Available online: September 26, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 2, Pages 139–146 (2018)
DOI: 10.1501/Commua1_0000000868
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

ON UNIVALENCE OF INTEGRAL OPERATORS

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Abstract. In this paper we consider functions of ψ_λ and we define integral operators denoted by $F_{\beta,\lambda}$ and $G_{\beta,\lambda}$ using by ψ_λ, then we proved sufficient conditions for univalence of these integral operators.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by S the subclass of A consisting of the functions $f \in A$ which are univalent in U.

Let ψ_λ defined by $\psi_\lambda(z) = (1-\lambda) f(z) + \lambda z f^{'}(z)$ for $z \in U, f \in A$ and $0 \leq \lambda \leq 1$. We consider the integral operators

$$F_{\beta,\lambda}(z) = \left[\beta \int_0^z u^{\beta-1} \psi_\lambda^{'}(u)\, du\right]^{\frac{1}{\beta}} \qquad (z \in U), \tag{1.1}$$

$$G_{\beta,\lambda}(z) = \int_0^z \left[\psi_\lambda^{'}(u)\right]^{\beta} du \quad (z \in U) \tag{1.2}$$

for $\psi_\lambda \in A, 0 \leq \lambda \leq 1$ and for some complex numbers β. In the present paper, we obtain new univalence conditions for the integral operators $F_{\beta,\lambda}$ and $G_{\beta,\lambda}$ to be in the class S.

Recently the problem of univalence of some generalized integral operators have discussed by many authors such as: (see [2]-[8], [10],[14]-[16])

Received by the editors: March 08, 2017, Accepted: June 19, 2017.

2010 *Mathematics Subject Classification.* Primary 30C45.

Key words and phrases. Analytic functions, univalent functions, univalance conditions, integral operators.

2. Preliminary Results

To discuss our problems for univalence of integral operators $F_{\beta,\lambda}$ and $G_{\beta,\lambda}$, we recall here some results.

Theorem 1. *Let $\alpha \in \mathbb{C}, \operatorname{Re}\alpha > 0$ and $f \in A$. If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in U$, then for any complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\, du\right]^{\frac{1}{\beta}}$$

is in the class S [12].

Theorem 2. *Let $f \in A$. If for all $z \in U$*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

then the function f is univalent in U [1].

Theorem 3. *If the function g is regular and $|g(z)| < 1$ in U, then for all $\eta \in U$ and $z \in U$ the following inequalities hold:*

$$\left|\frac{g(\eta)-g(z)}{1-\overline{g(z)}g(\eta)}\right| \leq \left|\frac{\eta - z}{1-\overline{z}\eta}\right| \tag{2.1}$$

and

$$\left|g'(z)\right| \leq \frac{1-|g(z)|^2}{1-|z|^2}.$$

In here, the equalities hold only in the case $g(z) = \varepsilon\frac{z+u}{1+\overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$ [9].

Remark 1. *For $z = 0$ and all $\eta \in U$, from inequality (2.1) we obtain*

$$\left|\frac{g(\eta)-g(0)}{1-\overline{g(0)}g(\eta)}\right| \leq |\eta|$$

and, hence

$$|g(\eta)| \leq \frac{|\eta| + |g(0)|}{1+|g(0)|\,|\eta|}.$$

Considering $g(0) = a$ and $\eta = z$, then

$$|g(z)| \leq \frac{|z|+|a|}{1+|a|\,|z|}$$

for all $z \in U$ [9].

Theorem 4. *Let β be a complex number,* $\operatorname{Re}\beta \geq 1$ *and* $f \in A$, $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$. *If there exist a constant* $K \in (0, m(r)]$, *where*

$$m(r) = \frac{1 - 2|a_2| r \left(1 - r^2\right) + \sqrt{\left[1 - 2|a_2| r \left(1 - r^2\right)\right]^2 + 8|a_2| r^3 \left(1 - r^2\right)}}{2r^2 \left(1 - r^2\right)}$$

$r = |z|, r \in (0, 1)$ *such that*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq K$$

for all $z \in U^* = U - \{0\}$, *then the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta - 1} f'(u)\, du \right]^{\frac{1}{\beta}}$$

is regular and univalent in U^* [11].

