

# On universal Severi varieties of low genus $K3$ surfaces

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## Abstract

We prove the irreducibility of universal Severi varieties parametrizing irreducible, reduced, nodal hyperplane sections of primitive  $K3$  surfaces of genus  $g$ , with  $3 \leq g \leq 11$ ,  $g \neq 10$ .

## Introduction

F. Severi was one of the first algebraic geometers who stressed the importance, for moduli and enumerative problems, of studying the families  $V_{d,g}$  of irreducible, nodal, plane curves of degree  $d$  and geometric genus  $g$  (see e.g. [29, Anhang F]). In particular he proved that the varieties  $V_{d,g}$  all have the *expected dimension*  $3d + g - 1$  (equal to the dimension of the linear system of all curves of degree  $d$  minus the number of nodes) and asserted, but did not succeed to prove, that they are irreducible, a result due to J. Harris in [16]. For this reason, the  $V_{d,g}$ 's have been called *Severi varieties*.

The notion of Severi variety can be extended to families of nodal curves on any surface, and analogous irreducibility problems naturally arise. These are in general hard questions, even for rational surfaces. For instance, irreducibility is known to hold for Hirzebruch surfaces [32] and for rational curves on Del Pezzo's surfaces [31], with one notable (and understood) exception for Del Pezzo's surfaces of degree 1. On the other hand, for surfaces with positive canonical bundle, Severi varieties have in general a quite unpredictable behaviour: examples are given in [6] of surfaces with reducible Severi varieties, and even with components of Severi varieties of dimension different from the expected one.

In this note we concentrate on Severi varieties on  $K3$  surfaces, which, as in the planar case, are quite interesting in relation with modular and enumerative problems. Here one can consider *universal* Severi varieties parametrizing irreducible, reduced, nodal curves on primitive  $K3$  surfaces, i.e. those polarised by an indivisible, ample line bundle of genus  $g$ . Conjecturally, all these varieties should be irreducible (see §1.3). This however seems, at the moment, to be quite hard to prove. Our result is Theorem 2.1 asserting the irreducibility in the range  $3 \leq g \leq 11$ ,  $g \neq 10$ . This partially affirmatively answers a question posed by the second author in [13] and it is, as far as we know, the first irreducibility result for Severi varieties of  $K3$  surfaces. In a nutshell, the proof relies on two facts. First, in the asserted range, the hyperplane sections with a given number  $\delta$  of nodes of the  $K3$ 's in question, embedded in  $\mathbf{P}^g$  as surfaces of degree  $2g - 2$ , fill up the whole component of nodal degenerate canonical curves with  $\delta$  nodes in the Hilbert scheme (see Proposition 2.6 or [14]). Secondly, using a degeneration technique due to Pinkham [24], we prove that all components of a certain flag Hilbert scheme pass through some *cone points*, where, on the other hand, we are able to prove smoothness of the flag Hilbert scheme, which is then irreducible at those points. Both ideas are inspired by [9, 11].

In §1 we recall general facts about  $K3$  surfaces (see §1.1); some basics about Severi varieties on them, like existence and dimensions (see §1.2); about universal Severi varieties, recently considered in [14] (see §1.3), and related moduli problems (see §1.4). Statement and proof of Theorem 2.1 are in §2. We take a Hilbert schematic viewpoint, which we set up in §2.1. We recall then Pinkham's technique of degeneration to cones in §2.2, and the use of graph curves to compute cohomology of normal bundles in §2.3. Applying this machinery, the proof, presented in §2.4, turns out to be quite simple.

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## 1 $K3$ surfaces and their Severi varieties

### 1.1 Generalities

A  $K3$  surface  $X$  is a smooth complex projective surface with  $\Omega_X^2 \cong \mathcal{O}_X$  and  $h^1(X, \mathcal{O}_X) = 0$ . A *primitive  $K3$  surface of genus  $g$*  is a pair  $(X, L)$ , where  $X$  is a  $K3$  surface, and  $L$  is an indivisible, nef line bundle

on  $X$ , such that  $|L|$  is without fixed component and  $L^2 = 2g - 2$  (hence  $g \geq 1$ ). Given such a pair,  $|L|$  is base point free, and the morphism  $\varphi_{|L|}$  determined by this linear system is birational if and only if  $L^2 > 0$  and  $|L|$  does not contain any hyperelliptic curve (hence  $g \geq 3$ ). In the latter case, the image of  $\varphi_{|L|}$  is a surface of degree  $2g - 2$  in  $\mathbf{P}^g$ , with canonical singularities, and whose general hyperplane section is a *canonical curve* of genus  $g$  (see [27]).

