{\mu\nu,\rho\sigma}. \end{aligned} \tag{27}$$

Hence, we can now perform the path integration over $k_{\mu\nu}$ without hesitation

$$Z = \int \mathcal{D}f_{\mu\nu} e^{iS_{\rm PF}}, \tag{28}$$

where $S_{\rm PF}$ is the Pauli-Fierz massive gravity action with the *correct* sign:[12)]

$$S_{\rm PF} = \frac{1}{\kappa^2}\int d^D x \left[\frac{1}{2}f^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}f^{\rho\sigma} - \frac{m^2}{4}(f^{\mu\nu}f_{\mu\nu} - f^2)\right]. \tag{29}$$

Let us consider deliberately what we have done above. It seems that the action (12), or equivalently the action (11), is equivalent to the Pauli-Fierz massive gravity action (29) at least at the linearized level since, as seen in Eq. (21), the tensor field

$k_{\mu\nu}$ does not interact with the other tensor one $f_{\mu\nu}$ at all so that we can integrate $k_{\mu\nu}$ away after the gauge-fixing. However, in the higher-order approximation level, there appear interaction terms between $k_{\mu\nu}$ (in other words, $h_{\mu\nu}$) and $f_{\mu\nu}$, so that it is impossible to perform the path integration over $k_{\mu\nu}$ to arrive at the Pauli-Fierz action (29). Then, we can only show that the action (12) *is* equivalent to the action S'_f in (21) with many of interaction terms involving the tensor fields $k_{\mu\nu}$ and $f_{\mu\nu}$, which is essentially an interacting theory of two symmetric tensor fields where one is a massless tensor field with the *wrong* sign and the other is a massive tensor one with the *correct* sign. Incidentally, let us mention the cases of three ($D = 3$)- and four ($D = 4$)-dimensional space-time. In the three-dimensional case, it is easy to see that the present analysis naturally reduces to the work by Bergshoeff et al.[13] On the other hand, in the case of four dimensions, we find it impossible to construct the tensor $K_{\mu\nu}$ satisfying the condition. This fact can be also seen in the presence of the pole at $D = 4$ in Eq. (7).

To give a definite answer to a question whether or not our massive gravity model is really physically plausible, we have to investigate the property of unitarity of the higher-derivative action (12) directly. Actually, we will find that the model is unitary only in three dimensions while in the other dimensions we have non-unitary massless spin-2 modes which come from the *wrong* sign in front of the Einstein-Hilbert action. Thus, it is impossible to generalize the three-dimensional massive gravity model by Bergshoeff et al.[13] to higher space-time dimensions.

To this aim, let us notice that each term in the action (12) is expressed by the spin projection operators as

$$\begin{aligned} -\sqrt{-g}R &= -\frac{1}{4}h^{\mu\nu}[P^{(2)} - (D-2)P^{(0,s)}]_{\mu\nu,\rho\sigma}\Box h^{\rho\sigma}, \\ \xi\sqrt{-g}R_{\mu\nu}R^{\mu\nu} &= \xi\frac{1}{4}h^{\mu\nu}[P^{(2)} + DP^{(0,s)}]_{\mu\nu,\rho\sigma}\Box^2 h^{\rho\sigma}, \\ \lambda\sqrt{-g}R^2 &= -\xi\frac{D}{4}h^{\mu\nu}P^{(0,s)}_{\mu\nu,\rho\sigma}\Box^2 h^{\rho\sigma}, \end{aligned} \tag{30}$$

where we have defined as $\xi \equiv \frac{1}{m^2}$ and $\lambda \equiv -\frac{1}{m^2}\frac{D}{4(D-1)} \equiv -\frac{D}{4(D-1)}\xi$. A nice feature in the present theory is that with the coefficients in front of the higher-derivative terms, the scalar ghost mode which exists in the spin projection operator $P^{(0,s)}$ is canceled out as can be seen

$$\xi\sqrt{-g}R_{\mu\nu}R^{\mu\nu} + \lambda\sqrt{-g}R^2 = \xi\frac{1}{4}h^{\mu\nu}P^{(2)}_{\mu\nu,\rho\sigma}\Box^2 h^{\rho\sigma}. \tag{31}$$

Taking De Donder's gauge conditions for diffeomorphisms again, the quadratic Lagrangian part of the action (12) (up to the overall constant $\frac{1}{\kappa^2}$) reads

$$\mathcal{L} = \frac{1}{2}h^{\mu\nu}\mathcal{P}_{\mu\nu,\rho\sigma}h^{\rho\sigma}, \tag{32}$$

where the operator $\mathcal{P}$ is defined as

$$\mathcal{P}_{\mu\nu,\rho\sigma} = \Box\Big[\frac{1}{2}(-1+\xi\Box)P^{(2)} - \frac{1}{2\alpha}P^{(1)} + \frac{2(D-2)\alpha - (D-1)}{4\alpha}P^{(0,s)} - \frac{1}{4\alpha}P^{(0,w)}$$

$$+ \frac{\sqrt{D-1}}{4\alpha} P^{(0,sw)} + \frac{\sqrt{D-1}}{4\alpha} P^{(0,ws)} \Bigg]_{\mu\nu,\rho\sigma} . \tag{33}$$