Theorem 5. *Let* $\beta \in \mathbb{C}$ *and* $g \in A$. *If*

$$\left| \frac{g''(z)}{g'(z)} \right| < 1$$

for all $z \in U$ *and the constant* $|\beta|$ *satisfies the condition*

$$|\beta| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\left(1 - |z|^2\right) |z| \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \right]}$$

then the function

$$G_\beta(z) = \int_0^z \left[g'(u) \right]^\beta du$$

is univalent in U [13].

3. Main Results

Theorem 6. *Let* $\beta \in \mathbb{C}$, $\operatorname{Re}\beta \geq 1$ *and* ψ_λ *a regular function in* U, $\frac{\psi_\lambda(z)}{z} \neq 0$ *for all* $z \in U$. *If there exist a constant* $K \in (0, m(r)]$, *where*

$$m(r) = \frac{1 - 2(1+\lambda)|a_2| r \left(1 - r^2\right) + \sqrt{\left[1 - 2(1+\lambda)|a_2| r \left(1 - r^2\right)\right]^2 + 8(1+\lambda)|a_2| r^3 \left(1 - r^2\right)}}{2r^2 \left(1 - r^2\right)} \tag{3.1}$$

$r = |z|, r \in (0, 1)$ *such that*

$$\left| \frac{\psi_\lambda''(z)}{\psi_\lambda'(z)} \right| \leq K$$

for all $z \in U^*$, *then the function (1.1) is regular and univalent in* U^*.

Proof. Let's consider the function $g(z) = \frac{1}{K}\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}$ where K is a real positive constant. Applying Theorem3 and Remark1 to the function g, we obtain

$$\left|\frac{1}{K}\frac{\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| \leq \frac{|z| + \frac{2(1+\lambda)|a_2|}{K}}{1 + \frac{2(1+\lambda)|a_2|}{K}|z|}, \quad z \in U^*$$

and hence, we have

$$\left(1-|z|^2\right)\left|\frac{z\psi_\lambda''(z)}{\psi_\lambda'(z)}\right| \leq K\left(1-|z|^2\right)|z|\frac{|z| + \frac{2(1+\lambda)|a_2|}{K}}{1 + \frac{2(1+\lambda)|a_2|}{K}|z|}. \tag{3.2}$$

Let's consider the inequality

$$K \leq \frac{1}{\left(1-|z|^2\right)|z|\frac{|z|+\frac{2(1+\lambda)|a_2|}{K}}{1+\frac{2(1+\lambda)|a_2|}{K}|z|}}. \tag{3.3}$$

Considering $|z| = r, r \in (0,1)$ and $2|a_2| = p, p > 0$, the inequality (3.3) becomes

$$K \leq \frac{K + (1+\lambda)pr}{(1-r^2)r[Kr + (1+\lambda)p]}. \tag{3.4}$$

We note that

$$\left(1-r^2\right)r[Kr + (1+\lambda)p] > 0 \tag{3.5}$$

for every $K > 0, p > 0, r \in (0,1)$ and $0 \leq \lambda \leq 1$. Using (3.5) the inequality (3.4) becomes

$$r^2\left(1-r^2\right)K^2 + \left[(1+\lambda)pr\left(1-r^2\right) - 1\right]K - (1+\lambda)pr \leq 0.$$

Let us consider the equation

$$r^2\left(1-r^2\right)K^2 + \left[(1+\lambda)pr\left(1-r^2\right) - 1\right]K - (1+\lambda)pr = 0, \tag{3.6}$$

with the unknown K. From (3.6) we obtain

$$K_{1,2} = \frac{1-(1+\lambda)pr\left(1-r^2\right) \pm \sqrt{\left[1-(1+\lambda)pr\left(1-r^2\right)\right]^2 + 4(1+\lambda)pr^3\left(1-r^2\right)}}{2r^2\left(1-r^2\right)}. \tag{3.7}$$