For all  $g \geq 2$ , we can consider the *moduli stack*  $\mathcal{B}_g$  of primitive  $K3$  surfaces of genus  $g$ , which is smooth, of dimension 19 (see [1, 23]). For  $(X, L)$  very general in  $\mathcal{B}_g$ , the Picard group of  $X$  is generated by the class of  $L$ , and  $L$  is very ample if  $g \geq 3$ .

## 1.2 Severi varieties

Given a  $K3$  surface  $(X, L)$  of genus  $g$  and two integers  $k$  and  $h$ , consider

$$V_{k,h}(X, L) := \{C \in |kL| \text{ irreducible and nodal with } g(C) = h\},$$

where  $g(C)$  is the *geometric genus* of  $C$ , i.e. the genus of its normalization, so that  $C$  has  $g - h$  nodes.  $V_{k,h}(X, L)$ , called the  $(k, h)$ -*Severi variety* of  $(X, L)$  (or simply *Severi variety* if there is no danger of confusion). It is a functorially defined, locally closed subscheme of the projective space  $|kL|$  of dimension  $1 + k^2(g - 1) =: p_a(k)$ , which is the arithmetic genus of the curves in  $|kL|$ . We will drop the index  $k$  if  $k = 1$  and we may drop the indication of the pair  $(X, L)$  if there is no danger of confusion.

**Theorem 1.1** *Let  $k \geq 1$  and  $0 \leq h \leq p_a(k)$ . The variety  $V_{k,h}$ , if not empty, is smooth of dimension  $h$ . If  $(X, L)$  is general in  $\mathcal{B}_g$ , then  $V_{k,h}$  is not empty.*

The first assertion is classical and standard in deformation theory (see [29] and, more recently, e.g. [6, 13, 30]). The second part is a consequence of the main theorem in [7] (see also Mumford's theorem in [1, pp. 365–367]).

If  $(X, L)$  is general,  $V_{k,0}$  is reducible, consisting of a finite number of points (for the degree of  $V_{k,0}$ , see [4, 33]). One might instead expect that if  $(X, L)$  is general and  $h \geq 1$ , then  $V_{k,h}$  is irreducible. This is trivially true for  $h = p_a(k)$  and not difficult for  $h = p_a(k) - 1$  (the reader may easily figure out why), but is complicated as soon as  $h$  gets lower. This conjecture, if true, certainly will not be easy to prove. As a first approximation, one may propose a weaker irreducibility conjecture concerning *universal Severi varieties* (see [13]), which we now recall.

## 1.3 Universal Severi varieties

For any  $g \geq 2$ ,  $k \geq 1$  and  $0 \leq h \leq p_a(k)$ , one can consider a stack  $\mathcal{V}_{k,h}^g$  (see [14, Proposition 4.8]), called the *universal Severi variety*, which is pure and smooth of dimension  $19 + h$ , endowed with a morphism  $\phi_{k,h}^g : \mathcal{V}_{k,h}^g \rightarrow \mathcal{B}_g^\circ$ , where  $\mathcal{B}_g^\circ$  is a suitable dense open substack of  $\mathcal{B}_g$ . The morphism  $\phi_{k,h}^g$  is smooth on all components of  $\mathcal{V}_{k,h}^g$ , and its fibres are described in the following diagram:

$$\begin{array}{ccc} \mathcal{V}_{k,h}^g & \supset & V_{k,h}(X, L) \\ \phi_{k,h}^g \downarrow & & \downarrow \\ \mathcal{B}_g^\circ & \ni & (X, L) \end{array}$$

Thus a point of  $\mathcal{V}_{k,h}^g$  can be regarded as a pair  $(X, C)$  with  $(X, L) \in \mathcal{B}_g$  and  $C \in V_{k,h}(X, L)$ .