Then, the inverse of the operator $\mathcal{P}$ is calculated as

$$\mathcal{P}^{-1}_{\mu\nu,\rho\sigma} = \frac{1}{\Box} \Bigg[\frac{2}{-1+\xi\Box} P^{(2)} - 2\alpha P^{(1)} + \frac{2}{D-2} P^{(0,s)} + \frac{2(D-1) - 4\alpha(D-2)}{D-2} P^{(0,w)}$$
$$+ \frac{2\sqrt{D-1}}{D-2} P^{(0,sw)} + \frac{2\sqrt{D-1}}{D-2} P^{(0,ws)} \Bigg]_{\mu\nu,\rho\sigma} . \tag{34}$$

Using it, the propagator for $h_{\mu\nu}$ takes the form

$$\langle 0 | T(h_{\mu\nu}(x) h_{\rho\sigma}(y)) | 0 \rangle = i \mathcal{P}^{-1}_{\mu\nu,\rho\sigma} \delta^{(D)}(x-y). \tag{35}$$

Now we are willing to investigate the unitarity of the theory. One of the easiest way is to see the imaginary part of the residue of the tree-level amplitudes at the poles where the external sources are conserved, transverse stress-energy tensor. Then, the longitudinal operator $\omega_{\mu\nu}$ in the spin projector operators does not contribute, so only the projection operators $P^{(2)}$ and $P^{(0,s)}$ survive. Thus, the amplitude A takes the form in the momentum space

$$A = i T^{*\mu\nu} \left[\frac{2}{p^2 + \frac{1}{\xi}} P^{(2)} - \frac{2}{p^2} \left(P^{(2)} - \frac{1}{D-2} P^{(0,s)} \right) \right]_{\mu\nu,\rho\sigma} T^{\rho\sigma}$$
$$= i \left[\frac{2}{p^2 + \frac{1}{\xi}} \left(|T_{\mu\nu}|^2 - \frac{1}{D-1} |T^{\mu}_{\mu}|^2 \right) - \frac{2}{p^2} \left(|T_{\mu\nu}|^2 - \frac{1}{D-2} |T^{\mu}_{\mu}|^2 \right) \right] . \tag{36}$$

Since the stress-energy tensor $T_{\mu\nu}$ is now conserved and transverse, we can expand it in terms of the polarization vector ε^i_μ with $i = 1, 2, \cdots, D-2$ as $T_{\mu\nu} = t_{ij} \varepsilon^i_\mu \varepsilon^j_\nu$. Then the amplitude A can be rewritten as

$$A = i \left[\frac{2}{p^2 + \frac{1}{\xi}} \left(|t_{ij}|^2 - \frac{1}{D-1} |t^i_i|^2 \right) - \frac{2}{p^2} \left(|t_{ij}|^2 - \frac{1}{D-2} |t^i_i|^2 \right) \right] . \tag{37}$$

It is now straightforward to evaluate the imaginary part of the residue of the amplitude at the poles. First, at the massless pole corresponding to the massless graviton, we have

$$\mathrm{Im}\,\mathrm{Res}(A)|_{p^2=0} = -2 \left(|t_{ij}|^2 - \frac{1}{D-2} |t^i_i|^2 \right) . \tag{38}$$

This is obviously vanishing for $D = 3$ while it becomes negative for $D > 4$. This fact implies that there is no dynamical massless graviton in three dimensions whereas the massless graviton becomes a *ghost* for $D > 4$. On the other hand, at the massive pole corresponding to the massive graviton

$$\mathrm{Im}\,\mathrm{Res}(A)|_{p^2=-\frac{1}{\xi}} = 2 \left(|t_{ij}|^2 - \frac{1}{D-1} |t^i_i|^2 \right) , \tag{39}$$

which is positive for both $D = 3$ and $D > 4$. Therefore, the massive graviton with mass 'm' is a dynamical field with the positive norm. Accordingly, it is worth while to emphasize that the gravitational theory defined by the action (12) is free from the ghost and describes an interacting unitary massive gravity theory only in three space-time dimensions whereas it is not unitary in more than four dimensions. This fact is also certified by the observation that the action (21) includes the Einstein-Hilbert action with the *wrong* sign so that the corresponding massless graviton mode has a negative norm, which is non-dynamical only in three dimensions. It is remarkable that the ghost does not show up in three dimensions even if there is a propagator like $\frac{1}{\Box(\Box - \frac{1}{\xi})}$ which can be seen in Eq. (34).

In this short article, we have clarified unitarity of a massive gravity model in three space-time dimensions by Bergshoeff et al.[13] Although we can formally construct a sort of dual action (11) which connects the higher-derivative action (12) and the Pauli-Fierz massive gravity action (29) at the quadratic level via path integration, it has turned out that it is necessary to analyze the higher-derivative action (12) in some detail. We have pointed out that even if it seems to be possible to generalize the three-dimensional theory with a particular higher-derivative terms,[13] to higher dimensions except in four dimensions, only the three-dimensional theory provides a unitary theory of the massive graviton. This is because in three space-time dimensions the non-unitary massless graviton mode is not dynamical while in the other higher dimensions it becomes dynamical, thereby violating the unitarity of the theory.

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