For every $p > 0$, $r \in (0,1)$ and $0 \leq \lambda \leq 1$ the following inequality holds

$$\left[1-(1+\lambda)pr\left(1-r^2\right)\right]^2 + 4(1+\lambda)pr^3\left(1-r^2\right) > 0. \tag{3.8}$$

Using (3.7) and (3.8) it results that K_1, K_2 are real solutions. Considering $a = 1-r^2$, $a \in (0,1)$ and $b = pr$, $b > 0$ from (3.7) we get

$$K_{1,2} = \frac{1-(1+\lambda)ab \pm \sqrt{[1-(1+\lambda)ab]^2 + 4(1+\lambda)ab(1-a)}}{2a(1-a)}. \tag{3.9}$$

□

We have the following cases:

Case 1. *For* $|a_2| > \frac{1}{2(1+\lambda)r(1-r^2)}$ *it results that* $1-(1+\lambda)\,ab<0$, *so that*

$$K_1 = \frac{1-(1+\lambda)\,ab - \sqrt{\left[1-(1+\lambda)\,ab\right]^2 + 4\,(1+\lambda)\,ab\,(1-a)}}{2a\,(1-a)}$$

is real negative solution. Clearly,

$$K_2 = \frac{1-(1+\lambda)\,ab + \sqrt{\left[1-(1+\lambda)\,ab\right]^2 + 4\,(1+\lambda)\,ab\,(1-a)}}{2a\,(1-a)}$$

is real positive solution. In this case, for $K \in (0, K_2]$ *the inequality (3.3) is verified.*

Case 2. *For* $|a_2| < \frac{1}{2(1+\lambda)r(1-r^2)}$ *it results that* $1-(1+\lambda)\,ab>0$.

Let's prove that $K_1 < 0$. Supposing that $K_1 > 0$, we obtain $4\,(1+\lambda)\,ab\,(1-a) < 0$ the fact which is false. It results that $K_1 < 0$. We note that $K_2 > 0$, and the inequality (3.3) is verified for $K \in (0, K_2]$.

Case 3. *For* $|a_2| = \frac{1}{2(1+\lambda)r(1-r^2)}$ *using (3.9) we obtain*

$$K_{1,2} = \frac{\pm\sqrt{(1+\lambda)\,ab\,(1-a)}}{a\,(1-a)}$$

and the inequality (3.3) is verified only for $K \in (0, K_2]$ *where*

$$K_2 = \frac{\sqrt{(1+\lambda)\,ab\,(1-a)}}{a\,(1-a)}.$$

Considering equality (3.1) in conclusion for $|a_2|$, r *stable and* $K \in (0, m\,(r)]$, *the inequality (3.3) is verified and using (3.2) it results that*

$$\left(1-|z|^2\right)\left|\frac{z\psi_\lambda^{''}(z)}{\psi_\lambda^{'}(z)}\right| \le 1, z \in U^*. \tag{3.10}$$

From (3.10) and Theorem1 in the case $\alpha = 1$ *we obtain that the function* $F_{\beta,\lambda}(z)$ *is regular and univalent in* U^*.

Theorem 7. *Let* β *be a complex number and the function* $\psi_\lambda \in A, \psi_\lambda(z) = (1-\lambda)\,f(z) + \lambda z f^{'}(z)$ *for* $f \in A$ *and* $0 \le \lambda \le 1$. *If*

$$\left|\frac{\psi_\lambda^{''}(z)}{\psi_\lambda^{'}(z)}\right| < 1 \tag{3.11}$$

for all $z \in U$ *and the constant* $|\beta|$ *satisfies the condition*

$$|\beta| \le \frac{1}{\max\limits_{|z|\le 1}\left[\left(1-|z|^2\right)|z|\,\frac{|z|+2(1+\lambda)|a_2|}{1+2(1+\lambda)|a_2||z|}\right]} \tag{3.12}$$

then the function $G_{\beta,\lambda}$ *is univalent in* U.