One can conjecture that *all universal Severi varieties  $\mathcal{V}_{k,h}^g$  are irreducible*. This does not imply the irreducibility of the *pointwise Severi varieties*  $V_{k,h}(X, L)$ , even if  $(X, L)$  is general in  $\mathcal{B}_g$ . The conjecture rather means that the monodromy of the morphism  $\phi_{k,h}^g$  transitively permutes the components of the fibre  $V_{k,h}(X, L)$ , for  $(X, L) \in \mathcal{B}_g$  general. This makes sense even if  $h = 0$ , when the pointwise Severi variety  $V_{k,0}(X, L)$  is certainly reducible.

In addition to its intrinsic interest, this conjecture is motivated by the results in [13], where it is shown that (a weak version of) it implies the non-existence of rational map  $f : X \dashrightarrow X$  with  $\deg(f) > 1$  for a general  $K3$  surface  $(X, L)$  of a given genus  $g$ . Very recently a proof of this result, based on quite delicate degeneration argument, has been proposed by Xi Chen [8].

## 1.4 The moduli map

There is a natural *moduli map*  $\mu_{k,h}^g : \mathcal{V}_{k,h}^g \rightarrow \mathcal{M}_h$ , where  $\mathcal{M}_h$  is the *moduli stack* of curves of genus  $h$ . The case  $k = 1, h = g$  has been much studied. It is related to the behaviour of the *Wahl map*  $w_C : \bigwedge^2 H^0(C, \omega_C) \rightarrow H^0(C, \omega_C^3)$  of a smooth curve  $C$  of genus  $g$ , to extension properties of canonical curves and to the classification of *Fano varieties of the principal series* and of *Mukai varieties*. We will not dwell recalling all results on this subject, deferring the reader to the current literature (see, in chronological order, [20, 34, 3, 21, 11, 12, 22, 9, 10, 5]). Only recently the nodal case  $h < g, k = 1$ , received the deserved attention. We recall the following theorem.

**Theorem 1.2** *Assume  $3 \leq g \leq 11$  and  $0 \leq h \leq g$ . For any irreducible component  $\mathcal{V}$  of  $\mathcal{V}_h^g$ , the moduli map  $\mu_{h|\mathcal{V}}^g : \mathcal{V} \rightarrow \mathcal{M}_h$  is dominant, unless  $g = h = 10$ .*

The case  $h = g$  is in the series of papers [20, 21, 22] (see also [5]). The rest is in [14].

**Remark 1.3** As stated in [14], the theorem applies only for  $h \geq 2$ . The case  $h = 0$  is trivially true. The proof in [14] applies to the case  $h = 1$  if  $3 \leq g \leq 11$  and  $g \neq 10$  as well. The case  $h = 1, g = 10$  is not covered by the original argument (see also [14, last lines of the proof of Theorem 5.5]), but can also be fixed. We do not dwell on this here.

In the recent paper [15], the moduli map  $\mu_{k,h}^g$  has been studied also for  $g \geq 13$ , any  $k$  and  $h$  sufficiently large with respect to  $g$ , proving that, as one may expect,  $\mu_{k,h}^g$  is generically finite to its image in these cases. The remaining cases for  $g, h, k$  are very interesting and still widely open.

## 2 The main theorem

The aim of this paper is to prove the following result, which affirmatively answers the conjecture in §1.3 in some cases.

**Theorem 2.1** *For  $3 \leq g \leq 11, g \neq 10$  and  $0 \leq h \leq g$ , the universal Severi variety  $\mathcal{V}_h^g$  is irreducible.*

By adopting a Hilbert schematic viewpoint and inspired by [11], we find a flag Hilbert scheme  $\mathcal{F}_{g,h}$ , with a rational map  $\mathcal{F}_{g,h} \dashrightarrow \mathcal{V}_h^g$  dominating all components of  $\mathcal{V}_h^g$ , and we prove that  $\mathcal{F}_{g,h}$  is irreducible (see Theorem 2.2). To show this, we exhibit smooth points of  $\mathcal{F}_{g,h}$  which are contained in all irreducible components of  $\mathcal{F}_{g,h}$  (see §2.4).