Proof. The function $G_{\beta,\lambda}$ defined by (1.2) is regular in U. Let us consider the function

$$p(z)=\frac{1}{|\beta|}\frac{G''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)} \tag{3.13}$$

where the constant $|\beta|$ satisfies the inequality (3.12). The function p is regular in U and from (1.2) and (3.13) we have

$$p(z)=\frac{\beta}{|\beta|}\frac{\psi''_{\lambda}(z)}{\psi'_{\lambda}(z)}. \tag{3.14}$$

Using (3.14) and (3.11) we obtain

$$|p(z)|<1$$

for all $z\in U$ and $|p(0)|=2(1+\lambda)|a_2|$. When Remark1 applied to the function p, it gives

$$\frac{1}{|\beta|}\frac{G''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)}\leq\frac{|z|+2(1+\lambda)|a_2|}{1+2(1+\lambda)|a_2||z|} \tag{3.15}$$

for all $z\in U$. From (3.15) we get

$$\left(1-|z|^2\right)\left|\frac{zG''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)}\right|\leq|\beta|\left(1-|z|^2\right)|z|\frac{|z|+2(1+\lambda)|a_2|}{1+2(1+\lambda)|a_2||z|}$$

for all $z\in U$. Hence we have

$$\left(1-|z|^2\right)\left|\frac{zG''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)}\right|\leq|\beta|\max_{|z|\leq1}\left(1-|z|^2\right)|z|\frac{|z|+2(1+\lambda)|a_2|}{1+2(1+\lambda)|a_2||z|}. \tag{3.16}$$

From (3.16) and (3.12) we obtain

$$\left(1-|z|^2\right)\left|\frac{zG''_{\beta,\lambda}(z)}{G'_{\beta,\lambda}(z)}\right|\leq1$$

for all $z\in U$. From Theorem2, it follows that the function $G_{\beta,\lambda}$ defined by (1.2) is univalent in U. □

Remark 2. *Taking $\lambda=0$ in Theorem6 and Theorem7, we obtain Theorem4 and Theorem5, respectively.*

If we take $\lambda=1$ in Theorem6 and Theorem7, we have the following corollaries.

Corollary 1. *Let β be a complex number, $\operatorname{Re}\beta\geq1$ and ψ_1 a regular function in $U,\psi_1(z)=zf'(z)$ and $\frac{\psi_1(z)}{z}\neq0$ for all $z\in U$. If there exist a constant $K\in(0,m(r)]$, where*

$$m(r)=\frac{1-4|a_2|r\left(1-r^2\right)+\sqrt{\left[1-4|a_2|r\left(1-r^2\right)\right]^2+16|a_2|r^3\left(1-r^2\right)}}{2r^2\left(1-r^2\right)},$$

$r=|z|, r\in(0,1]$ *such that*

$$\left|\frac{\psi_1^{''}(z)}{\psi_1^{'}(z)}\right|=\left|\frac{f''(z)}{f'(z)}\right|\leq K$$

for all $z\in U^*$, *then the function*

$$F_{\beta,1}(z)=\left[\beta\int_0^z u^{\beta-1}\psi_1^{'}(u)\,du\right]^{\frac{1}{\beta}}$$

is regular and univalent in U^*.

Corollary 2. *Let* β *be a complex number and the function* $\psi_1(z)=zf^{'}(z)$ *where* $f\in A$. *If*

$$\left|\frac{\psi_1^{''}(z)}{\psi_1^{'}(z)}\right|<1$$

for all $z\in U$ *and the constant* $|\beta|$ *satisfies the condition*

$$|\beta|\leq\frac{1}{\max\limits_{|z|\leq 1}\left[\left(1-|z|^2\right)|z|\,\frac{|z|+4|a_2|}{1+4|a_2||z|}\right]}$$

then the function

$$G_{\beta,1}(z)=\int_0^z\left[\psi_1^{'}(u)\right]^{\beta}du$$

is univalent in U.

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