### 2.1 The Hilbert schematic viewpoint

For any  $g \geq 3$ , we let  $\mathcal{B}_g$  be the component of the Hilbert scheme of surfaces in  $\mathbf{P}^g$  whose general point parametrizes a primitive K3 surface of genus  $g$ . An open subset of  $\mathcal{B}_g$  is a  $\mathrm{PGL}(g+1, \mathbf{C})$ -bundle over the open subset of  $\mathcal{B}_g$  corresponding to pairs  $(X, L)$  with very ample  $L$ . The variety  $\mathcal{B}_g$  is therefore irreducible of dimension  $g^2 + 2g + 19$ .

Let  $\mathcal{C}_g$  be the component of the Hilbert scheme of curves in  $\mathbf{P}^g$  whose general point parametrizes a *degenerate* canonical curve of genus  $g$ , i.e. a smooth canonical curve of genus  $g$  lying in a hyperplane of  $\mathbf{P}^g$ . An open subset of  $\mathcal{C}_g$  is a  $|\mathcal{O}_{\mathbf{P}^g}(1)| \times \mathrm{PGL}(g, \mathbf{C})$ -bundle over the open subset of  $\mathcal{M}_g$  parametrizing non-hyperelliptic curves, so  $\mathcal{C}_g$  is irreducible of dimension  $g^2 + 4g - 4$ .

Let  $\mathcal{F}_g$  be the component of the *flag Hilbert scheme* of  $\mathbf{P}^g$  (see [17, 28]) whose general point is a pair  $(X, C)$  with  $X \in \mathcal{B}_g$  general and  $C \in \mathcal{C}_g$  a general hyperplane section of  $X$ . An open subset of  $\mathcal{F}_g$  is a  $\mathbf{P}^g$ -bundle over an open subset of  $\mathcal{B}_g$ . As such it is irreducible of dimension  $g^2 + 3g + 19$ .

Let  $0 \leq h \leq g$ . We denote by  $\mathcal{C}_{g,h}$  the Zariski closure of the locally closed, functorially defined, subset of  $\mathcal{C}_g$  formed by irreducible, nodal, genus  $h$  curves. It comes with a moduli map  $c_{g,h} : \mathcal{C}_{g,h} \dashrightarrow \mathcal{M}_h$ , which is dominant. Up to projective transformations, the fibre over a curve  $C \in \mathcal{M}_h$  is a dense open subset of  $\mathrm{Sym}^\delta(\mathrm{Sym}^2(C))$ , with  $\delta = g - h$ , and is therefore irreducible. So  $\mathcal{C}_{g,h}$  is irreducible, of dimension  $g^2 + 4g - 4 - \delta$ .

We let  $\mathcal{F}_{g,h}$  be the inverse image of  $\mathcal{C}_{g,h}$  under the projection  $\mathcal{F}_g \rightarrow \mathcal{C}_g$ . We have a natural dominant map  $m_{g,h} : \mathcal{F}_{g,h} \dashrightarrow \mathcal{V}_h^g$ . Any irreducible component  $\mathcal{F}$  of  $\mathcal{F}_{g,h}$  dominates  $\mathcal{B}_g$  via the restriction of the projection  $\mathcal{F}_g \rightarrow \mathcal{B}_g$  (see §§1.2 and 1.3), and has dimension

$$\dim(\mathcal{F}) = \dim(\mathcal{F}_{g,h}) = g^2 + 3g + 19 - \delta,$$

where again  $\delta = g - h$ . We let  $p_{g,h} : \mathcal{F}_{g,h} \rightarrow \mathcal{C}_{g,h}$  be the natural projection and we use the shorter notation  $p_g$  for  $p_{g,g}$ .

Because of the existence of the dominant map  $m_{g,h}$ , the following implies Theorem 2.1.

**Theorem 2.2** *Let  $3 \leq g \leq 11$ ,  $g \neq 10$ , and  $0 \leq h \leq g$ . Then  $\mathcal{F}_{g,h}$  is irreducible.*

For the proof, we need to recall a few facts, collected in the next two subsections.

## 2.2 Degenerations to cones

The following lemma relies on a well known construction of Pinkham [24, (7.7)], and is based on the fact that smooth K3 surfaces are *projectively Cohen–Macaulay*, see [19, 27].

**Lemma 2.3** *Let  $(X, C) \in \mathcal{F}_{g,h}$  with  $X$  a smooth K3 surface. Let  $X_C$  be the cone over  $C$  from a point  $v$  in  $\mathbf{P}^g$  off the hyperplane in which  $C$  sits. Then one can flatly degenerate  $(X, C)$  to  $(X_C, C)$  inside the fibre  $F_C$  of  $p_{g,h}$  over  $C$ .*

**Proof.** Let  $H$  be the hyperplane containing  $C$ . Choose homogeneous coordinates  $(x_0 : \dots : x_g)$  such that  $v = (1 : 0 : \dots : 0)$  and  $H$  is given by  $x_0 = 0$ . Consider the projective transformation  $\omega_t$ ,  $t \neq 0$ , such that  $\omega_t(x_0 : \dots : x_g) = (tx_0 : x_1 : \dots : x_g)$ . Set  $X_t = \omega_t(X)$ . Then  $(X_t, C) \in F_C$  for all  $t \neq 0$ . Since  $X$  is projectively Cohen–Macaulay,  $X_C$ , with its reduced structure, is the flat limit of  $X_t$  when  $t$  tends to 0.  $\square$

The fibre  $F_C$  of  $p_{g,h}$  equals the fibre of  $p_g$ , whose tangent space at the point  $(X, C)$  is isomorphic to  $H^0(X, N_{X/\mathbf{P}^g}(-1))$  (see e.g. [28, §4.5.2]). The next lemma computes this space at a *cone point*  $(X_C, C)$  (the proof is the same as in [24, Theorem 5.1], and relies on the fact that  $C$  is projectively Cohen–Macaulay, see [25, 18, 26]).

**Lemma 2.4** *Let  $C$  be a reduced and irreducible, not necessarily smooth, degenerate canonical curve in  $\mathbf{P}^g$ , of arithmetic genus  $g$ . Let  $X_C$  be the cone over  $C$  from a point in  $\mathbf{P}^g$  off the hyperplane in which  $C$  sits. For all  $i \geq 0$ , one has*

$$H^0(X_C, N_{X_C/\mathbf{P}^g}(-i)) \cong \bigoplus_{k \geq i} H^0(C, N_{C/\mathbf{P}^{g-1}}(-k)). \quad (2.1)$$

Next we need to bound from above the dimensions of the cohomology spaces appearing in the right-hand-side of (2.1). We use semi-continuity, and a special type of canonical curves for which they can be computed.

## 2.3 Canonical graph curves

A *graph curve* of genus  $g$  is a stable curve of genus  $g$  consisting of  $2g - 2$  irreducible components of genus 0 (see [2, 11]). A graph curve has  $3g - 3$  nodes (three nodes for each component), and it is determined by the dual *trivalent* graph, consisting of  $2g - 2$  nodes and  $3g - 3$  edges. If  $C$  is a graph curve and its dualizing sheaf  $\omega_C$  is very ample, then  $C$  can be canonically embedded in  $\mathbf{P}^{g-1}$  as a union of  $2g - 2$  lines, each meeting three others at distinct points. This is a *canonical graph curve*.

**Proposition 2.5** [11] *For  $3 \leq g \leq 11$ ,  $g \neq 10$ , there exists a genus  $g$  canonical graph curve  $\Gamma_g$  in  $\mathbf{P}^{g-1}$ , sitting in the image of  $p_g$ , such that the dimensions of the spaces of sections of negative twists of the normal bundle are given in the following table:*

$h^0(N_{\Gamma_g/\mathbf{P}^{g-1}}(-k)) \setminus g$	3	4	5	6	7	8	9	11
$k = 1$	10	13	15	16	16	15	14	12
$k = 2$	6	5	3	1	0	0	0	0
$k = 3$	3	1	0	0	0	0	0	0
$k = 4$	1	0	0	0	0	0	0	0
$k \geq 5$	0 for every $g$							

hence

$$\sum_{k \geq 1} h^0(\Gamma_g, N_{\Gamma_g/\mathbf{P}^{g-1}}(-k)) = 23 - g. \quad (2.2)$$

## 2.4 Proof of the main theorem

Here we prove Theorem 2.2 and therefore also Theorem 2.1. The first step is the following:

**Proposition 2.6** *Let  $g$  and  $h$  be two integers such that  $3 \leq g \leq 11$ ,  $g \neq 10$ , and  $0 \leq h \leq g$ . Let  $\mathcal{F}$  be a component of  $\mathcal{F}_{g,h}$  and let  $(X, C) \in \mathcal{F}$  be a general point. Then all components of the fibre  $F_C$  of  $p_{g,h}$  over  $C$  have dimension  $23 - g$ , and the restriction of  $p_{g,h}$  to  $\mathcal{F}$  is dominant onto  $\mathcal{C}_{g,h}$ .*

**Proof.** Note that  $X$  is general in  $\mathcal{B}_g$  (see §2.1). As we saw,  $F_C$  equals the fibre of  $p_g$ , whose tangent space at  $(X, C)$  is isomorphic to  $H^0(X, N_{X/\mathbf{P}^g}(-1))$ . Its dimension does not depend on the hyperplane section  $C$  of  $X$ . So this is like computing the tangent space to the fibre of  $p_g$  at a general point  $(X, \bar{C})$  of  $\mathcal{F}_g$ , with  $\bar{C}$  general in  $\mathcal{C}_g$  by the case  $h = g$  of Theorem 1.2.

By degenerating to the cone point  $(X_{\bar{C}}, \bar{C})$ , by Lemma 2.4, and by upper-semi-continuity, we have  $h^0(X, N_{X/\mathbf{P}^g}(-1)) \leq \sum_{k \geq 1} h^0(\bar{C}, N_{\bar{C}/\mathbf{P}^{g-1}}(-k))$ . By further degenerating to one of the graph curves in Proposition 2.5, and taking into account (2.2), we have  $h^0(X, N_{X/\mathbf{P}^g}(-1)) \leq 23 - g$  (this argument has been extracted from [9, §5.3]). So this is an upper bound for the dimension of  $F_C$  at  $(X, C)$ . Since

$$23 - g = \dim(\mathcal{F}) - \dim(\mathcal{C}_{g,h})$$

(see §2.1), this equals the dimension of  $F_C$  at  $(X, C)$ , and the restriction of  $p_{g,h}$  at  $\mathcal{F}$  is dominant.  $\square$

With a similar argument we can finish the:

**Proof of Theorem 2.2.** Let  $\mathcal{F}_i$ ,  $1 \leq i \leq 2$ , be distinct components of  $\mathcal{F}_{g,h}$ . Let  $C \in \mathcal{C}_{g,h}$  be a general point. By Proposition 2.6, there are points  $(X_i, C) \in \mathcal{F}_i$ , and they can be assumed to be general points on two distinct components  $F_i$  of  $F_C$ ,  $1 \leq i \leq 2$ . By Lemma 2.3, both  $F_1$  and  $F_2$  contain the cone point  $(X_C, C)$ . We will reach a contradiction by showing that  $(X_C, C)$  is a smooth point of  $F_C$ .

Since  $C$  is general in  $\mathcal{C}_{g,h}$  and  $\mathcal{C}_{g,h}$  clearly contains the graph curves  $\Gamma_g$  of Proposition 2.5, by upper-semicontinuity  $h^0(X_C, N_{X_C/\mathbf{P}^g}(-1))$  is bounded from above by (2.2). This proves the asserted smoothness of  $F_C$  at  $(X_C, C)$ , concluding the proof.  $\square$

**Remark 2.7** (i) Proposition 2.6 gives a quick alternative proof of the part  $h < g$  of Theorem 1.2 when  $g \neq 10$ , which is based on the part  $h = g$ .

(ii) The argument does not work for  $g = 10$ . In fact, if  $C$  is any curve in the image of  $p_{10}$  lying on a smooth  $K3$  surface, one has  $h^0(X, N_{X/\mathbf{P}^g}(-1)) = 14$ , see [12, Lemma 1.2]. The analogue of Proposition 2.6 in this case is that all components of a general fibre of  $p_{10,h}$  have dimension 14. So the image of  $p_{g,h}$  has codimension 1 in  $\mathcal{C}_{g,h}$ . Luckily, and as one could expect, Theorem 1.2 ensures that the moduli map  $c_{g,h}$  dominates  $\mathcal{M}_h$  for  $0 \leq h \leq 9$ . However the argument in the final part of the proof of Theorem 2.2 falls short, since we do not know whether the image of  $p_{10,h}$  is irreducible, or all of its components contain a curve  $C$  for which the fibre  $F_C$  can be controlled.

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